A Note on Concavity, Homogeneity and Non-Increasing Returns to Scale

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September 2010

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Abstract

This paper provides a simple proof of the result that if a production function is homogeneous, displays non-increasing returns to scale, is increasing and quasiconcave, then it is concave. If the function is strictly quasiconcave or one-to-one, homogeneous, displays decreasing returns to scale and if either it is increasing or if $\mathbf{0}$ is in its domain, then it is strictly concave. Finally it is shown that we cannot dispense with these assumptions.

Keywords: Homogeneity, Concavity, Non-Increasing Returns to Scale and Production Function.

JEL Classification Numbers: D20, D24, C60

*I want to thank Miguel Espinosa for his valuable comments and suggestions for this paper.

Una nota acerca de concavidad, homogeneidad y retornos no-crecientes a escala[1]

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Septiembre 2010

Resumen

Este artículo presenta una demostración sencilla del resultado que toda función homogénea, con retornos no-crecientes a escala, creciente y cuasicóncava, es cóncava. Además si la función es o estrictamente cuasicóncava o uno-a-uno, homogénea, con retornos decrecientes a escala y es o creciente o tiene al vector $\mathbf{0}$ en su dominio, entonces es estrictamente cóncava. Finalmente se muestra que no es posible ignorar ninguno de los supuestos.

Palabras Clave: Homogeneidad, Concavidad, Retornos No-Crecientes a Escala y Función de Producción.

Clasificación JEL: D20, D24, C60

[1] Agradezco a Miguel Espinosa por sus valiosos comentarios y sugerencias para este artículo.

Introduction

Friedman (1973) explores "the relationship between decreasing or constant returns on the one hand, and concavity on the other" and provides a theorem for when a homogeneous production function satisfying some standard economic assumptions is concave. His proof however, as noted by Dalal (2000), "is not widely known, and because the proof is fairly complex, it may not be generally accessible". The same could be said about Bone (1989), who proves a similar result. Dalal (2000) offers an easier alternative proof, but his result only applies to positive functions. We extend the results by giving an easy proof that applies for non-negative functions and by showing that we cannot dispense with the assumptions we make.

Results and examples

Throughout this paper we will consider a function $f : X \subseteq \mathbb{R}^n_+ \longrightarrow \mathbb{R}$, where

$$\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n : x_i \geq 0 \, \text{for } i = 1, \ldots, n\}$$

is the non-negative orthant of $\mathbb{R}^n$. This can represent a production technology, where X is the input space, or a utility function, where X is the consumption set. We assume that X is a convex set.

We now define some basic properties of $f(\cdot)$:

- The function $f(\cdot)$ is called increasing if and only if $\mathbf{x} \geq \mathbf{y}$ implies that $f(\mathbf{x}) \geq f(\mathbf{y})$. Here $\mathbf{x} \geq \mathbf{y}$ if and only if $x_i \geq y_i$ for $i = 1, \ldots, n$.
- The function $f(\cdot)$ is quasiconcave if and only if for all $\mathbf{x}$, $\mathbf{y} \in X$ and for all $\lambda \in [0, 1]$ we have $f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) \geq \min\{f(\mathbf{x}), f(\mathbf{y})\}$. It is strictly quasiconcave if and only if for all $\mathbf{x} \neq \mathbf{y}$ and $\lambda \in (0, 1)$ we have $f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) > \min\{f(\mathbf{x}), f(\mathbf{y})\}$.
- The function $f(\cdot)$ is homogeneous of degree γ if and only if for all $t > 0$, $\mathbf{x} \in X \subseteq \mathbb{R}^n_+$, we have

  $$f(t\mathbf{x}) = t^\gamma f(\mathbf{x})$$

  If $0 < \gamma \leq 1$, homogeneity implies that for all $\mathbf{x} \in X \subseteq \mathbb{R}^n_+$, $f(t\mathbf{x})$ is a strictly increasing and weakly concave function of t. This property is referred as "ray concavity" in the literature.

