Fixed points for singlevalued operators with respect to tau-distance

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Abstract. In this paper we first recall the concept of τ-distance on a metric space. Then, we prove a fixed point theorem for singlevalued operators in terms of a τ-distance.

Key words: fixed point, τ-distance, singlevalued operator.

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1 INTRODUCTION

In 2001 T.Suzuki introduced the concept of τ-distance on a metric space. They gave some examples of τ-distance and improve the generalization of Banach contraction principle, Caristi's fixed point theorem, Ekeland's variational principle and the Takahashi's nonconvex minimization theorem, see [1], [2]. Also, some fixed point theorems for singlevalued operators on a complete metric space endowed with a τ-distance were established in T.Suzuki [3].

The suppose of this paper is the present a theorem for singlevalued operators in a complete metric space with respect to τ-distance.

2 Preliminaries

Definition 1.1 Let X be any space and $f : X \to X$ a singlevalued operator. A point $x \in X$ is called *fix point* for f if $x = f(x)$. The set of all fixed points of f is denoted by $Fix(f)$.

Definition 1.2

(1) A singlevalued operator f defined on a metric space (X, d) is said to be lower semicontinuous (lsc) at a point $t \in X$ if either $\liminf_{x\to t} f(x) = \infty$ or $\liminf_{x\to t} f(x) \geq f(t)$.

(2) A singlevalued operator f defined on a metric space (X, d) is said to be upper semicontinuous (usc) at a point $t \in X$ if either $\limsup_{x\to t} f(x) = -\infty$ or $\limsup_{x\to t} f(x) \leq f(t)$.

(3) A singlevalued operator f defined on a metric space (X, d) is said to be continuous at a point $t \in X$ if f is lower semicontinuous and upper semicontinuous in the same time at the point $t \in X$. If f is continuous in all $t \in X$ then f is continuous in (X, d).

The concept of τ-distance was introduced by T. Suzuki (see[1]) as follows:

Let (X,d) be a metric space, $p : X \times X \to [0, \infty)$ is called $\tau - distance$ on X if there exists a function $\eta : X \times \mathbb{R}_+ \to \mathbb{R}_+$ and the following are satisfied :

(τ_1) $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$;

(τ_2) $\eta(x, 0) = 0$ and $\eta(x, t) \geq t$ for all $x \in X$ and $t \in \mathbb{R}_+$, and η is concave and continuous in its the second variable;

(τ_3) $\lim_n x_n = x$ and $\lim_n sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ imply

$p(w,x) \le \lim_n inf(p(w,x_n))$ for all $w \in X$;

(τ_4) $\lim_n sup\{p(x_n,y_m)) : m \ge n\} = 0$ and $\lim_n \eta(x_n,t_n)$ imply $\lim_n \eta(y_n,t_n) = 0$;

(τ_5) $\lim_n \eta(z_n,p(z_n,x_n)) = 0$ and $\lim_n \eta(z_n,p(z_n,y_n)) = 0$ imply $\lim_n d(x_n,y_n) = 0$;

We may replace (τ_2) by the following $(\tau_2)'$:

$(\tau_2)'$ $inf\{\eta(x,t) : t > 0\} = 0$ for all $x \in X$, and η is nondecreasing in the second variable.

Let us give some examples of τ-distance (see[2]).

Exemple 1.1. Let (X,d) be a metric space . Then the metric "d" is a τ-distance on X.

Exemple 1.2. Let (X,d) be a metric space and p be a w-distance on X.Then p is also a τ-distance on X.

Exemple 1.3. Let (X,d) be a metric space and p be a w-distance on X, let $h : \mathbb{R}_+ \to \mathbb{R}$ be a nondecreasing function such that $\int_0^\infty \frac{1}{1+h(r)}dr = \infty$, and let $z_0 \in X$ be fixed.Then a function $q : X \times X \to \mathbb{R}_+$ defined by:

$$q(x,y) = \int_{p(z_0,x)}^{p(z_0,x)+p(x,y)} \frac{dr}{1+h(r)}, \text{ for all } x,y \in X$$

is a τ-distance. For the proof of the main result we need of the definition of the $p-Cauchy$ sequnce and the following lemmas (see [3]).

