Risk, limited liability and firm scope

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Abstract

This paper provides a new explanation for the relationship between firm scope, agent's effort and corporate risk. I set up a moral hazard in teams model with multiple agents and departments under the assumption that both the principal and the agents are protected by limited liability. Each agent exerts effort to reduce the probability of loss of his department. The *two-sided limited liability assumption* creates an externality between agents, since the bad performance of an agent could reduce the firm's expected profit, and decrease the expected payoff of a good performing agent within the same firm. This would lower the incentive for other agents to exert effort, which causes *'Contagious shirking'*. I prove for the optimal contract and derive conditions for effort to increase or decrease with scope, and explain why 'contagious effect' could better answer this question than diversification when firm scope is large.

*'Nothing can be so unjust as for a few persons abounding in wealth to offer a portion of their excess for the information of a company, to play with that excess, to lend the importance of their whole name and credit to the society, and then should the funds prove **insufficient to answer all demands**, to retire into the security of their unhazarded fortune, and leave the bait to be devoured by the poor deceived fish.'*

<div align="right">

---*The Times of London*, 1824.5.25

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1. Introduction

We witnessed the fall of many large firms during the recent financial crisis. AIG was forced to accept nationalization due to mistakes in investment contracts although its insurance business was still promising. The case is also true for Lehman Brothers and many others, that the default in one or several of its many departments may trigger the fall of a large firm. The traditional view that a firm with large scope could lower its risk through diversification is being challenged by these new empirical facts.

This paper provides a new explanation by using an agency approach to discuss the relationship between scope and risk. I set up a moral hazard in teams model with multiple projects and agents. The principal chooses the number of projects and hires one agent for each project. He then signs contracts with the agents. Each agent exerts unobservable effort in order to reduce the probability of loss in his project.

The key assumption of our model is *two-sided limited liability*. Limited liability for the agent implies that punishments cannot go to extremes in designing contracts, creating an insufficient incentive problem. Limited liability for the principal brings externalities to agents' performances, since the principal could default on wage payments when firm profit is low. So his ability to pay wages depends on the firm's profit which is affected by the agents' efforts. An agent's payoff is then correlated with the other agents' efforts, and the shirking of one agent negatively affects the incentives of others within the same firm, leading to *'contagious shirking'*. So, an agent's incentive is not only affected by the principal's incentive scheme, but also by the efforts of other agents: the shirking of other agents would make the principal's wages less effective in providing incentives.

I will prove that the contract we previously specified is the optimal one, and discuss the effect of firm expansion on agents' incentives, which is determined by the *externality* (either positive or negative) *an additional agent creates on the pre-existing agents*. When effort cost is high, an additional agent would exert less effort, and have a larger negative externality on the

whole team, which, in turn, makes the principal's wage less effective in providing incentives. The principal would then lower wage in the new contract which exacerbates the shirking problem. The magnitude of loss is another key determinant of the externality. So high effort cost and large losses would cause agent's effort to decline when the firm expands.

This paper mainly relates to three strands of literature: *firm scope*, *moral hazard in teams* and the discussions on distortions brought on by *limited liability*.

Originated from Coase (1937), followed by Alchian, Demsetz (1972), and Williamson (1985), the discussion of firm boundaries could be seen in the mass literature related to industrial organization and theory of the firm. These literature captured certain important features of a firm and set up relationships for agents within a firm to distinguish them from two agents working for different firms. The property rights view, for example, defined the firm as '*a collection of physical assets under common ownership*' and analyzed the role of ownership in providing incentives for ex ante relationship-specific investments in a world of incomplete contracts (Grossman and Hart 1986, Hart and Moore 1990). The agents in their model were interrelated by future transactions, whose gains depend on the two agents' ex-ante unverifiable relationship-specific investments.

Other models and views regarding firm scope will be discussed in detail in section V. This model, different from previous ones, focuses on the two sided limited liability characteristic of a firm. Agents in the same firm are correlated by the financial situation of the firm, which is affected by the agents' effort and will affect the expected payoff of every good performing agent.

The *'moral hazard in teams'* problem was first addressed by Holmström (1982), who highlighted the free-riding and competition problems associated with a multi-agent setting. Extensions of this treatment include: a team of risk averse agents (Rasmusen 1987), reputation and relational concerns (Rayo 2007), two sided moral hazard problems (Najjar 1997) and so forth.

In addition, many papers discussed the relationship between individual agent's effort and the number of agents (team size, span of control, firm scope, etc.). For example, in Aghion and Tirole's (1997) model, the principal and the agents exert effort to discover the payoffs of different possible actions. An increase in the number of agents would lower the principal's effort on every agent's project, and lowers his probability to discover the payoffs of different alternatives, which means, an agent's decision is less likely to be ruled out by the principal if he discovers his optimal choice. The agents would exert more effort as a response. If we change the relationship between

the agents' and the principal's effort from substitutes to complements, for example, the principal's effort is to monitor the agents, then an increase in firm scope would decrease agents' effort at the individual level (Qian 1994).

Compared with previous literatures on moral hazard in teams, our model has the following differences: under the setting of indirect externalities and the assumption of unobservable effort, the principal's wage (incentive scheme) acts like a '*magnifier*' to high effort cost when firm scope expands, which means, the changes in wage when firm scope expands would further cause agents' effort to decline. The relationship between the 'magnifier effect' and the 'contagious shirking effect' is: contagious shirking implies externalities, which is another determinant of agent's incentives besides wage. The negative externalities makes the principal's wage less effective in providing incentives for the agent, together with the trade off the principal faces (trade off between offering incentives and each successful project's profit), leads to this 'magnifier effect'.

Limited liability has long been a controversial topic in corporate law and governance. Despite its effect on mitigating the loss of investors, the investors could also default debt and wage payments in terms of bad states and leave the cost to the society (Halpern et. al. 1980). Firms with large risks would not fully consider the consequences of large losses and the goal of the principal is distracted from that of the social planner.

Limited liability for the agent, which creates incentive distortions by prohibiting large penalties, has been discussed by a vast growing literature starting from Holmström (1979) and Lewis (1980). The design for correction mechanisms has also become a topic of interest. Laux (2001) set up an agency model with a single agent and multiple projects under the assumption that only the agent is protected by limited liability. One manager could carry out multiple projects in the Laux model so as to reduce the inefficiencies and distortions brought up by limited liability[2]. Biais et. al. (2010) introduced a dynamic model on firm size under a unilateral limited liability setting. In their model, the principal's commitment to invest and liquidate provides incentives to the agent by changing their continuation utility. The nature of effort is similar here to our model: to reduce the probability of large losses; but the settings are quite different. Ours is a multi-agent static model concerning firm scope, while the previous one is dynamic and includes only a single agent.

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[2] This is because by combining projects to a single manager could relax the limited liability constraint.

The remainder of this article is organized as follows. Section 2 outlines the general structure of our model. Section 3 will provide the main theoretical results on the optimal contract and firm scope. We will make some extensions and provide our numerical results in section 4. Section 5 will present some detailed discussions on the relationship between our paper and the existing literature on theory of firm scope. Section 6 will give out the conclusions.

