Gaussian and non-Gaussian models for financial bubbles via econophysics

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Abstract

We develop a rational expectations model of financial bubbles and study how the risk-return interplay is incorporated into prices. We retain the interpretation of the leading Johansen-Ledoit-Sornette model: namely, that the price must rise prior to a crash in order to compensate a representative investor for the level of risk. This is accompanied, in our stochastic model, by an illusion of certainty as described by a decreasing volatility function. As the volatility function decreases crashes can be seen to represent a phase transition from stochastic to deterministic behaviour in prices. Our approach is first illustrated by a benchmark Gaussian model – subsequently extended to a heavy-tailed model based on the Normal Inverse Gaussian distribution. Our model is illustrated by an empirical application to the London Stock Exchange. Results suggest that the aftermath of the Bank of England's process of quantitative easing has coincided with a bubble in the FTSE 100.

Keywords: financial crashes, super-exponential growth, illusion of certainty, heavy tails, bubbles.

1 Introduction

Rational expectations models were introduced with the work of Blanchard and Watson to account for the possibility that prices may deviate from fundamental levels [1]. We take

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as our main starting point the somewhat controversial subject of log-periodic precursors to financial crashes [2]-[11], with a fundamental aim of our approach being relatively easy calibration of our model to empirical data. Additional background on log-periodicity and complex exponents can be found in [12]. A first-order approach in [3] and subsequent extensions in [13] state that prior to a crash the price must exhibit a super-exponential growth in order to compensate a representative investor for the level of risk. However, this approach concentrates solely on the drift function and ignores the underlying volatility fluctuations which typically dominate financial time series [14]. We undertake a similar approach to that in [3] but extend the original method by deriving a second-order condition which incorporates volatility fluctuations and enables us to combine insights from a rational expectations model with a stochastic model [15]-[16].

Our model gives two important characterisations of bubbles in economics. Firstly, a rapid super-exponential growth in prices. Secondly, an illusion of certainty as described by a decreasing volatility function prior to the crash. As the volatility function goes to zero bubbles and crashes can be seen to represent a phase transition from stochastic to purely deterministic behaviour in prices. This clarifies the oft cited link in the literature between phase transitions in critical phenomena and financial crashes. Further, this recreates the phenomenology of the Sornette-Johansen paradigm: namely that prices resemble a deterministic function prior to a crash. We explore a number of different applications of our model and the potential relevance to recent events is striking.

The layout of this paper is as follows. In Section 2 we introduce a benchmark Gaussian model. In Section 3 we extend the basic model to a heavy-tailed setting in order to account for leptokurtosis in financial returns. Section 4 gives an empirical application. Section 5 is a conclusion. A probability Appendix, included for the reader's convenience, can be found at the end of the paper.

2 Motivation: a simple Gaussian model

In this section we derive and solve a Gaussian model for financial bubbles, our approach later serving to motivate a non-Gaussian model in Section 3. An alternate formulation of the basic model in [3] leads naturally to a stochastic generalisation of the original model as follows. Let $P(t)$ denote the price of an asset at time t. Our starting point is the equation

$$P(t) = P_1(t)(1-\kappa)^{j(t)}, \tag{1}$$

where $P_1(t)$ satisfies

$$dP_1(t) = \mu(t)P_1(t)dt + \sigma(t)P_1(t)dW_t, \tag{2}$$

where W_t is a Wiener process and $j(t)$ is a jump process satisfying

$$j(t) = \begin{cases} 0 & \text{before the crash} \\ 1 & \text{after the crash.} \end{cases} \tag{3}$$

