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Abstract: In this paper we will generate queueing systems with *c* stations where the inter-arrival time and the *c* service times depend through a *c*+1 copula *C*. We will consider two models: first when the customer does not know the order of service times for the free service channels (he/she chooses the service channel randomly), and the second when he/she knows this order (he/she chooses the fastest free service channel).
The marginals can be exponential, Erlang or hyper-exponential.
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1. Introduction

The queueing systems have been studied among others by Kleinrock (see [10]) and Asmussen (see [2]). Some authors like Ciucu and Mihoc (see [2]) consider other queue discipline than FIFO, like SIRO, LIFO and priorities. Others consider the possibility that the server stops for a time if there is no customer on the queue (see [12,9]). But all of these models consider that the arrivals and the services are independent.

Definition 1. *A copula is a function* $C:[0,1]^n \to [0,1]$ *so that*

1) *If there is i so that* $x_i = 0$ *than* $C(x_1,...,x_n) = 0$.
2) *If* $x_j = 1$ *for all* $j \neq i$ *than* $C(x_1,...,x_n) = x_i$.
3) C *is increasing in each argument.*

We have the following theorem (see [15,13,14]).

Theorem 1 (Sklar). *Let* X_1, X_2,..., X_n *be random variables with the cumulative distribution functions* F_1, F_2,..., F_n, *and the common cdf* $H(x_1,...,x_n) = P(X_1 \leq x_1,...,X_n \leq x_n)$. *In this case there is a copula* $C(u_1,...,u_n)$ *so that* $H(x_1,...,x_n) = C(F_1(x_1),...,F_n(x_n))$. *The copula* C *is well defined on the chartesian product of the images of the marginals* F_1, F_2,..., F_n.

Definition 2 ([15,16,17]). *If* $n = 2$ *the copula* C *is Archimedean if* $C(u,u) < u$ *for any* $u \in (0,1)$ *and* $C(C(u,v),w) = C(u,C(v,w))$ *for any* $u,v,w \in [0,1]$. *If* $n > 2$ *the copula* C *is Archimedean if there is an* $n-1$ *Archimedean copula* C_1 *and a* $2-$*Archimedean copula* C_2 *so that* $C(u_1,...,u_n) = C_2(C_1(u_1,...,u_{n-1}),u_n)$.

Consider a function $\varphi:[0,1] \to R$ decreasing and convex with $\varphi(1) = 0$ and its pseudo-inverse g ($g(y)$ has the value x if there is x so that $\varphi(x) = y$ and 0 in the contrary case). We know (see [8,15]) that a copula C is Archimedean if and only if there is a function φ as above so that for any $x,y \in [0,1]$ we have

$$C(x,y) = g(\varphi(x) + \varphi(y)). \tag{1}$$

In [16,17] methods to simulate Archimedean copulas are presented, and in [4] some algorithms to simulate queueing systems with one channel with arrivals and services depending through copulas.
For generating such queueing systems where the copula is Archimedean we use in [4] the following theorem (see [8]).

Theorem 2. *If X and Y are uniform random variables connected by the Archimedean copula C given by φ like in* (1) *the random variables $Z_1 = \frac{\varphi(X)}{\varphi(X)+\varphi(y)}$ and $Z_2 = C(X,Y)$ are independent. Z_1 is a uniform random variable on $[0,1]$ and Z_2 has the cdf K, where $K(v) = v - \lambda(v)$ and $\lambda(v) = \frac{\varphi(v)}{\varphi'(v)}$ for any $v \in [0,1]$.*

For any $n-$copula C we have (see [1])

$$W(x_1,...,x_n) \le C(x_1,...,x_n) \le \min(x_1,...,x_n), \text{ where} \tag{2}$$

$$W(x_1,...,x_n) = \sum_{i=1}^{n} x_i - n + 1 \tag{2'}$$

is the lower Fréchet bound, and $\min$ is the upper Fréchet bound.

In [4] we have used in order to generate the copulas W and $\min$ the fact that if X and Y are connected by the copula $\min$ there is a function $f:[0,1] \to [0,1]$ increasing so that $f(X) = Y$. If the copula is W than f is decreasing, and if the copula is $\mathrm{Pr}\,od$ the variables are independent (see [15,13]). The Fréchet copulas (see [1,15,13]) are generated by the mixture method (see [4]). If we know a formula for the copula C we generate in [4] the uniform random variables X and Y connected by the copula C using the following theorem.

Theorem 3 ([4,5]). *If X and Y are uniform random variables with their common cdf given by the copula C, the conditional random variable $X|Y$ has the cdf $\frac{\partial C}{\partial x}$.*

In [6] analytical formulae can be found for the copulas that connect the number of customers in a Gordon and Newell queueing network, and their corresponding Spearman ρ and Kendall τ.

2. Generating Archimedean Copulas

First we will give a generalization of theorem 2.

Theorem 4. *If X_1, X_2..., X_n are uniform random variables connected by the Archimedean copula C given by φ like in* (1) *, the random variables $Z_1 = \frac{\varphi(X_1)}{\sum_{i=1}^{n}\varphi(X_i)}$ and $Z_2 = \sum_{i=1}^{n}\varphi(X_i)$ are independent. Z_1 has the cdf $\Psi_n(v) = 1-(1-v)^{n-1}$ for $v \in [0,1]$, and Z_2 has the cdf $F_n(v) = 1 - \sum_{i=0}^{n-1} \frac{(-1)^i \cdot x^i}{i!} \cdot g^{(i)}(x)$.*

Proof: Firstly we will prove that $\sum_{i=0}^{n} \varphi(X_i) \le \varphi(0)$ with the probability 1. Of course, if $\varphi(0) = \infty$ the above relation is obvious.

If $\varphi(0) < \infty$ we consider x_1, x_2,..., x_n such that $\sum_{i=0}^{n} \varphi(x_i) > \varphi(0)$. Using (1) we obtain $C(x_1,...,x_n) = 0$.

