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ABSTRACT

This report shows how to determine in analytic form the security prices implied by the market equilibrium model described by Fama in his book "Foundations of Finance", Chapter 8, Section III. The model assumes that all investors agree on the expected values and covariances of the random final prices at the end of an investment period. The investors interact so that the initial prices lead to an efficient portfolio with security weights proportional to the total initial value of the corresponding firm.

If we assume that the expected portfolio return or the risk-free rate is specified, we find that there is a one-dimensional continuum of sets of initial prices satisfying the conditions of the model. It is unclear how to resolve this ambiguity about which model is correct.

1. INTRODUCTION

Although it dates back to 1970 or earlier Fama's Efficient Market Hypothesis (EMH) still attracts a lot of interest. A key concept in the EMH is the price formation process, whereby investors at any time, properly taking account of all available information, together set appropriate prices of securities. In other words the prices are determined by applying a model of market equilibrium, sometimes called an asset pricing model.

Fama's published EMH articles (Fa70, Fa91, Fa98) tell us very little about models of market equilibrium. To obtain more details on this subject we turn to Fama's text "Foundations of Finance" (Fa76), which we denote by FOF. It might be thought that a 30-year old book would be of only historical interest, but this is presumably not Fama's view. The Internet tells us that, for a recent course (Fa05) at the University of Chicago, Fama expects students to study almost the entire book.

We therefore believe it worthwhile to examine an important section of FOF, Section III of Chapter 8 (pages 271 - 278), which treats a particular model of market equilibrium, the Sharpe-Lintner model, now commonly known as the Capital Asset Pricing Model (CAPM).

Fama assumes that there is a set of n securities that may held by investors for a period running from time 1 to time 2. At times 1 and 2 security i has prices $p_i, \tilde{v}_i$ respectively, scaled so that they represent the market capitalization of the corresponding firm (i.e. at time 1 there is one share outstanding with price p_i). Throughout it is assumed that the final price variables $\tilde{\mathbf{V}}' = \{\tilde{v}_1, \ldots, \tilde{v}_n\}$ are random with a certain joint probability distribution, agreed to by all investors. The initial prices $\mathbf{P}' = \{p_i, \ldots, p_n\}$ are non-random. (The prime means matrix transpose.)

The return of security i from time 1 to time 2 is the random variable

$$\tilde{r}_i = \frac{\tilde{v}_i}{p_i} - 1. \tag{1.1}$$

The investors know the expected values and covariances of $\tilde{\mathbf{V}}$. Given the initial prices, they therefore can deduce the covariances of the returns $\tilde{\mathbf{R}}' = \{\tilde{r}_1, \ldots, \tilde{r}_n\}$, and the expected returns $\mathbf{R}' = E(\tilde{\mathbf{R}}') = \{r_1, \ldots, r_n\}$. We denote the return covariance matrix by $\mathbf{C}$ with elements c_{ij}.

At time 1 an investor can form a portfolio of securities with security i having weight x_i. The weights must obey the normalization condition

$$\sum_{i=1}^{n} x_i = 1. \tag{1.2}$$

A portfolio with weights $\mathbf{X} = \{x_1, \ldots, x_n\}$ at time 1 will have expected return ρ and standard deviation σ, where

$$\rho = \mathbf{X'R} = \sum_{i=1}^{n} x_i r_i \tag{1.3}$$

$$\sigma^2 = \mathbf{X'CX} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i c_{ij} x_j \; . \tag{1.4}$$

Fama assumes that all investors will choose to invest in an efficient portfolio, defined as the unique portfolio, normalized weight vector $\mathbf{W} = \{w_1, \ldots, w_n\}$, with specified expected portfolio return ρ that has minimum standard deviation σ. As described below the value of the risk-free rate R_F is related to ρ and other quantities.