2. The Basic Model

At time 0, the principal pays a registration fee F to set up a conglomerate. He could choose a number of projects to undertake from the set $\Omega = \{\omega_s, s \in \Lambda\}$ [3]. Project ω could generate a verifiable monetary payoff (either positive or negative) at time 2, which is represented by a random variable X_ω. The distribution is given by: $X_\omega = 1$ (with probability δ); $X_\omega = -b$ (with probability $1 - \delta$ and $b > 0$). We will further assume that $\{X_\omega, \omega \in \Omega\}$ are i.i.d [4]. The principal then hires a project manager for each project and chooses a contract from a feasible set [5].

I assume that both the agents and the principal are risk neutral and are protected by limited liability or bankruptcy law. The agent and the principal's initial wealth could be normalized to 0, *two-sided limited liability* implies that their ex post participation constraint must be satisfied [6].

The principal specifies a wage level w for every good performing agent ex ante. The firm's only asset at time 2 is the cash flows from the projects it undertakes, so its liquidation value equals to the net profit it generates from all of its projects. Since it is a static model, we assume that the firm perishes after time 2, so that the firm always fully liquidates its assets by that time. In case of liquidation the employee's wages have higher priority [7] than the principal's return, so if the corporate value is larger than wN (N is the number of projects undertaken which generates positive profits), each of the good performing agents would receive a wage w and the principal

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[3] In equilibrium, F should equal to the maximum profit a conglomerate could generate if the later is finite.

[4] The definition of independency of an infinite family of random variables could be seen in the 3rd chapter of <A Course in Probability>. (by Kai Lai Chung)

[5] Note that some of the variables are unverifiable or non-describable, and could not be specified in a formal contract.

[6] We will further observe that in our model, changing the initial wealth level does not affect the main conclusions of our results. If the principal could choose to increase his commitment power by putting in more money ex ante as a collateral, the principal's optimal commitment level is below social first best, which implies he would not commit fully under every possible future contingency.

[7] According to the US Bankruptcy Law, Chapter 7.

claims the rest of the profits. If corporate profit is less than wN and larger than 0, each agent with good performance could only receive $1/N$ of the total profit, since the principal is protected by limited liability. If corporate profit is 0 or negative, both the agents and the principal's payoffs would be 0.

Remember that the contract specified above is only one of implementable contracts. We will also discuss other forms of contracts in the next session, and see why contracts in this form is optimal. The proof in section 3 will show that any contract which implements the same level of effort and satisfies the principal's commitment constraint are equivalent.

At time 1, each agent could exert a non-observable effort (for agent i, the effort level is denoted by a_i) to lower the probability of large loss of his project, where $\delta_i = a_i$, $a \in [0,1]$ [8]. Effort is costly for the agent with the cost function $g(a)$. g is assumed to be continuous and third order differentiable, with $g'(a) \geq 0$, $g''(a) \geq 0$, $\lim_{a \to 0^+} g'(a) = 0, \lim_{a \to 1^-} g'(a) = \infty$. [9]

Also, define the baseline effort level as: $a_0 = \sup\{a \mid 0 < a < 1, g'(a) = g''(a) = 0\}$, an effort level below a_0 will not incur any cost for the agent. We will now show respectively for different values of the large loss level b, the relationship between corporate scope, optimal wage levels, agents' effort and individual project risk. Under the contractual form we specified above, the agent's payoff could be expressed as:

$$Ep_j = E \min\{w, \frac{\chi\{X_j > 0\}\chi\{\sum_{i=1}^{n} X_i > 0\}\sum_{i=1}^{n} X_i}{N}\}$$ (χ is the indicator function)

The agent's problem is to choose the effort level to maximize his expected payoff. The principal would choose the wage level to maximize his expected payoff given the agent's response function. We could express them respectively as:

$$\max_w EP_n = \max_w E\sum_{i=1}^{n}(X_i - p_i)$$

Subject to: $\max_{a_j} Ep_j - g(a_j)$

[8] The nature of effort here is 'selfish'.
[9] This form of effort cost function is also adopted by Aghion and Tirole (1997), Hiriart and Martimort (2006),etc.

We will first consider the case when $b = 1$ and the number of projects is even $(n = 2k)$ and later show the relationship between $n = 2k$ and $n = 2k + 1$. Let $r_i = \min\{ w, \dfrac{2k - 2i}{2k - i} \}$, for $i = 0, 1, ..., k - 1$. For any positive integer k, the agent's expected payoff function could be expressed as: (exposing the symmetric condition for agents other than j)

$$Ep_j = a_j (\sum_{i=0}^{k-1} C_{2k-1}^i r_i a^{2k-1-i}(1-a)^i) - g(a_j) \qquad \text{--- (1)}$$

The FOC and the symmetric condition together imply:

$$g^{'}(a) = wa^{2k-1} + \sum_{i=1}^{k-1} C_{2k-1}^i r_i a^{2k-1-i}(1-a)^i \qquad \text{--- (2)}$$

which is equivalent to:

$$w = \frac{1}{a^{2k-1}}(-\sum_{i=1}^{k-1} C_{2k-1}^i r_i a^{2k-1-i}(1-a)^i + g^{'}(a)) \qquad \text{--- (2')}$$

The principal's payoff function could be written as:

$$EP_{2k} = 2ka^{2k}(1-w) + C_{2k}^1 a^{2k-1}(1-a)[(2k-2)-(2k-1)r_1] + ... + C_{2k}^{k-1} a^{k+1}(1-a)^{k-1}[2-(k+1)r_{k-1}]$$

$$\text{--- (3')}$$

Substituting w with (2'), by simple algebra and equivalent transformations, we could obtain the reduced form:

$$EP_{2k} = \sum_{i=0}^{k-1}(2k-2i)C_{2k}^i a^{2k-i}(1-a)^i - 2kag^{'}(a) \qquad \text{--- (3)}$$

Take the FOC with respect to agent's effort level we can get:

$$\text{FOC:} \ 0 = 2k^2 a^{2k-1} + (k-1)(2k-1)C_{2k}^1 a^{2k-2}(1-a) + ... + C_{2k}^{k-1}(k+1)a^k(1-a)^{k-1}$$

$$- (k-1)C_{2k}^1 a^{2k-1} - ... - C_{2k}^{k-1}(k-1)a^{k+1}(1-a)^{k-2} - kg^{'}(a) - kag^{''}(a)$$

The reduced form could be written as:

$$L(a) = R_{2k}(a) \qquad \text{--- (4)}$$

$$(L(a) = \frac{g^{'}(a) + ag^{''}(a)}{2} \ , R_{2k}(a) = \sum_{i=1}^{k} C_{2k-1}^{i-1} a^{2k-i}(1-a)^{i-1})$$

Define a family of i.i.d. random variables $\{Y_i, i = 1, 2, ...2k-1\}, Y_i = 1$ (with probability a);

$Y_i = 0$ (with probability $1 - a$).

$$\text{Denote } S_n = \sum_{i=1}^{n} Y_i \text{, then } R_{2k}(a) = P(S_{2k-1} \geq k) = P(\frac{S_{2k-1}}{2k-1} \geq \frac{1}{2}).$$

SOC: $kC_{2k-1}^{k}a^{k-1}(1-a)^{k-1} - \dfrac{2g^{''}(a) + ag^{'''}(a)}{2} \leq 0$, which is satisfied for the greatest solution

of the FOC[10].