Note that homogeneity of degree γ implies that $f(\mathbf{0}) = 0$ if $\mathbf{0} \in X$. To see this, take any $t > 0$. Then $f(\mathbf{0}) = f(t\mathbf{0}) = t^{\gamma} f(\mathbf{0})$. If $f(\mathbf{0}) \neq 0$ then we would need $t = 1$, but t is arbitrary. Now, for any $\mathbf{x} \in X \subseteq \mathbb{R}^n_+$ we have $\mathbf{x} \geq \mathbf{0}$. Since $f(\cdot)$ is increasing, this implies that $f(\mathbf{x}) \geq 0$. Finally, homogeneity gives us homotheticity: $f(\mathbf{x}) = f(\mathbf{y})$ implies $f(t\mathbf{x}) = f(t\mathbf{y})$ for all $t > 0$, $\mathbf{x}$, $\mathbf{y} \in X \subseteq \mathbb{R}^n_+$.

Now we can prove a useful theorem for increasing, homogeneous and quasiconcave functions.

Theorem 1. *If $f(\cdot)$ is quasiconcave, increasing and homogeneous of degree γ in $\mathbf{x}$, where $0 < \gamma \leq 1$, then $f(\cdot)$ is concave in $\mathbf{x}$.*

Proof. Consider two distinct vectors $\mathbf{x_1}$, $\mathbf{x_2} \in X \subseteq \mathbb{R}^n_+$ and define $y_1^{\gamma} = f(\mathbf{x_1})$ and $y_2^{\gamma} = f(\mathbf{x_2})$. If $\lambda = 0$ or $\lambda = 1$ then $f(\lambda\mathbf{x_1} + (1-\lambda)\mathbf{x_2}) = \lambda f(\mathbf{x_1}) + (1-\lambda) f(\mathbf{x_2})$ trivially. Consider then $\lambda \in (0,1)$. We will analyze all possible cases.

Case 1: $y_1 = y_2$.

We have

$$f(\lambda\mathbf{x_1} + (1-\lambda)\mathbf{x_2}) \geq \min\{f(\mathbf{x_1}), f(\mathbf{x_2})\} = y_1^{\gamma} = y_2^{\gamma} = \lambda f(\mathbf{x_1}) + (1-\lambda) f(\mathbf{x_2})$$

and the concavity definition is satisfied.

Case 2: $y_1 \neq y_2$ and $y_1 = 0$.

If $y_1 = 0$ then we have $f(\mathbf{x_1}) = 0$, $f(\mathbf{x_2}) \neq 0$ and

$$f(\lambda\mathbf{x_1} + (1-\lambda)\mathbf{x_2}) \geq f((1-\lambda)\mathbf{x_2}) = (1-\lambda)^{\gamma} f(\mathbf{x_2}) \geq (1-\lambda) f(\mathbf{x_2}) = \lambda f(\mathbf{x_1}) + (1-\lambda) f(\mathbf{x_2})$$

where the first inequality follows from $f(\cdot)$ increasing and the second follows because $x^{\gamma} \geq x$ if $x \in [0,1]$ and $0 < \gamma \leq 1$.

Case 3: $y_1 \neq 0$, $y_2 \neq 0$ and $y_1 \neq y_2$.

This is the case proved by Dalal (2000). Following his proof we have

$$f\left(\frac{\mathbf{x_1}}{y_1}\right) = f\left(\frac{\mathbf{x_2}}{y_2}\right) = 1$$

by homogeneity. Then by quasiconcavity

$$f\left(\lambda\frac{\mathbf{x_1}}{y_1} + (1-\lambda)\frac{\mathbf{x_2}}{y_2}\right) \geq \min\left\{f\left(\frac{\mathbf{x_1}}{y_1}\right), f\left(\frac{\mathbf{x_2}}{y_2}\right)\right\} = 1$$

Make $\lambda = \frac{ty_1}{ty_1+(1-t)y_2}$ for $t \in [0,1]$. We have that $\lambda \in [0,1]$ and substituting

$$f\left(\frac{t\mathbf{x_1} + (1-t)\,\mathbf{x_2}}{ty_1 + (1-t)\,y_2}\right) \geq 1$$

and by homogeneity we get

$$f\left(t\mathbf{x_1} + (1-t)\,\mathbf{x_2}\right) \geq \left(ty_1 + (1-t)\,y_2\right)^{\gamma}$$

because $\left(ty_1 + (1-t)\,y_2\right)^{\gamma} > 0$. If $0 < \gamma \leq 1$ the function $h(x) = x^{\gamma}$ is concave and we get $\left(ty_1 + (1-t)\,y_2\right)^{\gamma} \geq ty_1^{\gamma} + (1-t)\,y_2^{\gamma}$. Then we conclude that

$$f\left(t\mathbf{x_1} + (1-t)\,\mathbf{x_2}\right) \geq tf\left(\mathbf{x_1}\right) + (1-t)\,f\left(\mathbf{x_2}\right)$$

Now that all cases were considered and we can conclude that $f(\cdot)$ is concave. □

Friedman (1973) and Bone (1989) prove a result similar to Theorem 1 under the assumptions of "ray concavity" and "homotheticity". Their proof however is more involved. Note also that the proof provided by Dalal (2000) only covers Case 3. That is, it assumes that $f(\mathbf{x}) > 0$ for all $\mathbf{x}$.