When a crash occurs $\kappa\%$ is automatically wiped off the value of the asset. Prior to a crash $P(t) = P_1(t)$ and $X_t = \log(P(t))$ satisfies

$$dX_t = \tilde{\mu}(t)dt + \sigma(t)dW_t + \ln[(1-\kappa)]dj(t), \tag{4}$$

where $\tilde{\mu} = \mu(t) - \sigma^2(t)/2$. If a crash has not occurred by time t, we have that

$$\begin{aligned} E[j(t+dt) - j(t)] &= h(t)dt + o(dt), & (5) \\ \text{Var}[j(t+dt) - j(t)] &= h(t)dt + o(dt), & (6) \end{aligned}$$

where $h(t)$ is the hazard rate. We compare (4) with the prototypical Black-Scholes model for a stock price:

$$dX_t = \mu dt + \sigma dW_t, \tag{7}$$

where $\tilde{\mu} = \mu - \sigma^2/2$, and use (7) as our model for "fundamental" or purely stochastic behaviour in prices.

The first-order condition see e.g. [1], [3], suggests that $\tilde{\mu}(t)$ in (4) grows in order to compensate a representative investor for the risk associated with a crash. The instantaneous drift associated with (4) is

$$\tilde{\mu}(t) + (\ln(1-\kappa))h(t). \tag{8}$$

For (7) the instantaneous drift is $\tilde{\mu}$. Setting (8) equal to $\tilde{\mu}$, it follows that in order for bubbles and non-bubbles to co-exist

$$\tilde{\mu}(t) = \tilde{\mu} - (\ln(1-\kappa))h(t). \tag{9}$$

If we ignore volatility fluctuations by setting $\sigma(t) = \sigma$, then our pre-crash model for an asset price becomes

$$dX_t = (\tilde{\mu} - \ln(1-\kappa)h(t))dt + \sigma dW_t. \tag{10}$$

However, this is actually a rather poor empirical model [18], failing to adequately account for the volatility fluctuations in (4). Under a Markowitz interpretation, means represent returns and variances/standard deviations represent risk. Suppose that in (4) $\sigma(t)$ adapts in an analogous way to $\mu(t)$ so as to compensate a representative investor for bearing additional levels of risk. The instantaneous variance associated with (4) is

$$\sigma^2(t) + (\ln(1-\kappa))^2 h(t). \tag{11}$$

For (7) the instantaneous variance is σ^2. Setting (11) equal to σ^2, the second-order condition for co-existence of bubbles and non-bubbles becomes

$$\sigma^2(t) = \sigma^2 - (\ln(1-\kappa))^2 h(t). \tag{12}$$

(12) illustrates an *illusion of certainty* – a decrease in the volatility function – which arises as part of a bubble process. Intuitively, in order for a bubble to occur not only must returns increase but the volatility must also decrease. If this does not happen (7) with an instantaneous variance of σ^2 would represent a more attractive and less risky investment than a market described by (10) and bubbles could not occur. We use (7) as a model of a 'fundamental' or purely stochastic regime, as in Black-Scholes theory. From (12), our model for prices under a bubble regime becomes

$$dX_t = [\tilde{\mu} - \ln(1-\kappa)h(t)]dt + \sqrt{\sigma^2 - (\ln(1-\kappa))^2 h(t)}dW_t. \tag{13}$$

The simplest $h(t)$ considered in [3] is

$$h(t) = B(t_c - t)^{-\alpha}, \tag{14}$$

where it is assumed that $\alpha \in (0,1)$ and t_c is a critical time when the hazard function becomes singular, by analogy with phase transitions in statistical mechanical systems [19]. Here, we choose on purely statistical grounds

$$h(t) = \frac{\beta t^{\beta-1}}{\alpha^\beta + t^\beta}, \tag{15}$$

which is the form corresponding to a log-logistic distribution and is intended to capture the essence of the previous approach as the hazard rate has both a relatively simple form and, for $\beta > 1$, has a non-trivial mode at $t = \alpha(\beta - 1)^{\frac{1}{\beta}}$, with modal point $(\beta - 1)^{1-\frac{1}{\beta}}/\alpha$. For these reasons, the log-logistic distribution is commonly used in statistics [20]. The log-logistic distribution has probability density

$$f(x) = \frac{\beta\alpha^{\beta}x^{\beta-1}}{(\alpha^{\beta} + x^{\beta})^2}, \tag{16}$$