Therefore the pdf in this point is $\frac{\partial^n C}{\partial x_1 ... \partial x_n} = 0$, and from here the above relation.

The common cdf of $\varphi(X_1)$,..., $\varphi(X_n)$ is

$$P(\varphi(X_i) \le x_i) = P(X_i \ge g(x_i)) = 1 - \sum_{i=1}^{n} g(x_i) + \sum_{i=1}^{n} \sum_{j=1,\, j\ne i}^{n} g(x_i + x_j) + ... + (-1)^n \cdot g\left(\sum_{i=1}^{n} x_i\right)$$

Therefore the common pdf is $(-1)^n \cdot g^{(n)}\left(\sum_{i=1}^{n} x_i\right)$.

It results that the cdf of $Z_2 = \sum_{i=1}^{n} \varphi(X_i)$ is

$$F_n(v) = \int_0^v \int_0^{v-x_1} \dots \int_0^{v-\sum_{i=1}^{n-1} x_i} (-1)^n \cdot g^{(n)}\left(\sum_{i=1}^{n} x_i\right) dx_n \dots dx_2 dx_1 = \int_0^v \int_0^{v-x_1} \dots \int_0^{v-\sum_{i=1}^{n-2} x_i} (-1)^n \cdot g^{(n-1)}(v) dx_{n-1} \dots dx_2 dx_1 +$$

$$\int_0^v \int_0^{v-x_1} \dots \int_0^{v-\sum_{i=1}^{n-2} x_i} (-1)^{n-1} \cdot g^{(n-1)}\left(\sum_{i=1}^{n-1} x_i\right) dx_{n-1} \dots dx_2 dx_1 = F_{n-1}(v) - \frac{(-1)^{n-1} \cdot v^{n-1}}{(n-1)!} \cdot g^{(n-1)}(v) .$$

The formula from enunciation for F_2 is proved by computations, and for F_n we use the above recurrence formula and the mathematical induction.

Using the formula for F_n we obtain the formula for the pdf

$$f_n(x) = \frac{(-1)^n \cdot x^{n-1}}{(n-1)!} \cdot g^{(n)}(x) .$$

From this relation if $\varphi(0) = \infty$, respectively the fact that $\sum_{i=0}^{n} \varphi(X_i) \le \varphi(0)$ with the probability 1 if $\varphi(0) < \infty$ it results that $g^{(n)}(\varphi(0)) = 0$ for any $n \ge 0$.

In the same way we prove that the common cdf of $\varphi(X_1)$ and $\sum_{i=2}^{n} \varphi(X_i)$ is for $n \ge 2$

$$H_n(x, y) = 1 - g(x) + \sum_{i=0}^{n-2} \frac{(-1)^i \cdot y^i}{i!} \cdot \left(g^{(i)}(x + y) - g^{(i)}(y)\right).$$

Computing the second order derivative we obtain the common pdf

$$h_n(x, y) = \frac{\partial^2 H_n}{\partial x \partial y} = \frac{(-1)^{n-2} \cdot y^{n-2}}{(n-2)!} \cdot g^{(n)}(x + y) .$$

If $\varphi(0) = \infty$ we obtain

$$\Psi_n(v) = \int_0^{\infty} \int_0^{\frac{v}{1-v} \cdot y} \frac{(-1)^{n-2} \cdot y^{n-2}}{(n-2)!} \cdot g^{(n)}(x + y) dx dy = \int_0^{\infty} \frac{(-1)^{n-2} \cdot y^{n-2}}{(n-2)!} \cdot g^{(n-1)}\left(\frac{y}{1-v}\right) dy -$$

$$\int_0^{\infty} \frac{(-1)^{n-2} \cdot y^{n-2}}{(n-2)!} \cdot g^{(n-1)}(y) dy .$$

In the last integral we notice that the function which we integrate with the sign "-" is the pdf of Z_2. Therefore, using the substitution $y = (1 - v) \cdot z$ we obtain in this case the formula for Ψ_n from enunciation.

If $\varphi(0) < \infty$ we obtain

$$\Psi_n(v) = 1 - \int_0^{(1-v)\varphi(0)} \int_{\frac{v}{1-v} y}^{\varphi(0)-y} \frac{(-1)^{n-2} y^{n-2}}{(n-2)!} g^{(n)}(x + y) dx dy = 1 - \int_0^{(1-v)\varphi(0)} \frac{(-1)^{n-2} y^{n-2}}{(n-2)!} g^{(n)}(\varphi(0)) dy +$$

$$\int_0^{(1-v)\varphi(0)} \frac{(-1)^{n-2} y^{n-2}}{(n-2)!} g^{(n)}\left(\frac{y}{1-v}\right) dy .$$

We take now into account that $g^{(n)}(\varphi(0)) = 0$ and the above form of f_n. It results that the form of Ψ_n is also that from enunciation in this case.

For proving the independence we notice first that if $\varphi(0) < \infty$ we have with the probability 1 $(1 - Z_1)\varphi(0) \le Z_2$.

From this it results that we must prove the independence in the cases $\varphi(0)=\infty$, $\varphi(0)<\infty$ and $w<(1-v)\varphi(0)$.

The common cdf of Z_1 and Z_2 is

$$H_n(v,w)=\int_0^w\int_0^{\frac{v}{1-v}y}\frac{(-1)^{n-2}y^{n-2}}{(n-2)!}g^{(n)}(x+y)dxdy=\int_0^w\frac{(-1)^{n-2}y^{n-2}}{(n-2)!}g^{(n-1)}\left(\frac{y}{1-v}\right)dy-\int_0^w\frac{(-1)^{n-2}y^{n-2}}{(n-2)!}g^{(n-1)}(y)dy.$$

But the term from the second integral with the sign "-" is f_{n-1}. Using the substitution $y=(1-v)z$ as above, we conclude that in this cases we have $H_n(v,w)=\Psi_n(v)\cdot F_n(w)$.