Extending our discussion to any positive integer $n > 1$, the optimal effort level satisfies:

$$L(a) = P(\frac{S_{n-1}}{n-1} \geq \frac{1}{2}) = \frac{n - 2[\frac{n}{2}]}{2} C_{n-1}^{[\frac{n}{2}]} a^{n-[\frac{n}{2}]-1}(1-a)^{[\frac{n}{2}]} + \sum_{i=1}^{[\frac{n}{2}]} C_{n-1}^{i-1} a^{n-i}(1-a)^{i-1} \text{ }[11]$$

Since $\dfrac{n - 2[\frac{n}{2}]}{2}$ equals to 0 when n is even and $\dfrac{1}{2}$ when n is odd, so it is consistent with the

previous results. When $n = 1$, the agent's effort level satisfies: $g^{'}(a) = w$, the principal's payoff

is $a(1-w) = a(1 - g^{'}(a))$, which is a degeneration form of the above expressions.

Multiplicity exists for a large n since the corresponding relationship between effort level and

wage is not monotonic when n is large enough, we may witness multiple equilibriums under a

given wage. We will make the artificial refinement by making the assumption that if multiple

symmetric Nash Equilibriums exist for a given wage level, the equilibrium with the highest effort

level is always reached. This refinement would guarantee effort to be strictly increasing with wage,

but there are finite numbers of jumps if we plot a against w, a is not a continuous function of w.

Under this refinement, we could prove the following lemmas.

Lemma 1: Take the partial derivatives to the principal's payoff function with respect to a is

equivalent to the principal's FOC with respect to w. Under the largest solution of (4), the

principal's expected payoff is globally maximized.

Proof: See appendix

Lemma 2: FOC, optimal effort level, optimal wage is the same for $n = 2k$ and $n = 2k + 1$;

$EP_{2k} / EP_{2k+1} = 2k / (2k+1)$.

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[10] We will later see that this is the equilibrium reached after our refinement.
[11] [x] denotes the greatest integer no larger than x.

Proof: See appendix

We will next loose the assumption $b=1$ and find the function for the optimal effort for every scope n with a given integer $b(>0)$.

Lemma 3: The interim individual rationality constraint of the agent is always satisfied under our given contract.

Proof: See appendix

Proposition 1: If the firm operates in n projects, the optimal effort then satisfies:

$$\frac{n-[\frac{n}{b+1}](b+1)}{b+1}C_{n-1}^{[\frac{n}{b+1}]}a^{n-[\frac{n}{b+1}]-1}(1-a)^{[\frac{n}{b+1}]}+\sum_{i=1}^{[\frac{n}{b+1}]}C_{n-1}^{i-1}a^{n-i}(1-a)^{i-1}$$

$$=P(\frac{S_{n-1}}{n-1}\geq\frac{b}{b+1})=\frac{g^{'}(a)+ag^{''}(a)}{b+1} \qquad\text{--- (5)}$$

Denote solution for the above equation to be $a_{(n)}$, the principal's payoff function is:

$$EP_n=\sum_{i=0}^{[\frac{n-1}{b+1}]}[n-(b+1)i]C_n^i a_{(n)}^{n-i}(1-a_{(n)})^i-na_{(n)}g^{'}(a_{(n)}) \qquad\text{--- (6)}$$

$S_n=\sum_{i=1}^{n}Y_i$, where $\{Y_n,n\geq 1\}$ is a set of independent and identically distributed random variables with distribution: $Y_n=1$ with probability a ; $Y_n=0$ with probability $1-a$.

Proof: See appendix

3. Optimal Contract and Firm Scope

3.1 Optimal Contracts

We will prove that the contract we specified above is the optimal one, and a change in contractual form could not increase the principal's payoff. Since individual department's profit is the only relevant variable which is verifiable, the principal could write a contract which makes wage contingent on the performance of the n agents.

In general, a contract with n agents is in the form of: $g:\Theta\rightarrow R^n$, where Θ is the set of all possible state of the world at time 1. Each possible state is given by the value of the n random variables, namely $X=(X_1,X_2...X_n)\in\Theta$. $g=(g_1,g_2...g_n)$ is the payoff of respective agents.

The principal's limited liability brings the constraint at a given state is: $\sum_{i=1}^{n} g_i(X) \leq \sum_{i=1}^{n} X_i$; agent

i's limited liability constraint could be written as $g_i(X) \geq 0$. We will solve for the optimal

contract in a symmetric setting, which means, all the agents with the same observable result gets

the same expected payoff.

A contract which provides incentives for the agent to invest would maximize the gap between

an agent with good result and one with bad result, given that the agents are risk neutral. So, the

manager gets paid a wage $g_i(X) = w(R) > 0$ if and only if $X_i = 1$ and $\sum_{i=1}^{n} X_i > 0$ The wage in

this case satisfies: $Nw(R) \leq R$, where $N = \#\{i : X_i = 1\}$.

We suppose the initial contract specifies a manager with positive profit could obtain a

wage r_i if i of the total n departments are running deficits $(0 \leq i \leq [\frac{n}{2}])$. $r_i(n-i) \leq n-2i$ must be

satisfied to make the principal's ex ante commitment credible because of the limited liability

constraint. We could derive the following conditions under a refined symmetric Nash Equilibrium

with n agents:

FOC for the agent:

$$0 = (\sum_{i=0}^{[\frac{n-1}{2}]} C_{n-1}^i r_i a^{n-1-i}(1-a)^i) - g'(a)$$

Payoff for the principal:

$$EP_n = \sum_{i=0}^{[\frac{n-1}{2}]} (n-2i)C_n^i a^{n-i}(1-a)^i - nag'(a)$$

which we could generalize that it is irrelevant to r_i. FOC for the principal:

$$L(a) = P(\frac{S_{n-1}}{n-1} \geq \frac{1}{2}) = \frac{n-2[\frac{n}{2}]}{2} C_{n-1}^{[\frac{n}{2}]} a^{n-[\frac{n}{2}]-1}(1-a)^{[\frac{n}{2}]} + \sum_{i=1}^{[\frac{n}{2}]} C_{n-1}^{i-1} a^{n-i}(1-a)^{i-1}$$

SOC for both is satisfied if we plug in the FOC. From these results, we could generalize the

proposition below:

Proposition 2: All contracts which satisfies:

1) An agent with bad performance never get paid;

2) $r_i(n-i) \le n-2i$ for every integer no greater than n ;

3) Effort level a for every agent is implementable.

must achieve the same payoff, which satisfies:

$$EP_n = \sum_{i=0}^{[\frac{n-1}{2}]} (n-2i)C_n^i a^{n-i}(1-a)^i - nag'(a)$$

The optimal contract is the a which maximizes EP_n.