on the positive half-line. The cumulative distribution function is

$$F(x) = 1 - \frac{\alpha^{\beta}}{\alpha^{\beta} + x^{\beta}} \tag{17}$$

The model (13) with $h(t)$ given by (15) has the solution

$$X_t = X_0 + \tilde{\mu}t + v \ln\left(1 + \frac{t^{\beta}}{\alpha^{\beta}}\right) + \int_0^t \sqrt{\sigma^2 - v^2 \frac{\beta t^{\beta-1}}{\alpha^{\beta} + t^{\beta}}} dW_u. \tag{18}$$

where $v = -\ln(1 - \kappa)$ with $v > 0$. From (18) the conditional densities can be written as

$$X_t | X_s \sim N(\mu_{t|s}, \sigma^2_{t|s}), \tag{19}$$

where

$$\mu_{t|s} = X_s + \tilde{\mu}(t - s) + v \ln\left(\frac{\alpha^{\beta} + t^{\beta}}{\alpha^{\beta} + s^{\beta}}\right), \tag{20}$$

$$\sigma^2_{t|s} = \sigma^2(t - s) - v^2 \ln\left(\frac{\alpha^{\beta} + t^{\beta}}{\alpha^{\beta} + s^{\beta}}\right). \tag{21}$$

Under the fundamental equation (7) these expressions are simply $\mu_{t|s} = X_s + \tilde{\mu}(t - s)$ and $\sigma^2_{t|s} = \sigma^2(t - s)$. Thus, we see that under the bubble model the incremental distributions demonstrate a richer behaviour over time.

The fundamental or purely stochastic non-bubble model (7) corresponds to the case that $\kappa = 0$, or equivalently that $v = 0$. We can test for bubbles by testing the null hypothesis $v = 0$ (no bubble) against the alternative hypothesis $v > 0$ (bubble). This can be simply done using a (one-sided) t-test since maximum likelihood estimates, and estimated standard errors, can be easily calculated numerically from (19). A range of further implications of our bubble model can be derived as we describe below.

Crash-size distribution. Suppose that prices are observed up to and including time t and that a crash has not occurred by time t. The crash-size distribution resists an analytical description but a Monte Carlo algorithm to simulate the crash-size C is straightforward and reads as follows:

1. Generate u from $U \sim$ Log-logistic(α, β) with the constraint $u \geq t$.
2. $C \sim \kappa e^Z$,
where

$$Z \sim N\left(X_t + \tilde{\mu}(u-t) + v \ln\left(\frac{\alpha^\beta + u^\beta}{\alpha^\beta + t^\beta}\right), \sigma^2(u-t) - v^2 \ln\left(\frac{\alpha^\beta + u^\beta}{\alpha^\beta + t^\beta}\right)\right) \tag{22}$$

We note that simulating u from the log-logistic distribution is straight-forward and from (17) possible via inversion using

$$F^{-1}(x) = \alpha\left(\frac{x}{1-x}\right)^{\frac{1}{\beta}} \quad \text{or} \quad F^{-1}(x) = \left(\frac{\alpha^\beta + t^\beta}{1-x} - \alpha^\beta\right)^{\frac{1}{\beta}} \quad \text{with constraint } u \geq t.$$

Post-crash increase in volatility. Before a crash equation (18) applies and the volatility is given by

$$\sigma^2(t) = \sigma^2 - \frac{v^2 \beta t^{\beta-1}}{\alpha^\beta + t^\beta}. \tag{23}$$

After a crash, the volatility reverts to its fundamental level σ^2. Equation (23) thus predicts a post-crash increase in volatility according to

$$\sigma^2(t) \ \propto \ \frac{v^2 \beta t^{\beta-1}}{\alpha^\beta + t^\beta}. \tag{24}$$

For $\beta = 1$ (24) corresponds to the model of post-financial-crash volatility decay in [21].

