On the existence of most-preferred alternatives in complete lattices

Kukushkin, Nikolai S.

16 December 2010

On the existence of most-preferred alternatives in complete lattices

Nikolai S. Kukushkin*

December 16, 2010

Abstract

If a preference ordering on a complete lattice is quasisupermodular, or just satisfies a rather weak analog of the condition, then it admits a maximizer on every subcomplete sublattice if and only if it admits a maximizer on every subcomplete subchain. *JEL* Classification Numbers: C 61; D 11.

Key words: lattice optimization; quasisupermodularity

1 Introduction

The familiar observation that an upper semicontinuous function attains its maximum on every compact set resolves the question of the existence of optimal choices in many situations, but not in all. First, sometimes preferences *have* to be described by discontinuous relations, e.g., lexicographic combinations of several scalar characteristics. Second, there may be no "natural" topology hence the question of whether the preferences are (semi)continuous may become intolerably vague.

This paper follows the approach of Milgrom and Shannon (1994): we assume an internal order structure on the set of alternatives; in most cases, it is a complete lattice. All assumptions about the preferences are formulated in terms of the order rather than topology. The existence of optimal choices is obtained under much milder conditions than in Theorem A4 of Milgrom and Shannon (1994). In particular, we prove that a quasisupermodular function defined on a complete lattice attains its maximum on every subcomplete sublattice if and only if it attains its maximum on every subcomplete subchain.

In Section 2, basic definitions and some useful results are reproduced. Section 3 starts with a very weak analog of Milgrom and Shannon's (1994) "upper semicontinuity on chains."

*Russian Academy of Sciences, Dorodnicyn Computing Center, 40, Vavilova, Moscow 119333 Russia
E-mail: ququns@inbox.ru

Theorem 2 shows it to be necessary and sufficient for the existence of optima in every subcomplete subchain of a chain-complete poset; there is an obvious, if vague, similarity with Theorem 4.1 of Smith (1974). Then we partition the quasisupermodularity property into four mutually independent "quarters," first introduced in Kukushkin (2008b, 2009). Theorems 5 and 6 show that two of these conditions are sufficient to reduce the problem of the existence of optima on a lattice to that on subchains; Example 7 shows that the other two are insufficient for that even if imposed together. In Section 4, two stronger conditions on subchains (though still weaker than upper semicontinuity) are introduced; when coupled with either of the remaining quarters of quasisupermodularity, they ensure the existence of optimal choices although this time without any claim to necessity. A few concluding remarks are in Section 5.

2 Basic notions

For every set A, we denote $\mathfrak{B}_A$ the set of all nonempty subsets of A. Given a binary relation $\succ$ on A and $X \in \mathfrak{B}_A$, we denote

$$M(X, \succ) := \{x \in X \mid \nexists y \in X\, [y \succ x]\}, \tag{1}$$

the set of *maximizers* of $\succ$ on X. The interpretation is that an agent has (strict) preferences described by relation $\succ$ over the whole A, but may be faced with the necessity to choose from a subset $X \in \mathfrak{B}_A$, in which case any point from $M(X, \succ)$ will do. For such choice to be possible, we need $M(X, \succ) \neq \emptyset$, at least, for "plausible" X.

Typically, A is a *partially ordered set* (a *poset*) with the order $>$; most often, a *lattice*. The exact definitions are assumed commonly known.

Theorem A (Zorn's Lemma). *If a poset X has the property that every chain $Y \in \mathfrak{B}_X$ has an upper bound in X, then $M(X, >) \neq \emptyset$.*

A poset A is *chain-complete* if $\sup X$ and $\inf X$ exist for every chain $X \in \mathfrak{B}_A$. If A is a chain-complete poset and $X \in \mathfrak{B}_A$, we call X a *chain-subcomplete subset* if $\sup Y$ and $\inf Y$ belong to X for every chain $Y \in \mathfrak{B}_X$; if X itself is a chain, we call it a *subcomplete subchain*. The set of all (nonempty) subcomplete subchains of a chain-complete poset A is denoted $\mathfrak{C}_A$.

A lattice is *complete* if the *greatest lower bound* or *meet*, $\bigwedge X$, and the *least upper bound* or *join*, $\bigvee X$, exist for every $X \in \mathfrak{B}_A$. If A is a complete lattice, $X \in \mathfrak{B}_A$ is a *subcomplete sublattice* of A if $\bigwedge Y$ and $\bigvee Y$ belong to X for all $Y \in \mathfrak{B}_X$. Given a complete lattice A, the set of all (nonempty) subcomplete sublattices is denoted $\mathfrak{L}_A$.