We cannot get rid of the homogeneity assumption, as illustrated by Friedman (1973). Since concavity implies quasiconcavity, this is a necessary condition.

We now provide an example where $f(\cdot)$ is not increasing and therefore Theorem 1 fails, even though all other conditions for it hold.

Fact 1. *Consider the function*

$$f(x,z) = \begin{cases} z^{\gamma} & \textit{if } x \geq z \\ 0 & \textit{otherwise} \end{cases}$$

where $0 < \gamma \leq 1$, $x \in \mathbb{R}_+$ and $z \in \mathbb{R}_+$. This function is homogeneous of degree γ and quasiconcave. However it is not increasing, not concave and not strictly quasiconcave.

Proof. It is easy to check that this function is homogeneous of degree γ. Take $t > 0$. Then

$$f(tx,tz) = \begin{cases} t^{\gamma}z^{\gamma} & \text{if } tx \geq tz \\ 0 & \text{otherwise} \end{cases} = t^{\gamma}\begin{cases} z^{\gamma} & \text{if } x \geq z \\ 0 & \text{otherwise} \end{cases} = t^{\gamma}f(x,z)$$

We need to check that $f(\cdot)$ is quasiconcave. Let $U_c = \left\{(x,z) \in \mathbb{R}^2_+ : f(x,z) \geq c\right\}$ be the upper contour set for $c \geq 0$. Note that $U_0 = \mathbb{R}^2_+$ is a convex set. Now consider $c > 0$. Then $(x,z) \in U_c \iff f(x,z) \geq c \iff z^\gamma \geq c \wedge x \geq z$. Thus

$$U_c = \left\{(x,z) \in \mathbb{R}^2_+ : z \geq c^{\frac{1}{\gamma}}\right\} \cap \left\{(x,z) \in \mathbb{R}^2_+ : x \geq z\right\}$$

is the intersection of two convex sets and therefore it is convex. Note that U_c is not strictly convex. Then $f(x,z)$ is quasiconcave but not strictly quasiconcave.

Now take any $x_1 > z_1 > 0$ and let $(x_2, z_2) = (x_1, x_1 + \epsilon)$ for any $\epsilon > 0$. Then $(x_2, z_2) \geq (x_1, z_1)$ but $f(x_2, z_2) = 0 < z_1^\gamma = f(x_1, z_1)$. Then the function is not increasing.

Finally we show that $f(\cdot)$ is not continuous. Let $x > 0$ and note that $f\left(x, x + \frac{1}{n}\right) = 0$ for all $n \in \mathbb{N}^*$, but $f(x,x) = x^\gamma > 0$. Since $f(\cdot)$ is not continuous, it is not concave. $\square$

Note that this example also shows that the conditions imposed by Theorem 1 are not enough to secure strict concavity, even when $\gamma < 1$. We provide a more economically appealing example of this fact.

Let $g(\mathbf{x}) = \min\{x_1, x_2\}$. It is well known that this function is increasing, concave but not strictly quasiconcave. Now consider the function $h(x) = x^\gamma$ for $\gamma \in (0,1)$. The function $h(x)$ is strictly increasing and strictly concave. Define now

$$f(\mathbf{x}) = h(g(\mathbf{x})) = (\min\{x_1, x_2\})^\gamma$$

Since $h(x)$ is strictly monotone, the upper contour sets of $f(\mathbf{x})$ are exactly the same upper contour sets of $g(\mathbf{x})$. Then $f(\mathbf{x})$ is increasing, concave but not strictly quasiconcave. However it is homogeneous of degree $\gamma \in (0,1)$ and quasiconcave in $\mathbf{x}$. Thus we cannot get strict concavity using only the conditions imposed in Theorem 1.

The problem is that strict concavity implies strict quasiconcavity. However the conditions of Theorem 1 do not guarantee the strict quasiconcavity of $f(\cdot)$. If we have this additional requirement, as in Friedman (1973), we can easily get strict concavity.