In the case $0\le\varphi(0)<w<\infty$ the above relation is obvious, and the theorem is proved.

Using theorem 4 we can build the following algorithm analogous to those for $n=2$ in [4] for the simulation of the uniform variables X_1, X_2,..., X_n connected by the Archimedean copula C if we know the functions φ and g.

Algorithm coparh

Generate U uniform on $[0,1]$.

$S\leftarrow F_n^{-1}(U)$

for $i=1$ **to** $n-1$ **do begin**

Generate U uniform on $[0,1]$.

$R\leftarrow 1-(1-U)^{1/(n-i)}$

$T\leftarrow R*S$

$S\leftarrow S-T$

$X[i]\leftarrow g(T)$

end

$X[n]\leftarrow g(S)$

$output(X[1],...,X[n])$

end.

The most difficult problem in the above algorithm is to compute $F_n^{-1}(U)$ because, using theorem 4 we have to use the derivative of the order $n-1$ of g. When we run our $C++$ program we can have an overlay run error when we compute $g^{(n)}(x)$ for large n (30 for instance). The explanation consists in the existence of some factors in the case of Clayton copula, or terms in the case of Frank, Gumbel-Hougaard or log-copula cases that become large for large n. But these factors or terms can be compensated by $\frac{x^n}{n!}$ in

$$f_n(x)=\frac{(-1)^n\cdot x^{n-1}}{(n-1)!}\cdot g^{(n)}(x). \tag{3}$$

Therefore we have to compute (if it is possible) $\mathrm{f_n}$ directly, or to obtain a recurrence formula for f_n.

The simplest case is the case of the Clayton family, where for an $\theta>0$

$$C(u,v)=\left(u^{-\theta}+v^{-\theta}-1\right)^{-\frac{1}{\theta}}. \tag{4}$$

For $\theta=0$ we obtain the copula $\Pr od$ (independence case), and for $\theta\to\infty$ we obtain the upper Fréchet bound $\min$.

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}}=\frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u)=-u^{-\theta-1}$, and from here

$$\varphi(u) = \frac{u^{-\theta} - 1}{\theta}, \text{ and} \tag{4'}$$

$$g(w) = (\theta w + 1)^{-\frac{1}{\theta}}. \tag{4''}$$

We obtain

$$g^{(n)}(w) = (-1)^n \cdot (\theta w + 1)^{-\frac{1}{\theta} - n} \cdot \prod_{i=1}^{n-1} (i\theta + 1)$$

for $n \geq 1$, and from here

$$f_n(x) = (\theta x + 1)^{-\frac{1}{\theta} - 1} \cdot \left(\frac{x}{\theta x + 1} \right)^{n-1} \cdot \prod_{i=1}^{n-1} \frac{i\theta + 1}{i}. \tag{4'''}$$

Other family of Archimedean copulas presented in [8,11,13] and simulated in [4] is the Frank family. In this case for $\theta \in \mathsf{R}^*$ we have

$$C(u, v) = -\frac{1}{\theta} \cdot \ln\left(\frac{e^{-\theta(u+v)} - e^{-\theta u} - e^{-\theta v} + e^{-\theta}}{e^{-\theta} - 1} \right). \tag{5}$$

We obtain also the copula $\Pr od$ for $\theta = 0$ and the copula $\min$ for $\theta \to \infty$. For $\theta \to -\infty$ we obtain the lower Fréchet bound W.

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u) = \frac{\theta e^{-\theta u}}{e^{-\theta u} - 1}$, and from here

$$\varphi(u) = \ln \frac{1 - e^{-\theta}}{1 - e^{-\theta \cdot u}}, \text{ and} \tag{5'}$$

$$g(w) = -\frac{1}{\theta} \ln\left(\gamma e^{-w} + 1\right), \text{ where } \gamma = e^{-\theta} - 1. \tag{5''}$$

In this case we have also $\varphi(0) = \lim_{v \to 0} \varphi(v) = \infty$, but the computation of $g^{(n)}(w)$ is difficult. We consider $f_0(x) = g(x)$ and $f_1(x) = -g'(x) = \frac{\gamma}{\theta} \cdot \frac{e^{-x}}{\gamma e^{-x} + 1}$.

From $e^{-\theta \cdot g} = \gamma \cdot e^{-x} + 1$ we obtain

$$g'' = \theta (g')^2 - g'.$$

We compute the derivative of the order $n-1$ in both sides of the above relation, we use the Leibnitz formula for $n-\text{th}$ derivative of the product and the formula (3). Finally we obtain

$$f_{n+1}(x) = \frac{x}{n} \cdot f_n(x) + \frac{\theta x}{n} \cdot \sum_{k=0}^{n-1} f_{k+1}(x) \cdot f_{n-k}(x) \text{ for } n \geq 1. \tag{5'''}$$

In the case of the Gumbel-Hougaard family (see [8,13,14,13]) we have for $\theta \geq 1$ and $\beta = \frac{1}{\theta}$

$$C(u, v) = e^{-\left((-\ln u)^{\theta} + (-\ln v)^{\theta} \right)^{\beta}}. \tag{6}$$

For $\theta = 1$ we obtain the copula $\Pr od$ and for $\theta \to \infty$ we obtain the copula $\min$.