The above discussion assumes that the initial prices $\mathbf{P}$ are known. The market equilibrium described in the above section of FOF outlines how these prices are determined. The key observation is that, if all investors hold portfolios with the same weight vectors, and all outstanding shares are held by these investors, then the initial security prices p_i must be proportional to the weights w_i of the efficient portfolio. Thus Fama writes in FOF page 275

The tangency (i.e. efficient) *portfolio is ... different for different sets of security prices at time 1 and different values of* R_F. *A market equilibrium - a set of security prices that clears the securities market and a value of* R_F *that clears the borrowing-lending market - requires that the tangency portfolio be the value-weighted version of the market portfolio.*

That is, there must be a positive quantity η such that

$$p_i = \eta w_i, \quad i = 1, \ldots, n \; . \tag{1.5}$$

Now, as Fama shows in FOF Section II of Chapter 8 (pages 258 - 267), there is an analytic expression for the weight vector $\mathbf{W}$ of the efficient portfolio in terms of covariance matrix $\mathbf{C}$, expected returns $\mathbf{R}$ and expected portfolio return ρ that follows from

$$\mathbf{CW} = \lambda \mathbf{R} + \mu \mathbf{1}, \tag{1.6}$$

where $\mathbf{1'} = \{1, \ldots, 1\}$, and λ, μ are derived from the conditions

$$\mathbf{1'W} = 1 \tag{1.7}$$

$$\mathbf{R'W} = \rho \; . \tag{1.8}$$

Although he does not pursue the question further in FOF, it appears that Fama is suggesting that the initial price vector $\mathbf{P}$ should be determined by substituting (1.5) for $\mathbf{P}$ wherever it appears in $\mathbf{C}$ and $\mathbf{R}$ and then solving ((1.6), (1.7) and (1.8) for $\mathbf{W}$, ultimately leading to $\mathbf{P}$.

This report obtains the following results concerning Fama's suggestion.

- For given expected values and covariances of final prices $\tilde{\mathbf{V}}$, and given expected portfolio return ρ, we obtain an analytic description of the solutions for the initial prices $\mathbf{P}$ for any number of securities $n \geq 3$;
- Because there is one more unknown but the same number of equations as in the standard efficient portfolio problem, there is now a continuum of solutions $\mathbf{P}$, which lie along a segment of a line in the n-dimensional $\mathbf{P}$-space.

The details of the analysis are found in Section 2. A brief discussion concerning R_F appears in Section 3.

2. ANALYSIS OF FAMA'S MARKET EQUILIBRIUM

Here we set out the procedure outlined above for solving equations (1.5) - (1.8) for the weight vector $\mathbf{W}$. We will also find an expression for the parameter η that will lead to the initial price vector $\mathbf{P}$.

We use the term "weight domain" to denote the part of $\mathbf{W}$-space where all the weights are positive. Only solutions in this domain are acceptable, since all the prices $\mathbf{P}$ must be positive.

DERIVATION OF THE BASIC EQUATIONS

Let us denote the expected values and covariance matrix of the random final prices $\tilde{\mathbf{V}}$ respectively by $\mathbf{V}' = \{v_1, \ldots, v_n\}$ and $\mathbf{G}$, elements g_{ij}. Define the scaled expected price ratio vector $\mathbf{Q}' = \{q_1, \ldots, q_n\}$ by

$$q_i = \frac{v_i}{w_i}, \quad i = 1, \ldots, n, \tag{2.1}$$

so that, using (1.1) and (1.5), we may write the expected value relation

$$\mathbf{R} = \frac{1}{\eta} \mathbf{Q} - \mathbf{1}. \tag{2.2}$$

Similarly the return covariance matrix $\mathbf{C}$ may be written, after substituting (1.5) $\mathbf{P} = \eta \mathbf{W}$,

$$c_{ij} = \frac{1}{\eta^2 w_i w_j} g_{ij}, \quad i, j = 1, \ldots, n. \tag{2.3}$$

Substituting (2.2) into (1.8) with (1.7) leads to

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$$\rho = \mathbf{R}'\mathbf{W} = \frac{1}{\eta}\sum_{i=1}^{n} v_i - 1 \quad \text{or} \quad \eta = \frac{V}{1+\rho} \tag{2.4}$$

with $V = \sum_{i=1}^{n} v_i$. Thus we know the scaling factor η.