Theorem 2. *If $f(\cdot)$ is strictly quasiconcave, increasing and homogeneous of degree γ in $\mathbf{x}$, where $0 < \gamma < 1$, then $f(\cdot)$ is strictly concave in $\mathbf{x}$.*

Proof. Consider two distinct vectors $\mathbf{x_1}$, $\mathbf{x_2}$ and define $y_1^\gamma = f(\mathbf{x_1})$ and $y_2^\gamma = f(\mathbf{x_2})$. Let $\lambda \in (0,1)$. We again consider all possible cases.

Case 1: $y_1 = y_2$.

We have

$$f(\lambda \mathbf{x_1} + (1-\lambda)\mathbf{x_2}) > \min\{f(\mathbf{x_1}), f(\mathbf{x_2})\} = \lambda f(\mathbf{x_1}) + (1-\lambda) f(\mathbf{x_2})$$

by strict quasiconcavity.

Case 2: $y_1 \neq y_2$ and $y_1 = 0$.

As shown in the proof of Theorem 1 we have

$$f(\lambda \mathbf{x_1} + (1-\lambda)\mathbf{x_2}) \geq f((1-\lambda)\mathbf{x_2}) = (1-\lambda)^{\gamma} f(\mathbf{x_2}) > (1-\lambda) f(\mathbf{x_2}) = \lambda f(\mathbf{x_1}) + (1-\lambda) f(\mathbf{x_2})$$

where the first inequality follows from $f(\cdot)$ increasing and the second inequality is strict because $0 < \gamma < 1$.

Note that if $\mathbf{x_1} = \mathbf{0}$ then $y_1 = 0$ and

$$f(\lambda \mathbf{x_1} + (1-\lambda)\mathbf{x_2}) = f((1-\lambda)\mathbf{x_2}) = (1-\lambda)^{\gamma} f(\mathbf{x_2}) > \lambda f(\mathbf{x_1}) + (1-\lambda) f(\mathbf{x_2})$$

where we did not need the fact that $f(\cdot)$ is increasing.

Case 3: $y_1 \neq 0$, $y_2 \neq 0$ and $y_1 \neq y_2$.

Then we get

$$f(\lambda \mathbf{x_1} + (1-\lambda)\mathbf{x_2}) \geq (\lambda y_1 + (1-\lambda) y_2)^{\gamma}$$

When $0 < \gamma < 1$ the function $h(x) = x^{\gamma}$ is strictly concave and we get $(\lambda y_1 + (1-\lambda) y_2)^{\gamma} > \lambda y_1^{\gamma} + (1-\lambda) y_2^{\gamma}$. Then we conclude that $f(\lambda \mathbf{x_1} + (1-\lambda)\mathbf{x_2}) > \lambda f(\mathbf{x_1}) + (1-\lambda) f(\mathbf{x_2})$. $\square$

Note that in the proof of Theorem 2 we only required the strict quasiconcavity of $f(\cdot)$ for Case 1 (when $y_1 = y_2$). If $f(\cdot)$ were to be one-to-one we would have that $y_1 = y_2$ implies $\mathbf{x_1} = \mathbf{x_2}$ and thus this case would be ruled out. Note also that if $\mathbf{0} \in X \subseteq \mathbb{R}_+^n$ and $f(\cdot)$ is one-to-one then for Case 2 ($y_1 \neq y_2$ and $y_1 = 0$) we have that $f(\mathbf{x_1}) = 0$ implies $\mathbf{x_1} = \mathbf{0}$ and we can get rid of the condition "$f(\cdot)$ is increasing". These facts give us the next Corollary.

Corollary 1. *Let $f : X \subseteq \mathbb{R}_+^n \longrightarrow \mathbb{R}$ be one-to-one, quasiconcave and homogeneous of degree γ in $\mathbf{x}$, where $0 < \gamma < 1$.*

If $f(\cdot)$ is increasing or if $\mathbf{0} \in X$ then $f(\cdot)$ is strictly concave in $\mathbf{x}$.

References

Bone, J. (1989): "A Note on Concavity and Scalar Properties in Production," *Bulletin of Economic Research*, 41(3), 213–217.

Dalal, A. (2000): "Strict Concavity with Homogeneity and Decreasing Returns to Scale," *Atlantic Economic Journal*, 28(3), 381–382.

Friedman, J. W. (1973): "Concavity of Production Functions and Non-Increasing Returns to Scale," *Econometrica*, 41(5), 981–984.