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u) = -\frac{\theta(-\ln u)^{\theta-1}}{u}$, and from here

$$\varphi(u) = (-\ln u)^{\theta}, \text{ and} \tag{6'}$$

$$g(x) = e^{-x^{\beta}}. \tag{6''}$$

Tacking into account that $\ln g = -x^{\beta}$ we obtain first

$$\frac{g'}{g} = -\beta \cdot x^{\beta-1}.$$

We differentiate in both sides by x and we multiply next by $x \cdot g^2$, and finally we obtain

$$x \cdot g'' \cdot g = (\beta - 1) \cdot g' \cdot g + x \cdot (g')^2.$$

From here we obtain

$$g'' = (\beta - 1)\frac{g'}{x} + \frac{(g')^2}{g}.$$

If we differentiate in both members of (7) we obtain

$$x \cdot g \cdot g''' = (\beta - 2) \cdot g \cdot g'' + \beta \cdot (g')^2 + x \cdot g' \cdot g''$$

if we do this operation only once, and if we do it $n-1$ times we obtain first

$$x \cdot (g \cdot g'')^{(n-1)} + (n-1) \cdot (g \cdot g'')^{(n-2)} =$$
$$(\beta - 1) \cdot (g \cdot g')^{(n-1)} + x \cdot \left((g')^2\right)^{(n-1)} + (n-1) \cdot \left((g')^2\right)^{(n-2)}.$$

If we differentiate once each term in the above formula for which the derivative order is $n-1$ we obtain

$$x \cdot (g \cdot g''')^{(n-2)} = x \cdot (g' \cdot g'')^{(n-2)} + (\beta - n) \cdot (g \cdot g'')^{(n-2)} + (\beta + n - 2) \cdot \left((g')^2\right)^{(n-2)},$$

and, using the Leibnitz formula for n order derivative of the product, we obtain analogously to the case of the Frank family for $n \geq 3$

$$x \cdot g \cdot g^{(n+1)} = -x \cdot \sum_{k=1}^{n-2} C_{n-2}^{k} \cdot g^{(k)} \cdot g^{(n+1-k)} + x \cdot \sum_{k=0}^{n-2} C_{n-2}^{k} \cdot g^{(k+1)} \cdot g^{(n-k)} +$$
$$(\beta - n) \cdot g \cdot g^{(n)} + (\beta - n) \cdot \sum_{k=1}^{n-2} C_{n-2}^{k} \cdot g^{(k)} \cdot g^{(n-k)} +$$
$$(\beta + n - 2) \cdot \sum_{k=0}^{n-2} C_{n-2}^{k} \cdot g^{(k+1)} \cdot g^{(n-k-1)}.$$

We obtain the recurrence formula

$$\begin{cases} f_0(x) = g(x) = e^{-x^{\beta}} \\ f_1(x) = -g'(x) = \beta \cdot x^{\beta-1} \cdot f_0(x) \\ f_2(x) = x \cdot g''(x) = (1-\beta) \cdot f_1(x) + \frac{x \cdot f_1^2(x)}{f_0(x)} \\ f_3(x) = -\frac{x^2}{2} \cdot g'''(x) = \frac{(2-\beta) \cdot f_2(x)}{2} - \frac{\beta \cdot x \cdot f_1^2(x)}{2 \cdot f_0(x)} + \frac{x \cdot f_1(x) \cdot f_2(x)}{2 \cdot f_0(x)} \\ f_{n+1}(x) = -\frac{x}{n(n-1)} \cdot \sum_{k=0}^{n-3} \frac{(n-k-1)(n-2k-3)}{k+1} \cdot \frac{f_{k+1}(x) \cdot f_{n-k}(x)}{f_0(x)} + \\ \frac{x}{n(n-1)} \cdot \sum_{k=0}^{n-3} \left((n-\beta)(n-k) - n - \beta + 2\right) \cdot f_{k+1}(x) \cdot f_{n-k-1}(x) + \frac{(n-\beta) \cdot f_0(x) \cdot f_n(x)}{n} + \\ \frac{x \cdot f_2(x) \cdot f_{n-1}(x)}{n \cdot (n-1)} - \frac{x \cdot (n+\beta-2) \cdot f_1(x) \cdot f_{n-1}(x)}{n \cdot (n-1)} \text{ for } n \geq 3 \end{cases} \quad . \qquad (6''')$$

The Gumbel-Barnett copula is

$$C(u,v) = u \cdot v \cdot e^{-(\theta(\ln u)(\ln v))}, \text{ with } 0 < \theta \leq 1. \qquad (8)$$

We notice that we have also the copula product (independence) for $\theta \to 0$.

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u) = -\frac{1}{u(1-\theta \ln u)}$, and from here

$$\varphi(u) = \frac{\ln(1-\theta \ln u)}{\theta}, \text{ and} \tag{8'}$$

$$g(x) = e^{\frac{1-e^{\theta x}}{\theta}}. \tag{8''}$$

From $1-\theta \ln g = e^{\theta x}$ we obtain by derivation

$$\frac{g'}{g} = -e^{\theta x}.$$

By derivation we obtain

$$g \cdot g'' = \theta \cdot g \cdot g' + (g')^2.$$

If we differentiate the above relation $n-1$ times we obtain first

$$g \cdot g^{(n+1)} = -\sum_{k=1}^{n-1} C_{n-1}^k \cdot g^{(k)} \cdot g^{(n+1-k)} + \theta \cdot g \cdot g^{(n)} + \theta \cdot \sum_{k=1}^{n-1} C_{n-1}^k \cdot g^{(k)} \cdot g^{(n-k)} + \sum_{k=0}^{n-1} C_{n-1}^k \cdot g^{(k+1)} \cdot g^{(n-k)},$$

and finally the recurrence formula

$$\begin{cases} f_0(x) = g(x) = e^{\frac{1-e^{\theta x}}{\theta}} \\ f_1(x) = -g'(x) = e^{\theta x} \cdot f_0(x) \\ f_2(x) = x \cdot g''(x) = -\theta \cdot x \cdot f_1(x) + \dfrac{x \cdot f_1^2(x)}{f_0(x)} \\ f_{n+1}(x) = -\dfrac{1}{n \cdot f_0(x)} \cdot \sum_{k=1}^{n-1} \dfrac{n-k}{k} \cdot f_k(x) \cdot f_{n+1-k}(x) - \dfrac{\theta x}{n} \cdot f_n(x) - \\ \dfrac{\theta \cdot x}{n \cdot f_0(x)} \cdot \sum_{k=1}^{n-1} \dfrac{1}{k} \cdot f_k(x) \cdot f_{n-k}(x) + \dfrac{x}{n} \cdot \sum_{k=1}^{n-1} f_{k+1}(x) \cdot f_{n-k}(x) \ \ for\ n \geq 2 \end{cases}. \tag{8'''}$$

The Ali-Mikhail-Haq copula is

$$C(u,v) = \frac{u \cdot v}{1-\theta(1-u)(1-v)}, \text{ with } -1 \leq \theta \leq 1. \tag{9}$$

We notice that we have the copula product (independence) for $\theta = 0$.