We now substitute (2.2) and (2.3) into (1.6), which becomes

$$\sum_{j=1}^{n} \frac{1}{\eta^2 w_i w_j} g_{ij} w_j = \lambda \left[\frac{v_i}{\eta w_i} - 1 \right] + \mu, \quad i = 1,\dots,n, \tag{2.5}$$

or, multiplying by w_i, dividing by $(\mu - \lambda)$ and rearranging,

$$\mathbf{W} = \frac{1}{\eta^2(\mu - \lambda)}\mathbf{G1} - \frac{\lambda}{\eta(\mu - \lambda)}\mathbf{V}. \tag{2.6}$$

Defining $s_i = \sum_{j=1}^{n} g_{ij}$, $\vartheta = \frac{1}{\eta^2(\mu-\lambda)}$ and $\phi = \frac{-\lambda}{\eta(\mu-\lambda)}$, (2.6) becomes

$$\mathbf{W} = \vartheta\mathbf{S} + \phi\mathbf{V}, \tag{2.7}$$

where by $\mathbf{S}' = \{s_1,\dots,s_n\}$.

A useful relation between ϑ and ϕ is obtained with the help of the constraint (1.7) and We multiply (2.7) on the left by $\mathbf{1}'$ to produce

$$\mathbf{1}'\mathbf{W} = 1 = \vartheta\mathbf{1}'\mathbf{S} + \phi\mathbf{1}'\mathbf{V} = \vartheta S + \phi V, \tag{2.8}$$

where $S = \sum_{i=1}^{n} s_i$. Note that the other constraint (1.8) tells us nothing useful about ϑ and ϕ - it has already produced the expression (2.4) for the previously unknown η.

SOLUTION

If we use (2.8) we may rewrite (2.7) in the form

$$\mathbf{W} = \vartheta\mathbf{S} + \phi\mathbf{V} = \vartheta\left[\mathbf{S} - \frac{S}{V}\mathbf{V}\right] + \frac{\mathbf{V}}{V} = \vartheta\tilde{\mathbf{S}} + \hat{\mathbf{V}}, \tag{2.9}$$

where

$$\tilde{\mathbf{S}} = \mathbf{S} - \frac{S}{V}\mathbf{V} \quad \text{and} \quad \hat{\mathbf{V}} = \mathbf{V}/V. \tag{2.10}$$

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Note that the sum of the elements of $\tilde{\mathbf{S}}$ is zero, but the sum for $\hat{\mathbf{V}}$ is unity, so that $\mathbf{W}$ given by (2.9) is properly normalized for any choice of ϑ. Thus (2.9) provides an acceptable solution to the problem of finding $\mathbf{W}$ for any ϑ such that $\mathbf{W}$ is in the weight domain, i.e. $w_i > 0, \quad i = 1,\ldots,n$. The set of acceptable $\mathbf{W}$ lies on the segment of the line, passing through $\hat{\mathbf{V}}$ in the direction of the vector $\tilde{\mathbf{S}}$, that is in the weight domain.

We can see that solutions always exist since the choice $\vartheta = 0$ leads to $\mathbf{W} = \hat{\mathbf{V}}$, which is in the weight domain, being normalized with all components positive.

We note that the solutions for $\mathbf{W}$ depend only on components of the vectors $\mathbf{S}$ and $\mathbf{V}$, that is rows of the covariance matrix and expected values of the final prices. In addition, the solution for the price vector $\mathbf{P}$ depends on the scaling factor η that depends on the expected portfolio return ρ.

RISK-FREE RATE

To every efficient portfolio there corresponds a value of the risk-free rate R_F. This value may be written in various ways in terms of the quantities introduced above. For example it may be shown that

$$R_F = -\frac{\mu}{\lambda} = \rho - \frac{\sigma^2}{\lambda} = \rho + \frac{\eta\vartheta}{\phi}. \tag{2.11}$$

Thus, apart from components of the vectors $\mathbf{S}$ and $\mathbf{V}$, which we can regard as fixed, R_F depends on our choice of ρ and ϑ.

3. DISCUSSION

In the quotation above Fama implies that the value of R_F is required to specify the market equilibrium. Our analysis indicates that this is not sufficient to determine both ρ and ϑ. We are uncertain as to how to proceed to obtain a unique market equilibrium.

Comments are welcome.

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