Proof: See above illustrations

The contract we discussed in session 2 satisfies the above conditions, since

$$p_j(X) = \min\{w, \frac{\chi\{X_j > 0)\chi\{\sum_{i=1}^{n} X_i > 0\}\sum_{i=1}^{n} X_i}{N}\}$$, where w is a prior specified wage level and

p_j is the payment to agent j ($w = w(\hat{a})$, where $\hat{a} = \arg\max EP_n(a)$). We will use this form of contract in the following discussions, while w is the wage level when all projects are running positive profit.

3.2 Firm Scope

In this sub-session, we will again assume $b = 1$ and examine the relationship between firm scope and agent's effort cost function, which could be extended to other values of b. We will give out and prove for some sufficient conditions for effort to increase/decrease with scope. We will present numerical findings in later sections. Proposition 3 gives out a sufficient condition for effort to decrease with scope, when effort cost is high.

Proposition 3: If $g'(a) + ag''(a) = 2$ has a solution smaller than 0.5, which means effort cost is very high, then effort level would converge to the baseline level a_0 when n goes to infinity. (We will denote the solution of $g'(a) + ag''(a) = 2$ by a^*)

Proof: See appendix

The intuitions of the assumptions are as follows. The solution of $g'(a) + ag''(a) = 2$ is small together with the monotonic increasing characteristic of the LHS imply that the cost of effort is increasing rapidly. In this case, the agent's effort level would strictly decrease with scope due to externalities despite the adjustment of wage by the principal. Wage could either increase or

decrease. An intuitive result is that wages would tend to rise when effort cost is low, and would tend to decline when effort cost is high. The logic is that the principal faces a trade-off between the payoff in each successful project (depends negatively on wage) and the probability of success. We could generalize that wages would tend to increase if the agent is more responsive (effort cost is low) or the value of b is larger.

We will then examine the case when $a^* > 0.5$, which means, effort cost is lower. Proposition 4 will give out a sufficient condition for effort to be increase with scope. The conditions are:

Condition 1: For every positive integer k, there always exists an interval $(b_k, c_k) \in (0,1)$, such that:

$$kC_{2k-1}^k a^{k-1}(1-a)^{k-1} - \frac{2g^{''}(a) + ag^{'''}(a)}{2} > 0 \Leftrightarrow a \in (b_k, c_k)$$

Condition 2: $\lim_{a \to 0^+} (L(a) - R_{2k}(a)) \leq 0$

Proposition 4: If $a_{(1)} > 0.5$ and condition 1 and 2 hold, effort is increasing with firm scope when the firm's expected profit is positive. $\{a_{(n)}\}$ is an increasing sequence for odd integer n which satisfies: $EP_n(a_{(n)}) > 0$.

Proof: See appendix

Note that $EP_n(a_{(n)}) > 0$ must be satisfied, when we plot EP_n against a, there are two or more local maximum point when n is large, one with a high effort level the other with an extremely low effort level[12]. The high effort level could not be achieved unless EP_n for the high effort level is above the low one. $EP_n(a_{(n)}) > 0$ since EP_n low is approximately 0.

4. Extensions and Numerical Findings

4.1 Benchmark: Social Optimal Scope and Wage

We will next consider the social optimal case where the firm scope and wage are both chosen by a social planner when both the agent and the conglomerate are protected by limited liability. The social optimal effort level satisfies:

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[12] See numerical findings on under-scope.

$$\max_{0<a<1} a + (-b)(1-a) - g(a) \qquad\qquad \text{--- (7)}$$

FOC: $g'(a) = 1 + b > 1$

Denote the solution to the FOC to be $\bar{a}$.

SOC: $-g''(a) \leq 0$

Lemma 4: The effort implemented under any feasible contract is less than the social optimal level. The social planner's objective function is strictly increasing with effort when $a < \bar{a}$. The social optimal scope is the n inducing the highest effort level when $w = 1$.

Proof: See appendix

From the discussion of the optimal contract in the previous section, we could generalize the following result:

Lemma 5: If $a > 0$, $w = 1 \Leftrightarrow EP_n = 0$ under the optimal contract.

Proof: See appendix

Denote the maximum solution for $EP_n = 0$ by a_n^*, the social optimal n satisfies:

$$n = \arg\max a_n^*$$

Proposition 5: The social first best choice of scope and wage is $w = 1$ and $n = 1$, which is independent of the effort cost function.

Proof: See appendix

4.2 Numerical Results on Principal's Optimal Scope

We will then list out several numerical findings based on specific function forms of g in this sub-session and also provide several unproved conjectures. We will focus on two functional forms

1) $g(a) = -\mu \ln(1-a)$

From the FOC, $\dfrac{\mu}{(1-a)^2} = \sum_{i=1}^{k} C_{2k-1}^{i-1} a^{2k-i}(1-a)^{i-1} \leq 1$, μ has to be small enough to ensure positive profit for the firm. It is easy to prove that $\mu < 1$. For certain values of μ, effort level could suddenly drop to 0 at some critical point of n. For this cost function, the SOC could be written as: $kC_{2k-1}^k a^{k-1}(1-a)^{k+2} < \mu$, the LHS reaches its maximum at $a = \dfrac{k-1}{2k+1}$, which means

condition 1 is satisfied. The same is true for condition 2 if we allow $a \to 0^+$ in the FOC. So, the results and corollaries in proposition 5 holds.

A. Social Optimal Scope=Principal Optimal Scope=1

$\mu = 0.3$. $a_{(1)} = 0.452$, $a_{(n)} = 0$ for $n > 1$. Social optimal scope=principal optimal scope=1.

B. Under-Scope

$\mu = 0.2$, $a_{(1)} < a_{(3)} < a_{(5)} = 0.612$, $a_{(n)} = 0$ for $n > 5$. The principal's payoff reaches its maximum when $n = 1$, $EP_1 = 0.3056$. The principal's optimal scope is less than social optimal scope.

C. Non-Converging

$\mu = 0.16$, $a_{(n)}$ is strictly non-decreasing with respect to n, and would approach its ceiling when n goes to infinity. The principal's payoff is also increasing with scope. Over-scope could not exist under this functional form. Based on simple calculations, we could reach the conclusion that: $a_{(3)} > 0 \Leftrightarrow a_{(1)} > 0.5$, which means, over-scope is impossible.

2) $\quad g'(a) = \mu \exp\{\frac{1}{1-a} - \frac{\lambda}{a}\}$

This cost function violates condition 1 and 2. In this case, there always exists a positive value of effort level a to generate the principal positive profits. However, there could still be a sharp decline in effort level due to multiplicity problems, which means, the effort level for the principal's payoff to reach its maximum may not be unique, especially when n is large.