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u) = -\frac{1}{u(1-\theta(1-u))}$, and from here

$$\varphi(u) = \frac{1}{1-\theta} \cdot \ln\left(\theta + \frac{1-\theta}{u}\right), \text{ and} \tag{9'}$$

$$g(x) = \frac{1-\theta}{e^{(1-\theta)x} - \theta}. \tag{9''}$$

From $\theta + \frac{1-\theta}{g} = e^{(1-\theta)x}$ we obtain first

$$\frac{1-\theta}{g} = e^{(1-\theta)x} - \theta .$$

If we differentiate once the above relation we obtain

$$\frac{g'}{g^2} = -e^{(1-\theta)x},$$

and by derivative of the above relation

$$g \cdot g'' = (1-\theta) \cdot g \cdot g' + 2 \cdot (g')^2 .$$

Analogously to the Gumbel-Barnett case, we obtain the recurrence formula

$$\begin{cases} f_0(x) = g(x) = \frac{1-\theta}{e^{(1-\theta)x}-\theta} \\ f_1(x) = -g'(x) = e^{(1-\theta)x} \cdot f_0^2(x) \\ f_2(x) = x \cdot g''(x) = -(1-\theta) \cdot x \cdot f_1(x) + \frac{2 \cdot x \cdot f_1^2(x)}{f_0(x)} \\ f_{n+1}(x) = -\frac{x}{n \cdot f_0(x)} \cdot \sum_{k=1}^{n-1} \frac{n-k}{k} \cdot f_k(x) \cdot f_{n+1-k}(x) - \frac{(1-\theta)\, x \cdot f_n(x)}{n} - \\ \frac{(1-\theta)\, x^2}{n \cdot f_0(x)} \cdot \sum_{k=1}^{n-1} \frac{1}{k} \cdot f_k(x) \cdot f_{n-k}(x) + \frac{2 \cdot x}{n \cdot f_0(x)} \cdot \sum_{k=0}^{n-1} f_{k+1}(x) \cdot f_{n-k}(x) \quad for\ n \ge 2 \end{cases} . \qquad (9''')$$

Of course, the above formulae are for $\theta < 1$. The particular case $\theta = 1$ (in this case we have $\frac{0}{0}$) must be treated separately. We have

$$C(u,v) = \frac{uv}{u+v-uv} . \qquad (10)$$

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u) = -\frac{1}{u^2}$, and from here

$$\begin{cases} \varphi(u) = \frac{1}{u} - 1 \\ g(x) = \frac{1}{x+1} \end{cases} . \qquad (11)$$

It results that $g^{(n)}(x) = \frac{(-1)^n \cdot n!}{(x+1)^{n+1}}$ and finally

$$\begin{cases} f_0(x) = g(x) = \frac{1}{x+1} \\ f_1(x) = -g'(x) = \frac{1}{(x+1)^2} \\ f_n(x) = \frac{n \cdot x^{n-1}}{(x+1)^{n+1}} \quad for\ n \ge 2 \end{cases} . \qquad (12)$$

If we consider the log-copula family for $\alpha > 0$, $\gamma > 0$, $\beta = (\alpha\gamma)^{\alpha+1}$ and $r = \frac{1}{\alpha+1}$ we have (see [8,15,13])

$$\varphi(u) = \left(\frac{1-\ln u}{\alpha \cdot \gamma}\right)^{\alpha+1} - \frac{1}{(\alpha \cdot \gamma)^{\alpha+1}} = \frac{(1-\ln u)^{\alpha+1}}{\beta} - \frac{1}{\beta}, \text{ and} \qquad (13)$$

$$g(x) = e \cdot e^{-(\beta x+1)^r} . \qquad (13')$$

If we apply the formula (1) we obtain using the above notations

$$C(u,v) = e \cdot e^{-\left((1-\ln u)^{\alpha+1} + (1-\ln v)^{\alpha+1} - 1\right)^r} . \qquad (13'')$$

Remark 1. *The copula* C *from* (13''') *does not depend on* γ. *If we use* $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{\varphi'(u)}{\varphi'(v)}$ *as in the* *previous family cases we obtain* $\frac{\varphi'(u)}{\varphi'(v)} = \frac{v(1-\ln u)^{\alpha}}{u(1-\ln v)^{\alpha}}$. *If we take* $\varphi'(u) = -\frac{(1-\ln u)^{\alpha}}{u}$ *we obtain the above copula with* $\gamma = \gamma_1$ *so that* $(\alpha \cdot \gamma_1)^{\alpha+1} = \alpha + 1$. *To obtain the log-copula for another* γ *we have to multiply first the value of* $\varphi'(u)$ *by* $\left(\frac{\gamma}{\gamma_1}\right)^{\alpha+1}$.