Fundamental values. The above model suggests a simple approach to estimate fundamental value. Under the fundamental dynamics (7)

$$P_F(t) := E(P(t)) = P(0)e^{\mu t}. \tag{25}$$

(25) leads to a simple approach to estimate fundamental value. This approach recreates the widespread phenomenology of approximate exponential growth in economic time series (see e.g. Chapter 7 in [22]).

*Estimated bubble component.*Define

$$H(t) = \int_0^t h(u)du. \tag{26}$$

Under the fundamental model $E(P(t))$ is given by (25). Under the bubble model, since

$X_t = \log(P_t)$ satisfies

$$X_t \sim \mathrm{N}\left(X_0 + \tilde{\mu}t + vH(t), \sigma^2 t - v^2 H(t)\right), \tag{27}$$

it follows that

$$P_B(t) := E(P(t)) = P(0)e^{\mu t + \left(v - \frac{v^2}{2}\right)H(t)}, \tag{28}$$

where $H(t)$ is given by

$$H(t) = \ln\left(1 + \frac{t^\beta}{\alpha^\beta}\right). \tag{29}$$

This motivates the following estimate for the proportion of observed prices which can be attributed to a speculative bubble:

$$1 - \frac{1}{T}\int_0^T \frac{P_F(t)}{P_B(t)}\, dt = 1 - \frac{1}{T}\int_0^T \left(1 + \frac{t^\beta}{\alpha^\beta}\right)^{-(v - v^2/2)} dt. \tag{30}$$

3 Heavy-tailed models via the NIG distribution

3.1 Purely stochastic or fundamental model

As a model for fundamental or purely stochastic behaviour in prices we choose the equation

$$dP(t) = \mu P(t)dt + \sigma\sqrt{U}P(t)dW_t, \tag{31}$$

where U is an unobserved random variable with an $\mathrm{IG}(1, 1/K)$ distribution (see the Appendix), which has mean 1 and is independent of the Wiener process W_t. This formulation retains the tractability of Gaussian stochastic calculus [23] but enables one to generate heavy-tailed non-Gaussian behaviour inline with stylized empirical facts [14], Chapter 7. The models in this section are based around the Normal Inverse Gaussian (NIG) distribution [24]-[25]. See the Appendix for the definition and for some additional facts about this distribution.

From (31) it follows that the log-price X_t evolves according to

$$dX_t = \left[\mu - \frac{\sigma^2 U}{2}\right] dt + \sigma\sqrt{U}dW_t. \tag{32}$$

From Result 1 in the Appendix it follows that

$$X_t \sim \text{NIG}\left(\mu = X_0 + \mu t, \alpha = \sqrt{\frac{1}{\sigma^2 t K} + \frac{1}{4}}, \beta = -\frac{1}{2}, \delta = \frac{\sigma\sqrt{t}}{\sqrt{K}}\right). \tag{33}$$

Further, the incremental distributions are given by

$$X_{t+\Delta} - X_t \sim \text{NIG}\left(\mu = \mu\Delta, \alpha = \sqrt{\frac{1}{\sigma^2 \Delta K} + \frac{1}{4}}, \beta = -\frac{1}{2}, \delta = \frac{\sigma\sqrt{\Delta}}{\sqrt{K}}\right). \tag{34}$$

We have that

$$E[X_{t+\Delta} - X_t] = \mu + \frac{\delta\beta}{\gamma} = \mu\Delta + \sigma\sqrt{\frac{\Delta}{K}}\left(-\frac{1}{2}\right)\sqrt{\sigma^2\Delta K} = \Delta\left(\mu - \frac{\sigma^2}{2}\right), \tag{35}$$

and

$$\begin{aligned} \text{var}[X_{t+\Delta} - X_t] &= \frac{\delta\alpha^2}{\gamma^3} = \sigma\sqrt{\frac{\Delta}{K}}(\sigma^2\Delta K)^{\frac{3}{2}}\left(\frac{1}{\sigma^2\Delta K} + \frac{1}{4}\right) = \sigma^4\Delta^2 K\left(\frac{1}{\sigma^2\Delta K} + \frac{1}{4}\right) \\ &= \sigma^2\Delta + o(\Delta). \end{aligned} \tag{36}$$

Hence it follows, as in the Gaussian case, that under the fundamental or purely stochastic regime X_t has instantaneous mean or drift given by $\mu - \sigma^2/2$ and instantaneous variance given by σ^2.