A. Over-Scope

$\lambda = 1.7, \mu = 1$, $a^* = 0.5202 > 0.5 > a_{(1)} = 0.459$. The agent's effort level is strictly decreasing with scope. Social optimal scope equals to 1 where a reaches its maximum value. The principal's optimal scope equals to 3, where there is an over-scope problem.

B. Under-Scope

$\lambda = 2, \mu = 1$, $a^* = 0.5568 > a_{(1)} = 0.5043 > 0.5$. The agent's effort level is strictly increasing until $n = 35$ after which we will witness a sharp decline. Social optimal scope equals to 35 where a reaches its maximum value 0.5138. The principal's optimal scope equals to 9, where there is an under-scope problem.

C. Non-Converging

$\lambda = 3, \mu = 1$, $a^* = 0.6112 > a_{(1)} = 0.5642 > 0.5$. The agent's effort level and the principal's payoff are both strictly increasing with n , which means, the social optimal scope and the principal optimal scope would be where n approaches infinity.

D. Social Optimal Scope=Principal Optimal Scope=1

$\lambda = 1.5, \mu = 1$, $0.5 > a^* = 0.4907 > a_{(1)} = 0.427$. The agent's effort level and the principal's payoff are both strictly decreasing with n , which means, the social optimal scope and the principal optimal scope both equal to 1.

So the results from our numerical observations are:

1) If for any integer n , there always exist a feasible effort (the wage corresponding at that scope is no greater than 1) $a > 0.5$, then effort level would be strictly increasing with scope and would be approaching its upper bound a^* . EP_n is also non-declining with scope. This is corresponding to the non-convergence case.

2) If $a_{(1)} > 0.5$, but there exists an integer m such that $a_{(m)} < 0.5$ (m is the smallest among those integers), then the principal's optimal scope no greater than $m-2$ and social optimal scope is $m-1$ or $m-2$. EP_n will first increase and then decrease with scope. This is corresponding to the under-scope case.

3) If $a^* > 0.5, a_{(1)} < 0.5$, effort level would be non-increasing with scope. EP_n will first increase and then decrease with scope, which means, the principal's optimal scope is greater than social optimal scope 1. This is corresponding to the over-scope case.

4) If $a^* < 0.5$, $a_{(1)} < 0.5$, effort level would be non-increasing with scope. EP_n would be declining when n is odd. The principal optimal scope also equals to 1.

4.3 Wage

Throughout this section, we will be focusing on the case when $b = 1$ under the contract we specified in session 2. We will refer "feasible effort level under scope n " to the set of values of a which satisfies $EP_n > 0$. The following corollaries about feasible effort are deduced from

proposition 2:

Corollary 1: $EP_n(a) > 0 \Leftrightarrow w_n(a) \in (0,1)$

Corollary 2: For every $a \in (0,1)$, there exists a unique w (not necessarily smaller than 1), such that the equation below holds:

$$g'(a) - \sum_{i=0}^{[\frac{n-1}{2}]} C_{n-1}^i r_i a^{n-1-i}(1-a)^i = 0 \qquad \text{--- (9)}$$

We will refer a wage w implements effort level a if (9) holds. Since only an implicit relationship between a and w is available, we will first give out results about wage in the under-scope case with simple functional form: $g(a) = -\mu\ln(1-a)$ and then list out several numerical findings about the relationship between wage and scope in the over-scope case with more complex functional form: $g'(a) = \mu\exp\{\dfrac{1}{1-a} - \dfrac{\lambda}{a}\}$.

4.3.1 Under-scope: $g(a) = -\mu\ln(1-a)$

In this section, we will first assume that the optimal scope for the principal and for the society are both finite and n is odd. Denote the principal's optimal scope to be n^* and social optimal scope to be n^{**}. From the definition of under-scope, we have the relations below:

$$a_{(n^*)} < a_{(n^{**})} \qquad EP_{n^*} > EP_{n^{**}} \qquad n^* < n^{**}$$

Lemma 6: A necessary condition for under-scope is there exists $n > 1$ such that:

$$\max_{a \in (0,1)} EP_n > 0 \Leftrightarrow 0 < \mu < \frac{1}{4} \Leftrightarrow a_{(1)} > \frac{1}{2}.$$

Proof: See appendix

Proposition 6: Under the conditions of lemma 6, $w_3(a_{(3)}) > w_1(a_{(1)})$. Generally, the optimal w is increasing with scope for any odd integer n satisfying: $\max_{a \in (0,1)} EP_n > 0$.

Proof: See appendix

So far, we have proved that for the simple functional form: $g(a) = -\mu\ln(1-a)$, wage is increasing with scope (for those n which satisfy $\max_{a \in (0,1)} EP_n > 0$) in case of under-scope.

4.3.2 Over-scope: $g^{'}(a) = \mu \exp\{\frac{1}{1-a} - \frac{\lambda}{a}\}$

Example: (over-scope) $\lambda = 1.7, \mu = 1$. Remember that effort level is non-increasing with scope.

$w_1(a_{(1)}) = 0.558$, $w_3(a_{(3)}) = 0.533$. The wage under social optimal scope is larger than principal optimal scope.

The morals we could address from the example above is that in case of over-scope, wages are declining with scope.

4.3.3 Optimal Wage as 'Magnifier'

Based on previous findings, we could generalize: wages would tend to increase with scope if effort cost is low, but decrease if effort cost is high. So, when effort cost is high for the agents, the principal's under-provision of incentives would further decrease the effort exerted by the agents. So we conclude that the principal's incentive scheme acts like a 'magnifier' when firm scope increases.

The intuition for this magnifier effect is: the agent is also affected by the performance of others within the organization besides the principal's wage. When effort cost is high, an additional agent would have more negative externality on the existing team, together, with the original effect of high effort cost, would make the principal's high wage less effective. The principal, facing a tradeoff between high effort (which requires high wages) and high profit from every successful project, would tend to pay a lower wage when the firm expands. This would exacerbate the insufficient incentive problem, which would lead to a further decline in effort.