From $\ln g(x) = 1 - (\beta x + 1)^r$ we obtain first

$$\frac{g'}{g} = -r \cdot \beta \cdot (\beta x + 1)^{r-1}.$$

Analogously to the Gumbel-Hougaard case we obtain

$$(\beta x + 1) \cdot g \cdot g'' = (r-1) \cdot \beta \cdot g \cdot g' + (\beta x + 1) \cdot (g')^2.$$

If we differentiate the above relation once we obtain

$$(\beta x + 1) \cdot g \cdot g''' = (r-2) \cdot \beta \cdot g \cdot g'' + (\beta x + 1) \cdot g' \cdot g'' + (r-1) \cdot \beta \cdot (g')^2.$$

If we differentiate $n-1$ times we obtain

$$(\beta x + 1) \cdot g \cdot g^{(n+1)} = -(\beta x + 1) \cdot \sum_{k=1}^{n-1} \frac{(n-1)!}{k! \cdot (n-k-1)!} \cdot g^{(k)} \cdot g^{(n-k-1)} -$$
$$(n-r) \cdot \beta \cdot g \cdot g^{(n)} + \beta \cdot \sum_{k=1}^{n-2} \frac{(n-1)!}{k! \cdot (n-k-2)!} \cdot g^{(k)} \cdot g^{(n-k)} +$$
$$(r-1) \cdot \beta \cdot \sum_{k=1}^{n-1} \frac{(n-1)!}{k! \cdot (n-k-1)!} \cdot g^{(k)} \cdot g^{(n-k)} + (\beta x + 1) \cdot \sum_{k=0}^{n-1} \frac{(n-1)!}{k! \cdot (n-k-1)!} \cdot g^{(k+1)} \cdot g^{(n-k)} +$$
$$\beta \cdot \sum_{k=0}^{n-2} \frac{(n-1)!}{k! \cdot (n-k-2)!} \cdot g^{(k+1)} \cdot g^{(n-k-1)}.$$

The recurrence formula for $g^{(n)}$ obtained analogously to the Gumbel-Hougaard case is

$$\begin{cases} f_0(x) = g(x) = e \cdot e^{-(\beta x+1)^r} \\ f_1(x) = -g'(x) = r \cdot \beta \cdot (\beta x + 1)^{r-1} \cdot f_0(x) \\ f_2(x) = x \cdot g''(x) = \frac{(1-r) \cdot \beta \cdot x \cdot f_1(x)}{\beta \cdot x + 1} + \frac{x \cdot f_1^2(x)}{f_0(x)} \\ f_3(x) = -\frac{x^2}{2} \cdot g'''(x) = \frac{(2-r) \cdot \beta \cdot x \cdot f_2(x)}{2 \cdot (\beta x + 1)} + \frac{x \cdot f_1(x) \cdot f_2(x)}{2 \cdot f_0(x)} + \frac{(1-r) \cdot \beta \cdot x^2 \cdot f_1^2(x)}{2 \cdot (\beta x + 1) \cdot f_0(x)} \\ f_{n+1}(x) = -\frac{x}{n \cdot f_0(x)} \cdot \sum_{k=1}^{n-1} \frac{n-k}{k} \cdot f_k(x) \cdot f_{n-k+1}(x) + \frac{(n-r) \cdot \beta \cdot x \cdot f_n(x)}{n \cdot (\beta x + 1)} - \\ \frac{\beta \cdot x^2}{n \cdot (\beta x + 1) \cdot f_0(x)} \cdot \sum_{k=1}^{n-1} \frac{n-k-1}{k} \cdot f_k(x) \cdot f_{n-k}(x) + \frac{(1-r) \cdot \beta \cdot x^2}{n \cdot (\beta x + 1) \cdot f_0(x)} \cdot \\ \sum_{k=1}^{n-1} \frac{1}{k} \cdot f_k(x) \cdot f_{n-k}(x) + \frac{x}{n \cdot f_0(x)} \cdot \sum_{k=0}^{n-1} f_{k+1}(x) \cdot f_{n-k}(x) - \\ \frac{\beta \cdot x^2}{n \cdot (\beta x + 1) \cdot f_0(x)} \cdot \sum_{k=0}^{n-2} f_{k+1}(x) \cdot f_{n-k-1}(x) \ \ for\ n \geq 3 \end{cases} . \tag{14}$$

The Nelsen Ten copula is

$$C(u,v) = \frac{u \cdot v}{\left(1 + \left(1 - u^{\theta}\right)\left(1 - v^{\theta}\right)\right)^{\frac{1}{\theta}}}, \text{ with } 0 < \theta \leq 1. \tag{15}$$

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u) = -\frac{1}{u\left(2-u^{\theta}\right)}$, and from here

$$\varphi(u) = \frac{1}{2\theta} \cdot \ln\left(2u^{-\theta} - 1\right), \text{ and} \tag{15'}$$

$$g(x)=\left(\frac{2}{e^{2\theta x}+1}\right)^{\frac{1}{\theta}}. \tag{15”}$$

From $\frac{2}{g^{\theta}}=e^{2\theta x}+1$ we obtain

$$g'(x)=-2\cdot g(x)+g^{\theta+1}(x).$$

We differentiate the above relation and we take into account that $g^{\theta+1}(x)=g'(x)+2\cdot g(x)$. We obtain

$$g''\cdot g=(\theta+1)\cdot(g')^2+2\cdot\theta\cdot g\cdot g'.$$

We differentiate the last formula $n-1$ times and we obtain

$$g\cdot g^{(n+1)}=-\sum_{k=1}^{n-1}C_{n-1}^{k}\cdot g^{(k)}\cdot g^{(n-k+1)}+(\theta+1)\cdot\sum_{k=0}^{n-1}C_{n-1}^{k}\cdot g^{(k+1)}\cdot g^{(n-k)}+$$
$$2\cdot\theta\cdot g^{(n)}+2\cdot\theta\cdot\sum_{k=1}^{n-1}C_{n-1}^{k}\cdot g^{(k)}\cdot g^{(n-k)}.$$