As was the case with the Gaussian model in Section 2, this simple NIG model also suggests a simple approach to estimating fundamental value. It follows from (33) and Result 2 in the Appendix that

$$P_F(t) := E(P(t)) = e^{X_0 + \mu t} = P(0)e^{\mu t}. \tag{37}$$

3.2 Leptokurtic bubble model

We formulate a heavy-tailed extension of the Gaussian bubble model in Section 2 as follows. We retain (1) but replace (2) with the equation

$$dP_1(t) = \mu(t)P_1(t)dt + \sigma(t)\sqrt{U}P_1(t)dW_t. \tag{38}$$

As before, we have that prior to a crash $P(t) = P_1(t)$ and from (38) that $X_t = \log(P(t))$ satisfies

$$dX_t = \left(\mu - \frac{\sigma^2(t)U}{2}\right) dt + \sigma(t)U dW_t + \ln[1-\kappa]dj(t). \tag{39}$$

Under the bubble model (39) we have that

$$E[X_{t+\Delta} - X_t|U] = \Delta\left[\mu(t) - \frac{\sigma^2(t)}{2}U\right] + \Delta\ln[1-\kappa]h(t) + o(\Delta). \tag{40}$$

Therefore

$$\begin{aligned} E[X_{t+\Delta} - X_t] &= E\left[\Delta\left[\mu(t) - \frac{\sigma^2(t)}{2}U\right]\right] + \Delta\ln[1-\kappa]h(t) + o(\Delta), && (41)\\ &= \Delta\left[\mu(t) - \frac{\sigma^2(t)}{2} + \ln[1-\kappa]h(t)\right] + o(\Delta). && (42)\end{aligned}$$

Similarly, we see that

$$\begin{aligned} \text{var}[X_{t+\Delta} - X_t] &= E[\text{var}[X_{t+\Delta} - X_t|U]] + \text{var}\left(E[X_{t+\Delta} - X_t|U]\right)\\ &= \Delta E[\sigma^2(t)U + v^2h(t)] + \Delta^2\text{var}\left(\mu - \frac{\sigma^2 U}{2}\right) + o(\Delta)\\ &= \Delta[\sigma^2(t) + v^2h(t)] + o(\Delta), && (43)\end{aligned}$$

where $v = -\ln[1-\kappa]$. Hence, it follows that under the bubble model the instantaneous mean is $\mu(t) - \sigma^2(t)/2 + vh(t)$ and the instantaneous variance is $\sigma^2(t) + v^2h(t)$. The mean-variance conditions for the co-existence of bubbles and non-bubbles become

$$\sigma^2 = \sigma^2(t) + v^2h(t),\ \sigma^2(t) = \sigma^2 - v^2h(t), \tag{44}$$

and

$$\mu - \frac{\sigma^2}{2} = \mu(t) - \frac{\sigma^2(t)}{2} - vh(t); \mu(t) = \mu + \left(v - \frac{v^2}{2}\right)h(t). \tag{45}$$

3.3 Statistical properties of the bubble model

As constructed, the bubble model in (38) has the following construction:

$$\begin{aligned} U &\sim \text{IG}\left(1, \frac{1}{K}\right),\\ X_t|U &\sim \text{N}\left(X_0 + \mu t + \left(v - \frac{v^2}{2}\right)H(t) - \frac{\left(\sigma^2 t - v^2H(t)\right)U}{2}, \left[\sigma^2 t - v^2H(t)\right]U\right).(46)\end{aligned}$$