4.3.4 Scope Selection Under a Fixed Wage

In this section, we will take wage as exogenously given and compare the principal's optimal scope with the social optimal one. Our analysis could also provide further insight into the magnifier effect of wage. Since an under-investment is commonly faced by agents in our model, we will assume that $0.5 < w < 1$. We will compare the individual effort level and the principal's payoff when $n = 1$ and $n = 3$ with effort cost function to be $g(a) = -\mu \ln(1-a)$, thus addressing the over-scope problem when letting the principal choose his optimal scope when we fix the wage. To make our discussion well-defined, we will assume that $\mu \in (0, 0.25)$, otherwise, principal would never generate positive profit when $n = 3$. The following expressions could be derived

from our basic model. When $n = 1$:

$$a = 1 - \frac{\mu}{w} \qquad \text{--- (11)}$$

$$EP_1(w) = (1-w)(1-\frac{\mu}{w}) \qquad \text{--- (12)}$$

When $n = 3, 0.5 < w < 1$:

$$\mu = wa^2(1-a) + a(1-a)^2 \qquad \text{--- (13)}$$

$$EP_3(w) = 3a^3(1-w) \qquad \text{--- (14)}$$

Proposition 7: The social optimal scope under a fixed wage is always $n = 1$. ($a_1 > a_3$)

Proof: See appendix

We will next focus on the existence of μ and w for $EP_3(w) > EP_1(w)$, which from (12) and (14), is equivalent to: $3a^3 > 1 - \frac{\mu}{w}$. By plugging in (13) we could get the equivalent expression:

$$\frac{a(1-a)^2}{w} > 1 - 2a^3 - a^2. \qquad \text{--- (15)}$$

When $w \to 1$, which means the wage is large enough, we could get the approximation:

$$3a^3 - a^2 + a - 1 > 0 \qquad \text{--- (16)}$$

Expression (16) could be satisfied when $a \to 1$, (13) implies that for every $a < 1$, there exists a value of μ, such that a is implementable under a wage approximate to but less than 1. From the above discussions, we could have the following proposition:

Proposition 8: When wages are fixed to be high enough and effort cost is low, the principal's optimal scope is larger than social optimal one, which indicates an over-scope problem.

Proof: See above illustrations.

Comparing the results of proposition 7 and 8 with proposition 6, we could clearly generalize: when effort cost is low, *the principal's optimal wage serves as a magnifier to further promote effort*. The underlying logic is: if the wage is fixed, then effort level should decline; but we witness an increase in effort by individual agent if we let the principal choose his optimal wage. The increase in effort is clearly spurred by the increase in wage.

4.4 Extreme Risk

In this sub-session, we will consider the case of extreme risks. We will assume that $b \gg 1$,

so that the default in anyone of the projects would wipe out the profit of the whole firm. We will first fix the corporate scope to be n. The principal and the agents have positive payoffs if and only if all the projects generate positive profits, and furthermore, all the agents will either get fully paid or get no payment at all. The principal's expected payoff given the agents' effort level and the wage level is:

$$EP_n = n(1-w)\prod_{i=1}^{n} a_i$$

Agent j's problem is to choose his effort level to maximize his expected payoff :

$$\max_{0<a_j<1} Ep_j(a_j) = w\prod_{i=1}^{n} a_i - g(a_j)$$

Take the first order conditions (FOC) with respect to agent j's effort level:

$$\frac{1}{a_j}w\prod_{i=1}^{n} a_i - g^{'}(a_j) = 0$$

The second order condition (SOC) is trivially satisfied since $g^{''}(a) \geq 0$. Imposing the symmetric conditions, we will get: $g^{'}(a) = wa^{n-1}$, where a stands for individual agent's effort level. The principal's maximization problem under a certain scope:

$$\max_w EP_n = \max_w a^n n(1-w).$$

Since w is a continuous function of a, EP_n is 0 when $a \to 0^+$ and goes to $-\infty$ when $a \to 1^-$, the point where EP_n achieving the maximum value satisfies: $\partial EP_n / \partial a = 0$. So the maximum profit could be written as: $\max_{a \in A} a^n n(1-w)$, $A = \{a | \partial EP_n / \partial a = 0\}$. a will not lie in the 'jumped areas', since a contradiction exists if we compare it with the EP_n at the same wage level with a higher effort. The above argument ensures that to take the FOC against a is rigorous.

FOC: $n^2 a^{n-1}(1-f_n(a)) - na^n f_n^{'}(a) = 0 \Leftrightarrow n(1-f_n(a)) - af_n^{'}(a) = 0$

which is also equivalent to:

$$na^{n-1} = ag^{''}(a) + g^{'}(a) \qquad \text{--- (17)}$$

SOC: $n(n-1)a^{n-2}(1-f_n) - 2na^{n-1}f_n^{'} - a^n f_n^{''} \leq 0$

which is equivalent to: $ag'''(a)+2g''(a) \geq n(n-1)a^{n-2}$, which is trivially satisfied if we plug in the FOC. We will next show that the above FOC is consistent with the results derived in section 1. For a fixed scope n, $b > n-1$ is equivalent to $b = n-1$ under limited liability, since the expected profit is the probability of all projects generating positive profits multiplies the number of projects; the loss in one project would wipe out all corporate profit. We have the expression below:

$$\frac{1}{n}(ag''(a)+g'(a)) = a^{n-1} = P(\frac{S_{n-1}}{n-1} > \frac{b}{b+1} > \frac{n-1}{n}) = P(S_{n-1} = n-1) \quad \text{--- (18)}$$

which is equivalent to the FOC when $b = n-1$.

5. More Discussions on Theory of the Firm

We will make our start by listing out several common features firms share, we will proceed by posing several questions correlated with these features: What is a firm? What determines its boundaries? How could agents within the same firm correlate with one another? How is this relation different from two agents belong to different firms? We will list out the answers to these questions based on previous literature as well as our model. By comparison, we could summarize our paper's contribution to the theory of the firm.

In the standard property right view, the firm could be seen as a collection of physical assets under common ownership, whereas the owner is endowed with both the decision authority and the rent over the assets he owned[13]. Different sectors of a firm interact mainly through transactions and other cooperative activities, such as the case of GM and Fisher Body (see Hart 1995). Externalities exist when the parties could improve the trading gain through ex ante relationship-specific investments[14]. Contractual problems arise when ex ante investment could not be verified, even though they are mutually observable. Firm boundaries would affect the decision authorities, hence the relative positions in ex post bargaining, and is determined by the complementarities and substitutability of physical assets, or more generally, the contribution of assets to a coalition (see Hart Moore 1990, Hart 1995)[15].

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[13] See Hart Moore (1990). The owner of the asset could enjoy larger share of the surplus.

[14] We usually refer to indirect externalities, for example, the buyer's investment would only increase his value, the seller's investment would only reduce its cost. For discussions on direct externalities, see Che and Hausch (1999).

[15] This could be extended to human capital, as in Rajan and Zingales (1998), the worker, who is given access, could make himself valuable by specialization investment, and he could have bargaining power based on his

Different from the Grossman-Hart-Moore paradigm, Hart and Holmström (2010) stressed the role of decision authority in handling disputes within a firm. The firm is different from market from its emphasis on authority. Different agents and departments are related through cooperation and coordination. There are conflicts between the gains from coordination and individual agent's private benefit, and integration would lead to too much coordination since the principal would put insufficient weight on the agents' private benefits. Firm boundaries are determined by the relative gains and losses from coordination and the weight which the principal places on agents' private benefits (determined by the dead-weight losses caused by shading).

Our paper views the firm as a legal person, who is protected by limited liability and has limited commitment over future payments. The agents within a firm relate to each other through indirect externalities, which is created by limited liability of the principal. Under the optimal contract, a well-performed agent would be paid a pre-specified wage if the firm's financial status permits such a payment; or receive the firm's value divided by the number of departments which performs well; or otherwise, if the firm's overall profit is non-positive, then he will not receive any payment. Therefore, the other agents' efforts could increase the marginal return of effort of an individual agent, and therefore, provide more incentive for him to work hard. Also, the shirking of one agent not only decreases the probability of success of his own department, but also decreases the incentives of other agents and discourages them to exert effort. The shirking of other agents could then have a negative feedback on that agent's incentive and further decrease his effort level. We call this '*contagious shirking'* .