We obtain the recurrence formula

$$\begin{cases} f_0(x)=g(x)=\left(\frac{2}{e^{2\theta x}+1}\right)^{\frac{1}{\theta}} \\ f_1(x)=-g'(x)=2\cdot f_0(x)-f_0^{\theta+1}(x) \\ f_2(x)=x\cdot g''(x)=\frac{(\theta+1)\cdot x\cdot f_1^2(x)}{f_0(x)}-2\cdot\theta\cdot x\cdot f_1(x) \\ f_{n+1}(x)=-\frac{x}{n\cdot f_0(x)}\cdot\sum_{k=1}^{n-1}\frac{n-k}{k}\cdot f_k(x)\cdot f_{n-k+1}(x)+ \\ \frac{(\theta+1)\cdot x}{n}\cdot\sum_{k=0}^{n-1}f_{k+1}(x)\cdot f_{n-k}(x)-\frac{2\cdot\theta\cdot x}{n}\cdot f_0(x)\cdot f_n(x)- \\ \frac{2\cdot\theta\cdot x^2}{n}\cdot\sum_{k=1}^{n-1}\frac{1}{k}\cdot f_k(x)\cdot f_{n-k}(x)\ \ for\ n\geq 2 \end{cases}. \tag{15’’’}$$

The inverse of the cdf F_n of Z_2 is computed in any point U using the bisection method as in [4].

3. Other Families of Copulas and Their Simulation

In [1,15,8] it is presented the Fréchet family of $2-$copulas

$$C(x,y)=\alpha\cdot\min(x,y)+(1-\alpha-\beta)\cdot \Pr od(x,y)+\beta\cdot W(x,y), \tag{16}$$

where $\min$ and W are the Fréchet bounds from (2) and $\Pr od$ is the product copula (characteristic for independent random variables). In the above formula we have $\alpha,\beta\geq 0$ and $\alpha+\beta\leq 1$. Because the lower Fréchet bound W is a $2-$copula, but it is not an $n-$copula, in the above formula we take $\beta=0$ for $n\geq 3$. Therefore we have the following algorithm to generate the uniform random variables X_1, X_2,..., X_n connected by a copula C from the Fréchet family with a given value $\alpha\in[0,1]$.

Algorithm Fréchet

input (n,α)

Generate $X[1]$ uniform on $[0,1]$.

$C\leftarrow X[1]$

for $i=2$ **to** n **do begin**

Generate U uniform on $[0,1]$.

if $U\leq\alpha$ **then**

$X[i] \leftarrow C$

else

Generate $X[i]$ uniform on $[0,1]$.

if $X[i] < C$ **then**

$C = \alpha * X[i] + (1-\alpha) * X[i] * C$

else

$C = \alpha * C + (1-\alpha) * X[i] * C$

end

output $(X[1],...,X[n])$

end.

In the above algorithm we have taken into account that if X and Y are connected by the copula $\min$ there is an increasing function between them. This function is the identity if the variables are uniform on $[0,1]$ (see [4,5]).

For a $2-$copula for which we know the analytical formula we have used theorem 3. For an $n-$copula we can consider $X_2,...,$ X_n connected by the same copula, but of the order $n-1$ and the marginals of $X_i | X_1$. By mathematical induction we obtain the following algorithm if we know the cdf $F_x(y)$ from theorem 3 and its inverse $F_x^{-1}(u)$.

Algorithm copulaf

input (n, F_x, F_x^{-1})

Generate $X[1]$ uniform on $[0,1]$.

for $i = 2$ **to** n **do begin**

Generate $X[i]$ uniform on $[0,1]$.

for $j = i-1$ **downto** 1 **do**

$X[i] \leftarrow F_{X[j]}^{-1}(X[i])$

end

output $(X[1],...,X[n])$

end.

We consider the cases of Farlie-Gumbel-Morgestern, Cuadras-Augé and of the Raftery families of copula [13,15,4].

4. Simulation of the Service Systems and of the Queueing Networks

First of all we define a class "copula" that has "tipcop" type so that 0 means a Clayton copula, 1 means Frank copula, and the last value is 10 for Fréchet family. The attribute "nrmarg" is the number of marginals ($c+1$ if we have a queueing system with c channels, respectively n if we have a queueing network with n nodes). "caz" is the case of the modeled copula: 0 means queueing systems and 1 means queueing networks. The parameters of the copula (usualy one, except the cases like log-copula, when the number of parameters is two) is given by the vector "parcop". The marginals are given by the integer vector of types "tipmarg" (tipmarg[i]=0 means for instance that the marginal i is exponential), and the matrix of parameters "parmarg". The vector "timp" means the generated times. If we are in the case of service system, the first marginal represents the inter-arrival time.

In the constructor the initialization is for a queueing system with one channel and exponential marginals. In the method "citire" we read in the case of service system the number of channels, the type and the parameters of the copula, and the types and the parameters of the marginals. Using the

operator "+=(int ind)" we generate all the new values for generated marginal times if *model*=0, respectively only the component ind in the contrary case. In fact "model" represents the model of copula as in [4]: 0 for self-service model, 1 for feedback model, and 2 for random feedback model.
The average number of customers in the system is computed using the formula (see [4,5])

$$\overline{N} = \frac{\sum_{j=1}^{m} N_j \cdot t_j}{ts}, \tag{17}$$

where ts is the total simulation time and the number of customers in the system is constant N_j during the period t_j $\left(\sum_{j=1}^{m} t_j = ts\right)$. In the same way we compute the average number of customers in the queue:

$$\overline{N_f} = \frac{\sum_{j=1}^{m} N_{fj} \cdot t_j}{ts}. \tag{17'}$$

The probabilities that there is no customer in the system, p_0, respectively that an arriving customer has to wait, p_a, are estimated using the formulae

$$\begin{cases} \hat{p}_0 = \dfrac{T(N=0)}{ts} \\ \hat{p}_a = \dfrac{T(N \geq c)}{ts} \end{cases}, \tag{18}$$

where $T(N=0)$ and $T(N \geq c)$ are the periods during the whole simulation time ts when we have no customers in the system, respectively when at least a customer waits.
We consider two possibilities for a customer whose service starts. First is the usual way to choose randomly an existing free channel, and second is when the customer chooses the channel with the minimum service time.