It follows from (46) and Result 1 in the Appendix that that X_t is NIG distributed with parameters

$$\begin{aligned}
\mu &= X_0 + \mu t + \left(v - \frac{v^2}{2}\right) H(t), \\
\alpha &= \sqrt{\frac{1}{\left(\sigma^2 t - v^2 H(t)\right) K} + \frac{1}{4}}, \\
\beta &= -\frac{1}{2}, \\
\delta &= \frac{\sqrt{\sigma^2 t - v^2 H(t)}}{\sqrt{K}},
\end{aligned} \tag{47}$$

where $H(t)$ is given by (29). Similar reasoning shows that we have that the conditional distribution of X_t given X_s is NIG distributed with parameters

$$\begin{aligned}
\mu &= X_s + \mu(t-s) + \left(v - \frac{v^2}{2}\right) \ln\left(\frac{\alpha^\beta + t^\beta}{\alpha^\beta + s^\beta}\right), \\
\alpha &= \sqrt{\frac{1}{\left(\sigma^2 (t-s) - v^2 \ln\left(\frac{\alpha^\beta + t^\beta}{\alpha^\beta + s^\beta}\right)\right)} + \frac{1}{4}}, \\
\beta &= -\frac{1}{2}, \\
\delta &= \frac{\sqrt{\sigma^2 (t-s) - v^2 \ln\left(\frac{\alpha^\beta + t^\beta}{\alpha^\beta + s^\beta}\right)}}{\sqrt{K}}.
\end{aligned} \tag{48}$$

Crash-size distribution. Suppose that prices are observed up to and including time t and that a crash has not occurred by time t. The crash-size distribution resists an analytical description but a Monte Carlo algorithm to simulate the crash-size C is straightforward and reads as follows:

1. Generate u from $U \sim$ Log-logistic(α, β) with the constraint $u \geq t$.

2. $C \sim \kappa e^Z$,

where Z is NIG distributed with parameters

$$\begin{aligned}
\mu &= X_t + \mu(u-t) + \left(v - \frac{v^2}{2}\right) \ln\left(\frac{\alpha^\beta + u^\beta}{\alpha^\beta + t^\beta}\right), \\
\alpha &= \sqrt{\frac{1}{\left(\sigma^2 (u-t) - v^2 \ln\left(\frac{\alpha^\beta + u^\beta}{\alpha^\beta + t^\beta}\right)\right)} + \frac{1}{4}}, \\
\beta &= -\frac{1}{2},
\end{aligned}$$

$$\delta = \frac{\sqrt{\sigma^2(u-t) - v^2 \ln\left(\frac{\alpha^\beta + u^\beta}{\alpha^\beta + t^\beta}\right)}}{\sqrt{K}} \tag{49}$$

Estimated bubble component. Under the fundamental model $E(P(t))$ is given by (37). Under the bubble model, since it follows from (47) and Result 2 in the Appendix that

$$P_B(t) := E(P(t)) = P(0)e^{\mu t + \left(v - \frac{v^2}{2}\right)H(t)}. \tag{50}$$

Continuing, we see that the estimated bubble component can be formulated in exactly the same way as in equation (30).

4 Empirical application

As an empirical application we look at daily prices of the FTSE 100 from March 2nd 2009 to October 29th 2010 to try and determine whether or not the Bank of England's policy of quantitative easing has coincided with, and possibly led to, a speculative bubble in the London Stock Exchange. As shown in Figure 1, even with such a relatively short data set, there appears to be some merit in using a heavy-tailed non-Gaussian model with the asymmetric NIG model offering a better fit than the normal distribution to the right tail of the empirical distribution of the log-returns.