This would provide an explanation for cultures in firms and other organizations, where the members' behaviors follow a certain norm, either do they work hard together or shirk together. Fischer and Huddart's (2008) model illustrated the importance of social norms in determining the optimal incentive scheme and the agent's behavior within an organization. In their model, they assumed that the agent's behavior cost is increasing with the level of adherence of his behaviour to social norm (average behavior). But their model does not explain why social norm exists. In our model, this sort of linkage of agents is brought up by *limited financial liability* of a firm, which is the key assumption in our model. In our selections of symmetric Nash Equilibriums, we also provide a rational which is consistent with the field experimental results of Uri Gneezy and Aldo

specialized human capital.

Rustichini (2000) which higher financial incentives do not necessarily lead to the more desirable actions.

The relationship between scope and effort is determined by the cost of effort and magnitude of losses. Rather than explain the question with 'diversification', we argue that the contagious effect (the externality of an additional agent) is the key determinant, which we have shown is more appropriate. Positive externalities imply 'contagious working' and negative ones imply 'contagious shirking'. If effort cost is high and the magnitude of losses are large, the second effect would cause effort to decrease when firm scope expands. If effort cost is low or the magnitude of losses are small, the first effect would cause effort to increase with scope.

Furthermore, it could even invalidates the principal's incentive schemes, since the expected marginal return to hard work is reduced. So the model also provides an explanation for why firms which requires costly unobservable effort are proned to choose a narrow scope. Our model also has some welfare and regulation implications on the regulation of managerial wages and firm scope.

6. Conclusion

Our model provides an explanation for the relationship between firm scope and corporate risk, especially for those phenomena that diversification could not explain. Our model coincides with some observations in the recent crisis, where a large firm went bankrupt because of the loss of one or several of its many departments.

The main difference of our setup with previous ones is the *two sided limited liability assumption*, especially, limited liability for the principal, which implies limited commitment ability on wage payments. The agent's incentive distortion brought up by limited liability leads him to shirk, and this deteriorates the other agents' incentives by lowering the expected profit of the firm, and hence the principal's ability to pay wages. This is because a good performing agent could not be well compensated if the whole firm is running a deficit. A vivid expression for this is 'contagious shirking', which is commonly observed in nowaday firms and organizations.

In our model, we proved that the equivalence between contracts which could implement the same effort level and satisfies the principal's commitment constraint on every contingency and also 0 payment to bad performing agents.

Under the optimal contract, whether the agent's effort is increasing or decreasing when the firm expands depends on the cost of effort and magnitude of losses. If effort cost is high, an additional agent would exert less effort, and have larger negative externality on the whole team, which makes the principal's wage less effective in providing incentives. The principal would lower wage in the new contract which exacerbates the shirking problem. So high effort cost would cause agent's effort to decline when the firm expands. Moreover, the magnitude of negative externalities also depend on the magnitude of losses, larger potential losses could further decrease effort in a large team where contagious shirking effect dominates.

Other conclusions include the social first best wage and scope; welfare implications on firm scope; how effective are the regulations on wage and scope; how to explain corporate cultures, and so forth.

A flaw of our model is that agents would always tend to exert less effort compared with social first best, so the social optimal scope is which induces the manager to exert the highest effort under the principal's optimal contract. In real world observations, there are cases when agents exert inefficient high efforts, and a social planner's goal in that case is to prevent managerial overloading brought up by expansion[16]. Another incomplete part of this paper is that it fails to give out the equivalent conditions of effort's increase (or decrease) when firm scope expands, but only gives out some sufficient conditions and numerical examples.

Another criticism might be that the principal could increase his commitment ability by throwing in more money initially and to provide a safe back for the agents. We confess that the optimal initial commitment level is not necessarily 0. But it is easy to prove that the optimal level is below $b(n-1)$, which indicates the principal would not fully commit to wage payment under every circumstance. This partial commitment is consistent with real world observations and therefore, the contagious effect still exists in an organization with two-sided limited liability.[17]

Besides, this model is just a preliminary step on how a firm's financial situation would affect its internal structure, level of riskiness and the agent's incentives. I will make several proposals for further research in this area.

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[16] Laux (2001) provides a model to explain this observation by assuming different projects under the control of a single manager.
[17] See Halpern, Trebilcock and Turnbull (1980).

First, we need to extend this static model to a dynamic one to study the more general case of a multiple agent setting where both the principal and the agents are protected by limited liability. The principal has more tools to provide incentives, the one agent with various project size case could be seen in Biais et al (2010), in their model, the principal could liquidate part of the asset as a punishment for large losses, and invest when observing good performance. The difference is that we should include multiple agents and assuming the principal is also protected by limited liability. The fixed project size assumption is also restrictive, since in reality the principal could manipulate on project size based on historical performance to provide incentives for the agents.

Secondly, individual department's performance is perfectly verifiable is also a restrictive assumption[18]. For example, if the profit of each department is observable for the principal but not observable for the agents (except for the one who is controlling the project), the agents' decision is affected by the principal's financial report. The principal could choose to fire the agent with bad performance (which is publically observable) or retain his position in order to hide current losses and cheat the other agents. He may have incentives to cheat in a dynamic model, since the seemingly good financial status may provide better incentives for the agents. This would lead to communication failure (the agents do not trust the financial report of the firm) and less incentive to exert effort in equilibrium. The wage's repayment path is also critical in this model: it could neither be too quick (the principal would run out of cash) nor too slow (contagious effect would lower incentives to exert effort). Solving for the optimal contract in the dynamic model and also the equilibriums in the cheat talk game as well as the optimal repayment path is left to future research.

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[18] The author is grateful to Yuliy Sannikov for this suggestion.

Appendix

Proof of lemma 1:

From (3), we could generalize that since $EP_{2k} \to 0$ when $a \to 0^+$ and $EP_{2k} \to -\infty$ when $a \to 1^-$, and EP_{2k} is a continuous function of a, the effort level which maximized the principal's payoff must satisfy: $\partial EP_{2k} / \partial a = 0$. If a could not correspond to any wage level after the refinement, then, there must be another $a^{'} > a$ corresponding to the same wage, which from the expression of EP_{2k}, strictly dominates a. So after the refinement process, the problem is well-defined.

Q.E.D.

Proof of lemma 2: The conclusion is equivalent to the reduced form:

$$\sum_{i=1}^{k} C_{2k-1}^{i-1} a^{2k-i}(1-a)^{i-1} = \frac{1}{2} C_{2k}^{k} a^{k}(1-a)^{k} + \sum_{i=1}^{k} C_{2k}^{i-1} a^{2k+1-i}(1-a)^{i-1}$$, which could be proven as trivially satisfied.

