Fixed points for singlevalued operators with respect to w-distance

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Abstract. In this paper we first recall the concept of w-distance on a metric space. Then, we prove a fixed point theorem for singlevalued operators in terms of a w-distance.

Key words: fixed point, w-distance, singlevalued operator.

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1 Introduction

In 1996 O. Kada, T. Suzuki and W. Takahashi introduced the concept of w-distance. They gave some examples of w-distance and have generalized Caristi's fixed point theorem, Ekeland's variational principle and the Takahashi's non-convex minimization theorem, see [1]. Also, some fixed point theorems for singlevalued operators on a complete metric space endowed with a w-distance were established in T.Suzuki [2] and J.Ume [4].

The concept of w-distance was introduced by O. Kada, T. Suzuki and W. Takahashi (see[1]) as follows:

Let (X,d) be a metric space, $w : X \times X \to [0, \infty)$ is called w-distance on X if the following axioms are satisfied :

(i) $w(x, z) \leq w(x, y) + w(y, z)$, for any $x, y, z \in X$;

(ii) for any $x \in X, w(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;

(iii) for any $\varepsilon > 0$, exists $\delta > 0$ such that $w(z, x) \leq \delta$ and $w(z, y) \leq \delta$ implies $d(x, y) \leq \delta$.

The suppose of this paper is the present a theorem for singlevalued operators in a complete metric space with respect to w-distance.

2 Preliminaries

Definition 1.1 Let X be any space and $f : X \to X$ a singlevalued operator. A point $x \in X$ is called *fix point* for f if $x = f(x)$. The set of all fixed points of f is denoted by $Fix(f)$.

Definition 1.2

(1) A singlevalued operator f defined on a metric space (X, d) is said to be lower semicontinuous (lsc) at a point $t \in X$ if either $\liminf_{x \to t} f(x) = \infty$ or $\liminf_{x \to t} f(x) \geq f(t)$.

(2) A singlevalued operator f defined on a metric space (X, d) is said to be upper semicontinuous (usc) at a point $t \in X$ if either $\limsup_{x \to t} f(x) = -\infty$ or $\limsup_{x \to t} f(x) \leq f(t)$.

(3) A singlevalued operator f defined on a metric space (X, d) is said to be continuous at a point $t \in X$ if f is lower semicontinuous and upper semicontinuous in the same time at the point $t \in X$. If f is continuous in all

$t \in X$ then f is continuous in (X, d).

Let us give some examples of w-distance (see [1]).

Exemple 1.3. Let (X, d) be a metric space . Then the metric "d" is a w-distance on X.

Exemple 1.4. Let X be a normed liniar space with norm $|| \cdot ||$. Then the function $w : X \times X \to [0, \infty)$ defined by $w(x, y) = max\{|\frac{1}{2}x - y|, \frac{1}{2}|x - y|\}$ for every $x, y \in X$ is a w-distance.

Exemple 1.5. Let (X,d) be a metric space and let $g : X \to X$ a continuous mapping. Then the function $w : X \times Y \to [0, \infty)$ defined by: $w(x, y) = max\{d(g(x), y), d(g(x), g(y))\}$ for every $x, y \in X$ is a w-distance.

For the proof of the main results we need the following lemma (see [3]).

Lemma 1.6. *Let X be a metric space with metric d, w be a w-distance in X, $\{x_n\}, \{y_n\}$ be two sequences in X, $\{\alpha_n\}, \{\beta_n\}$ be sequence in $[0, \infty)$ converging to 0 and $x, y \in X$. Then the following hold:*

(i) if $w(x_n, y) \leq \alpha_n$ and $w(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$

(ii) if $w(x_n, y_n) \leq \alpha_n$ and $w(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;

(iii) if $w(x_n, x_m) \leq \alpha_n$ for any $m, n \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;

(iv) if $w(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

3 Main result

Theorem *Let (X, d) a complete metric space, $w : X \times X \to [0, \infty)$ a w-distance in X and $f : X \to X$ a continuous operator, such that we have:*

(i) there exists $q < 1$ such that:

$$w(f(x), f(y)) \leq qw(x, y),$$

for every $x, y \in X$;

(ii) $\inf\{w(x, y)+qw(x, f(x))|x \in X\} > 0$, *for every $y \in X$ with $y \neq f(y)$.*

Then there exists $z \in X$ such that $z = f(z)$ and $w(z, z) = 0$.

Proof. Let $u_0 \in X$ such that $u_1 = f(u_0)$. Then for $u_2 = f(u_1)$ we have $w(u_1, u_2) \leq qw(u_0, u_1)$. Thus we can define the sequence $\{u_n\} \in X$ such that $u_{n+1} = f(u_n)$ and $w(u_n, u_{n+1}) \leq qw(u_{n-1}, u_n)$ for every $n \in \mathbb{N}$.

Then we have, for any $n \in \mathbb{N}$ and $q < 1$,

$$w(u_n, u_{n+1}) \leq qw(u_{n-1}, u_n) \leq ... \leq q^n w(u_0, u_1).$$

and hence for any $m, n \in \mathbb{N}$ with $m \geq 1$

$$w(u_n, u_{n+m}) \leq w(u_n, u_{n+1}) + w(u_{n+1}, u_{n+2}) + ... + w(u_{n+m-1}, u_{n+m})$$

$$\leq q^n w(u_0, u_1) + q^{n+1} w(u_0, u_1) + ... + q^{n+m-1} w(u_0, u_1)$$

$$\leq \frac{q^n}{1-q} w(u_0, u_1).$$

By the Lemma 1.6, $\{u_n\}$ is a Cauchy sequence. Then $\{u_n\}$ converges to some point $z \in X$. We fix $n \in \mathbb{N}$. Since $w(u, \cdot)$ is lower semicontinuous for any $u \in X$, we have:

$$w(u_n, z) \leq \lim_{m\to\infty} \inf w(u_n, u_{n+m}) \leq \frac{q^n}{1-q} w(u_0, u_1).$$

Assume that $z \neq f(z)$. Then, by hypothesis, we have:

$$0 < \inf\{w(x, z) + w(x, f(x))|x \in X\}$$

$$\leq \inf\{w(u_n, z) + w(u_n, u_{n+1})|n \in \mathbb{N}\}$$

$$\leq \inf\{\frac{q^n}{1-q} w(u_0, u_1) + q^n w(u_0, u_1)|n \in \mathbb{N}\} = 0$$

This is a contradiction. Therefore we have $z = f(z)$. Then we have

$$w(z, z) = w(f(z), f(z)) \leq qw(z, f(z)) = qw(z, z)$$

and hence $w(z, z) = 0$.□

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