Definition 1.4. Let (X,d) be a metric space and let p be a τ-distance on X. Then a sequence $\{x_n\}$ in X is called $p-Cauchy$ if there exists a function $\eta : X \times [0,\infty) \to [0,\infty)$ satisfying (τ_2)-(τ_5) and a sequence $\{z_n\}$ in X such that $\lim_n sup\{\eta(z_n,p(z_n,x_m)) : m \ge n\} = 0$.

Lemma 1.5. *Let (X,d) be a metric space and let p be a τ-distance on X. If a sequence $\{x_n\}$ in X satisfies $\lim_n sup\{p(x_n,x_m) : m > n\} = 0$, then $\{x_n\}$is a p-Cauchy sequence. Moreover, if a sequence $\{y_n\}$ in X satisfies $\lim_n p(x_n,y_n) = 0$, then $\{y_n\}$ is also a p-Cauchy sequence and $\lim_n d(x_n,y_n) = 0$.*

Lemma 1.6. *Let (X, d) be a metric space and let p be a τ-distance on X. If $\{x_n\}$ is a p-Cauchy sequence ,then $\{x_n\}$ is a Cauchy sequence. Moreover, if $\{y_n\}$ is a sequence satisfying $\lim_n sup\{p(x_n, y_m) : m > n\} = 0$, then $\{y_n\}$ is a p-Cauchy sequence and $\lim_n d(x_n, y_n) = 0$.*

3 MAIN RESULT

Theorem *Let (X, d) a complete metric space, $\tau : X \times X \to [0, \infty)$ a τ-distance in X and $f : X \to X$ a continuous operator, such that we have:*

(i) there exists $q < 1$ such that:

$$\tau(f(x), f(y)) \leq q\tau(x, y),$$

for every $x, y \in X$;

(ii) $\inf\{\tau(x, y) + q\tau(x, f(x)) | x \in X\} > 0$, for every $y \in X$ with $y \neq f(y)$. Then there exists $z \in X$ such that $z = f(z)$ and $\tau(z, z) = 0$.

Proof. Let $u_0 \in X$ such that $u_1 = f(u_0)$. Then for $u_2 = f(u_1)$ we have $\tau(u_1, u_2) \leq q\tau(u_0, u_1)$. Thus we can define the sequence $\{u_n\} \in X$ such that $u_{n+1} = f(u_n)$ and $\tau(u_n, u_{n+1}) \leq q\tau(u_{n-1}, u_n)$ for every $n \in \mathbb{N}$.

Then we have, for any $n \in \mathbb{N}$ and $q < 1$,

$$\tau(u_n, u_{n+1}) \leq q\tau(u_{n-1}, u_n) \leq ... \leq q^n\tau(u_0, u_1).$$

and hence for any $m, n \in \mathbb{N}$ with $m \geq 1$

$$\tau(u_n, u_{n+m}) \leq \tau(u_n, u_{n+1}) + \tau(u_{n+1}, u_{n+2}) + ... + \tau(u_{n+m-1}, u_{n+m})$$

$$\leq q^n\tau(u_0, u_1) + q^{n+1}\tau(u_0, u_1) + ... + q^{n+m-1}\tau(u_0, u_1)$$

$$\leq \frac{q^n}{1-q}\tau(u_0, u_1).$$

By the Lemma 1.5, $\{u_n\}$ is a p-Cauchy sequence and using Lemma 1.6 we have that the sequence $\{u_n\}$ is a Cauchy sequence .

Then $\{u_n\}$ converges to some point $z \in X$. We fix $n \in \mathbb{N}$. Since $\tau(u, \cdot)$ is lower semicontinuous for any $u \in X$, we have:

$$\tau(u_n, z) \leq \lim_{m \to \infty} \inf \tau(u_n, u_{n+m}) \leq \frac{q^n}{1-q}\tau(u_0, u_1).$$

Assume that $z \neq f(z)$. Then, by hypothesis, we have:

$$0 < \inf\{\tau(x, z) + \tau(x, f(x))|x \in X\}$$

$$\leq \inf\{\tau(u_n, z) + \tau(u_n, u_{n+1})|n \in \mathbb{N}\}$$

$$\leq \inf\{\frac{q^n}{1-q}\tau(u_0, u_1) + q^n\tau(u_0, u_1)|n \in \mathbb{N}\} = 0$$

This is a contradiction. Therefore we have $z = f(z)$. Then we have

$$\tau(z, z) = \tau(f(z), f(z)) \leq q\tau(z, f(z)) = q\tau(z, z)$$

and hence $\tau(z, z) = 0.\square$

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