Testing the null hypothesis of no bubble is a test of the hypothesis $v = 0$. This can be tested using a one-sided t-test – dividing the estimate $\hat{v}$ by its estimated standard error and comparing to a normal distribution. For this data set we obtain a t-statistic of 3.332 and a p-value of 0.000, giving strong evidence of a bubble. A plot of observed prices compared to estimated fundamental values is shown in Figure 2. Some degree of over-pricing is apparent although prices appear to have moved closer to estimates of fundamental value over the second half of 2010. In contrast, however, calculating the estimated bubble component in equation (30) is only estimated to be 0.006, suggesting that the speculative bubble component accounts for a relatively trivial amount, roughly 0.6%, of the observed prices.

In summary, the statistical test and the plot shown in Figure 2 give enough evidence to point to a bubble and to some level of over-pricing in the FTSE 100. However, the level of over-pricing does not seem particularly large and prices appear to have moved closer to estimated fundamental values over the second half of 2010. The level of over-pricing also seems much less than the recent UK housing bubble where a similar approach suggested that the speculative bubble component accounted for around 20% of the observed prices

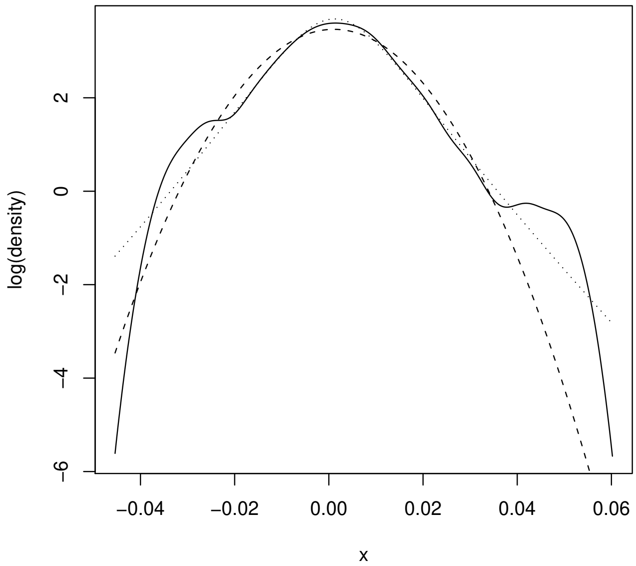

Figure 1: Distribution of log-returns. Plot of log kernel density estimate (solid line), together with best fits from a normal distribution (dashed line), and asymmetric NIG distribution (with $\beta = -1/2$) dots.

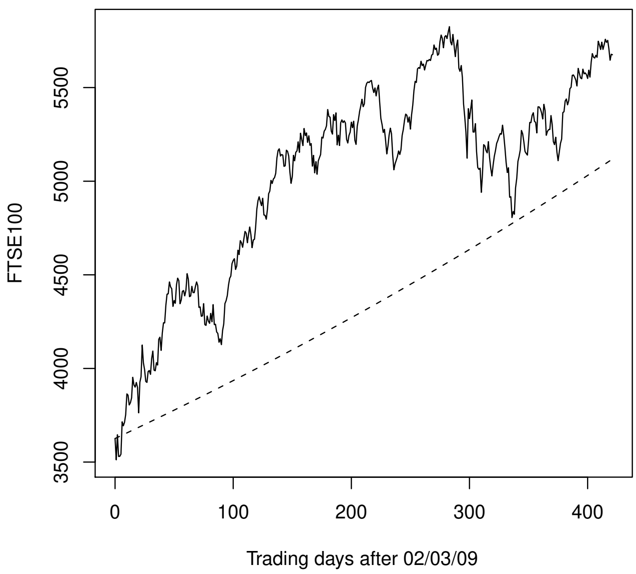

Figure 2: Plot of observed prices (solid line) together with estimated fundamental value (dashed line).

[26].