5. Applications

Example 1. *Consider the queueing system with three channels, inter-arrival times* $\exp(7)$ *and the service times for each channel* $\exp(2.5)$. *The theoretical results in the classical case of the independence are the following:*

1) *The probability of no units in the system:* $p_0 = 0.01597$.
2) *The probability that a customer that arrives has to wait:* $p_a = 0.87667$.
3) *The average number of units in the system:* $M(N) = 15.07348$.
4) *The average number of units in the queue:* $M(N_f) = 12.27348$.

The results for the above marginals and different types of copula are in the following table, where the condition to stop the program is that the simulation time tsim becomes at least the maximum simulation time 100. If we consider that the customer does not know the order of service times for free channels, hence the free channel is chosen randomly and we obtain the following results.

Copula type	$\hat{p}_0$	$\hat{p}_a$	$\overline{N}$	$\overline{N_f}$	tsim
Fréchet: $\theta = 0$	0.0264	0.7434	12.03687	9.35435	100.13159

Fréchet: $\theta = 0.5$	0.09411	0.71928	8.48935	6.54966	100.01805
Fréchet: $\theta = 1$	0.01304	0.71174	7.13639	4.37438	100.00166
Clayton: $\theta = 0.2$	0.03161	0.75541	13.42655	10.75626	100.04908
Clayton: $\theta = 1.67$	0.01752	0.69136	11.47095	8.79975	100.03114
Frank: $\theta = 0.05$	0.01225	0.77924	9.72765	7.89722	100.02877
Frank: $\theta = -0.05$	0.01228	0.54759	9.17179	7.49877	100.18701
Frank: $\theta = 5$	0.00522	0.81282	8.34361	5.55577	100.10293
Frank: $\theta = -5$	0.00483	0.5445	5.05532	3.29293	100.03008
Gumbel-Barnett: $\theta = 0.05$	0.03376	0.63966	12.88822	10.26863	100.08793
Gumbel-Barnett: $\theta = 1$	0.01783	0.7183	6.30056	3.56906	100.2205
Ali-Mikhail-Haq: $\theta = -1$	0.04206	0.75285	7.49417	4.84923	100.26359
Ali-Mikhail-Haq: $\theta = 0$	0.01127	0.8705	15.76395	13.89076	100.06537
Ali-Mikhail-Haq: $\theta = 1$	0.01918	0.75856	9.32661	6.58906	100.06032

If we consider that the customer for which the service starts knows at least the order of the service times of free channels, hence he/she chooses the channel with the minimum service time we obtain the following results.

Copula type	$\hat{p}_0$	$\hat{p}_a$	$\overline{N}$	$\overline{N_f}$	tsim
Fréchet: $\theta = 0$	0.07139	0.65454	11.62451	9.15974	100.09229
Fréchet: $\theta = 0.5$	0.11038	0.69031	7.52908	5.74493	100.02578
Fréchet: $\theta = 1$	0.01193	0.62173	6.1211	3.4599	100.00276
Clayton: $\theta = 0.2$	0.02213	0.69357	8.92108	6.21623	100.03023
Clayton: $\theta = 1.67$	0.01339	0.68199	6.68807	4.02216	100.10459
Frank: $\theta = 0.05$	0.01142	0.6084	4.08631	1.36552	100.07043
Frank: $\theta = -0.05$	0.00916	0.53599	4.11366	1.43095	100.04171
Frank: $\theta = 5$	0.00569	0.54827	4.05264	1.34468	100.27253
Frank: $\theta = -5$	0.00483	0.5445	4.05532	1.29293	100.03008
Gumbel-Barnett: $\theta = 0.05$	0.01225	0.77924	9.23038	6.42876	100.03353
Gumbel-Barnett: $\theta = 1$	0.01223	0.53065	3.96116	1.3486	100.02269
Ali-Mikhail-Haq: $\theta = -1$	0.05204	0.49112	5.1455	5.1455	100.10366
Ali-Mikhail-Haq: $\theta = 0$	0.10958	0.54469	5.6125	3.34918	100.02569
Ali-Mikhail-Haq: $\theta = 1$	0.03873	0.67888	8.32877	5.75197	100.19296

We notice that in the second case the customers optimize their services: the probability to wait, the average number of customers in the system and the average number of customers in the queue are small if they know the order of services for free channels.

6. Conclusions

For all the Archimedean copulas presented in [1,8,14] and [15], and simulated in [4] we have $\varphi(0) = \lim_{v \to 0} \varphi(v) = \infty$. The computation of $g^{(n)}(w)$ by analytical formula is possible only in the case of the Clayton family. But in the other cases we have obtained recurrence formulae that are useful in our $C++$ program. Due to the big numbers obtained if we use the recurrence formulae for $g^{(n)}$ we use instead the recurrence formulae for f_n: the high terms for $g^{(n)}$ are compensated by the factor $\frac{x^{n-1}}{(n-1)!}$, which tends to 0 for $n \to \infty$.

In the case of Fréchet family we have considered $\beta = 0$ (hence a mixture between the copula Prod and the copula $\min$). It is an open problem if we can consider also $0 < \beta < 1$ (we know only that we can not take $\beta = 1$).

We notice also the good approximations expressed by our $C++$ program. We know for instance that the limit case for Clayton family is the product copula (independence) for $\theta \to 0$, respectively the upper Fréchet bound (copula $\min$) for $\theta \to \infty$: see the estimated values for $\theta = 0.2$ and $\theta = 1.67$.

As models for queueing systems we have consider two models for the way the customer for which the service starts chooses the free channel: first when he/she does not know which is the free channel with the fastest service, hence he/she chooses one randomly, and the second when he/she knows, and chooses the fastest one. An open problem is to consider the models of self-service, the model of feedback and the model of random feedback as for only one station (see [4,5]).

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