Q.E.D.

Proof of lemma 3: Since the agent's payoff when effort level is 0 is 0. According to lemma 1, there exists a value a which satisfies the FOC and at which point, the agent's payoff is globally maximized. So the agent's expected payoff at that point must be greater than 0.

Q.E.D.

Proof of proposition 3: Denote the effort level corresponding to the principal's optimal contract under given scope n is $a_{(n)}$. Since $L(a_{(2k)}) < 1$ for any k, so $a_{(2k)} < 1/2$ for any k, and there exists $0 < s < 1/2$, such that $L(s) \geq 1$. From the Chebyshev Inequality:

$$L(a_{(2k)}) = R_{2k}(a_{(2k)}) \leq P(\left|\frac{S_{2k-1}}{2k-1} - a_{(2k)}\right| > \frac{1}{2} - a_{(2k)}) \leq \frac{1}{(s-1/2)^2} \frac{a_{(2k)}(1-a_{(2k)})}{2k-1} \to 0$$

when k goes to infinity.

So the effort level converges to the baseline level a_0.

Q.E.D.

Proof of proposition 4: $n = 2k+1$.

Condition 1 guarantees there are only two possible circumstances for the extremes of EP_n.

If a satisfies FOC and $a \in (b_k, c_k)$, then $EP_n(a)$ is a minimum value; otherwise ($a \notin (b_k, c_k)$), it is a maximum value. For a third order differentiable function, its maximum value must stay

between two minimum values, the inverse is also true. The only two possible circumstances are:

1) FOC has only one solution $a_{(n)}$, where EP_n reaches its maximum.

2) FOC has two solutions, the larger one is where EP_n reaches its maximum and the smaller one is where EP_n reaches its minimum.

According to the central limit theorem, $P(\dfrac{S_{2k-1}}{2k-1} \geq \dfrac{1}{2}) < P(\dfrac{S_{2k+1}}{2k+1} \geq \dfrac{1}{2})$ holds for all integer k if $a > 0.5$. We could then finish our proof using the recursive technique. If $a_{(n)} > 0.5$, since:

$$P(\frac{S_{2k+1}}{2k+1} \geq \frac{1}{2}) = R_{2k+2}(a) = \sum_{i=1}^{k+1} C_{2k+1}^{i-1} a^{2k+2-i}(1-a)^{i-1},$$

$L(a_{(n)}) = R_{2k}(a_{(n)}) < R_{2k+2}(a_{(n)})$. A maximum value of EP_n must be achieved to the right of $a_{(n)}$. From the uniqueness of the maximum value according to our previous discussions, $a_{(n+2)} > a_{(n)}$.

From lemma 2, $a_{(n+1)} = a_{(n+2)}$. Since $a_{(1)} > 0.5$, then $\{a_{(n)}\}$ so long as $EP_n(a_{(n)}) > 0$. $\quad$ Q.E.D.

Proof of lemma 4: From equation (2) in a general setting, we could generalize the inequality below for b equals to any positive value:

$$g'(a) = wa^{2k-1} + \sum_{i=1}^{[\frac{n-1}{b+1}]} C_{n-1}^i r_i a^{n-1-i}(1-a)^i \leq w(\sum_{i=0}^{[\frac{n-1}{b+1}]} C_{n-1}^i a^{n-1-i}(1-a)^i) \leq w \leq 1 < 1+b \quad \text{--- (8)}$$

From $g''(a) > 0$, we could generalize that $a < \bar{a}$. $1+b - g'(a) > 0$. From the monotonic relationship between wage and effort after the refinement, we could generalize that social optimal wage equals to 1 when it is chosen by a social planner. $\quad$ Q.E.D.

Proof of proposition 5: From the results generated from section 1, $g'(a_1) = 1$. $g'(a_n) < 1$ for any positive integer n greater than 1. So, the highest effort level is achieved when $n = 1$ and $w = 1$. $\quad$ Q.E.D.

Proof of lemma 6: We will prove: $\max_{a\in(0,1)} EP_n > 0 \Leftrightarrow 0 < \mu < \dfrac{1}{4}$.

"$\Leftarrow$": $EP_3 = 6a^3 - 3a^2 > 0 \Leftrightarrow a_{(3)} > \dfrac{1}{2}$, $a_{(3)} = \arg\max EP_3$.

which is equivalent to : $\mu = 2a_{(3)}(1-a_{(3)})^2 < \dfrac{1}{4}$ according to monotonic relations.

then $a_{(1)} = 1 - \sqrt{\mu} > \dfrac{1}{2}$.

"$\Rightarrow$" If $\mu > \dfrac{1}{4}$, then $a_{(n)} < \dfrac{1}{2}$. (Note that this relation holds only for specific function forms)

$\dfrac{1}{4} > \sum\limits_{i=0}^{k} C_{2k}^{i} r_i a^{2k-i}(1-a)^{i+1}$ for any $a < \dfrac{1}{2}$. No effort level is feasible. $\max_{a \in (0,1)} EP_n < 0$ for

any $n > 1$. $\hfill$ Q.E.D.

Proof of proposition 6: We will only give out the proof for former part. Denote w_n in short

of $w_n(a_{(n)})$.

$w_3 < \dfrac{1}{2} \Leftrightarrow w_3 = \dfrac{2 - 2a_{(3)}}{2 - a_{(3)}}$, which requires: $\dfrac{2}{3} < a_{(3)} < 1 \Rightarrow 0 < \mu < \dfrac{4}{27}$;

$w_3 > \dfrac{1}{2} \Leftrightarrow w_3 = \dfrac{1 - a_{(3)}}{a_{(3)}}$, which requires: $\dfrac{1}{2} < a_{(3)} < \dfrac{2}{3} \Rightarrow \dfrac{4}{27} < \mu < \dfrac{1}{4}$.

$w_1 = \sqrt{\mu} < \dfrac{1}{2}$

If $w_3 < w_1$, then $w_3 = \dfrac{2 - 2a_{(3)}}{2 - a_{(3)}} = \dfrac{\mu}{a_{(3)}^3 - 3a_{(3)}^2 + 2a_{(3)}} < \sqrt{\mu}$.

Since $\mu = 2a_{(3)}(1 - a_{(3)})^2$,

the inequality is equivalent to $2 < a_{(3)}(2 - a_{(3)})^2$, which never holds for $\dfrac{2}{3} < a < 1$, contradiction.

So $w_3 > w_1$. $\hfill$ Q.E.D.

Proof of proposition 7: From (11) and (13):

$a_1 > a_3 \Leftrightarrow \dfrac{\mu}{w} = 1 - a_1 < 1 - a_3 \Leftrightarrow a^2(1-a) + \dfrac{a(1-a)^2}{w} < 1 - a$

$\Leftrightarrow a^2 + \dfrac{a(1-a)}{w} < 1$

$a^2 + \dfrac{a(1-a)}{w} < a^2 + 2a(1-a) = 2a - a^2 \le 1$ $\hfill$ Q.E.D.

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