5 Conclusions

This paper builds on the now well-established analogy between financial crashes and phase transitions in critical phenomena. In a stochastic version of the original model of Johansen et al. (2000) crashes are seen to represent a phase transition from random to deterministic behaviour in prices. Crash precursors are a super-exponential growth accompanied by an"illusion of certainty", characterised by a decrease in the volatility function prior to the crash. A Gaussian model is introduced and then further extended to incorporate a heavy-tailed version of the model based around the NIG distribution. Under both settings a range of potential applications to economics were discussed. These include statistical tests for bubbles, crash-size distributions, predictions of a post-crash increase in volatility – related to Omori-style power laws in complex systems – and simple estimates of fundamental-value and speculative-bubble components. As an empirical application we test for whether a bubble is present in the FTSE 100 following the introduction of the Bank of England's policy of quantitative easing. Some evidence of a bubble and subsequent over-pricing is found. However, the level of over-pricing does not appear very large – particularly in comparison to the recent UK housing bubble – and prices appear to have converged towards estimated fundamental values during the latter half of 2010.

Probability appendix

Definition 1 *The inverse Gamma distribution is the probability distribution on* $[0, \infty)$ *with parameters* μ*,* λ *and probability density*

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}} e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}. \tag{51}$$

The mean is equal to μ *and the variance is equal to* μ^3/λ*.*

Definition 2 *The normal inverse Gaussian distribution is the probability density on* $(-\infty, \infty)$ *with parameters* μ*,* α*,* β*,* δ*. Define* $\gamma = \sqrt{\alpha^2 - \beta^2}$*,* $|\beta| < \alpha$*. The NIG distribution has probability density function given by*

$$f(x) = \frac{\alpha\delta K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{\pi\sqrt{\delta^2 + (x-\mu)^2}} e^{\delta\gamma + \beta(x-\mu)}. \tag{52}$$

where K_1 denotes the modified Bessel function of the second kind with integral representation

$$K_v(z) = \frac{e^{-z}}{\Gamma\left(v+\frac{1}{2}\right)}\sqrt{\frac{\pi}{2z}}\int_0^{\infty} e^{-t}t^{v-\frac{1}{2}}\left(1+\frac{t}{2z}\right)^{v-\frac{1}{2}} dt. \tag{53}$$

In addition to (53) we note, for later use, the following integral [14]

$$\int_0^{\infty} e^{-\frac{\alpha^2 t}{2}-\frac{\beta^2}{2t}} t^{-1-v} dt = 2\left(\frac{\alpha}{\beta}\right)^v K_v(\beta\alpha). \tag{54}$$

The mean of the NIG distribution is

$$\mu + \frac{\delta\beta}{\gamma}, \tag{55}$$

and the variance is

$$\frac{\delta\alpha^2}{\gamma^3}. \tag{56}$$

Further, the moment generating function of the NIG distribution, $E[\exp\{tX\}]$ is given by

$$M_X(t) = e^{\mu t+\delta\left(\gamma-(\beta+t)^2\right)}. \tag{57}$$

Result 1 (Mixture representation of the NIG distribution) *Suppose that X and U are random variables obeying the following construction:*

$$\begin{aligned} U &\sim IG(1, \frac{1}{K}) & (58)\\ X|U &\sim N\left(\mu - \frac{\sigma^2 U}{2}, \sigma^2 U\right), & (59)\end{aligned}$$

then the marginal distribution of X is NIG($\mu,\ \alpha,\ \beta,\ \delta$) where

$$\begin{aligned} \mu &= \mu, \\ \alpha &= \sqrt{\frac{1}{\sigma^2 K}+\frac{1}{4}}, \\ \beta &= -\frac{1}{2}, \\ \delta &= \frac{\sigma}{\sqrt{K}}. \end{aligned} \tag{60}$$

Result 2 *Suppose that X is NIG distributed with parameters given by (60). Then it* *follows that*

$$E(e^X) = e^{\mu}. \tag{61}$$

Proof
It follows from (57) that

$$\begin{aligned} E[e^X] &= e^{\mu+\delta(\gamma-\sqrt{\alpha^2-(-1/2+1)^2})} & (62)\\ &= e^{\mu+\delta(\gamma-\sqrt{\alpha^2-4})} = e^{\mu+\delta.0} = e^{\mu}. & (63)\end{aligned}$$

□

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