Theorem B (Veinott, 1989). *A lattice A is complete if and only if it is chain-complete as a poset. Then a sublattice of A is subcomplete if and only if it is chain-subcomplete.*

The preference relation $\succ$ is always assumed to be an *ordering*, i.e., irreflexive, transitive, and *negatively transitive*, $z \not\succ y \not\succ x \Rightarrow z \not\succ x$. Then the "non-strict preference" relation $\succeq$ defined by $y \succeq x \rightleftharpoons x \not\succ y$ is reflexive, transitive, and total. Orderings can also be defined in terms of representations in chains: $\succ$ is an ordering if and only if there is a chain $\mathcal{C}$ and a mapping $u\colon A \to \mathcal{C}$ such that

$$y \succ x \iff u(y) > u(x) \tag{2}$$

for all $x, y \in A$. Then $y \succeq x \iff u(y) \geq u(x)$, and $M(X, \succ) = \mathrm{Argmax}_{x \in X}\, u(x)$ for every $X \in \mathfrak{B}_A$.

The most usual assumption in game theory is that the preferences of a player are described by a *utility function* $u\colon A \to \mathbb{R}$. In a purely ordinal framework, it is natural to replace $\mathbb{R}$ with an arbitrary chain. Here we take an intermediate position: Henceforth, we consider preference orderings allowing representation (2) where $\mathcal{C}$ has the property that every $V \in \mathfrak{B}_{\mathcal{C}}$ contains a countable *cofinal* subset, i.e., a countable subset $W \subseteq V$ such that for every $v \in V$ there is $w \in W$ such that $w \geq v$. Obviously, $\mathbb{R}$ possesses the property; a bit less obvious example is $\mathbb{R}^m$ with a lexicographic order.

Given an ordering $\succ$ on A and $X \in \mathfrak{B}_A$, we call a sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ in X *maximizing* in X if (1) $x^{k+1} \succ x^k$ for all k; (2) for every $x \in X \setminus M(X, \succ)$, there is $k \in \mathbb{N}$ such that $x^k \succ x$.

Proposition 1. *For every ordering $\succ$ on A, the following statements are equivalent:*

1. *$\succ$ admits a representation* (2) *such that every nonempty subset of $\mathcal{C}$ contains a countable cofinal subset;*
2. *for every $X \in \mathfrak{B}_A$, either $M(X, \succ) \neq \emptyset$, or there exists a maximizing sequence in X.*

Proof. For completeness, we provide a straightforward proof. Let Statement 1 hold, $X \in \mathfrak{B}_A$, and $M(X, \succ) = \emptyset$. We denote $V := u(X)$ and $W = \{w^k\}_{k \in \mathbb{N}}$ a countable cofinal subset of V. Then we recursively construct an infinite increasing sequence $\langle v^k \rangle_{k \in \mathbb{N}}$ in V with the property that $v^k > w^h$ whenever $k > h$ in this way. First, $v^0 := w^0$. Having v^k defined, we set $m := \min\{h \in \mathbb{N} \mid h > k \ \&\ w^h > v^k\}$ and $v^{k+1} := w^m$; the assumption that $M(X, \succ) = \emptyset$, hence $\max V$ does not exist, ensures that the process never stops. For each $k \in \mathbb{N}$, we pick $x^k \in X$ for which $u(x^k) = v^k$. It is easily checked that the sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ is maximizing in X.

Conversely, if Statement 2 holds, we start with an arbitrary representation $u\colon A \to \mathcal{C}$ and set $Z := u(A)$; then $u\colon A \to Z$ is still a representation of $\succ$ in the sense of (2). Let $V \in \mathfrak{B}_Z$; we define $X := u^{-1}(V)$. If $M(X, \succ) \neq \emptyset$, then $u(M(X, \succ))$ is a singleton cofinal subset of V. Otherwise, there is a maximizing sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ in X. Denoting $w^k := u(x^k)$, we immediately see that $W := \{w^k\}_{k \in \mathbb{N}}$ is a countable cofinal subset of V. $\square$

We call a preference ordering *regular*, just for want of a better term, if it satisfies the conditions listed in Proposition 1.

Given an ordering $\succ$ on a poset A, we consider four auxiliary orders (irreflexive and transitive relations):

$$\begin{aligned}
y \underset{>}{\succ} x &\rightleftharpoons [y \succ x \;\&\; y > x];\\
y \underset{<}{\succ} x &\rightleftharpoons [y \succ x \;\&\; y < x];\\
y \underset{>}{\succeq} x &\rightleftharpoons [y \succeq x \;\&\; y > x];\\
y \underset{<}{\succeq} x &\rightleftharpoons [y \succeq x \;\&\; y < x].
\end{aligned}$$

3 Main characterization theorems

We call an ordering $\succ$ on a chain-complete poset A *mono-ω-transitive* if both following conditions hold:

$$\forall k \in \mathbb{N}\,[x^{k+1} \succ x^k \;\&\; x^{k+1} > x^k] \Rightarrow \sup\{x^k\}_k \succ x^0; \tag{3a}$$

$$\forall k \in \mathbb{N}\,[x^{k+1} \succ x^k \;\&\; x^{k+1} < x^k] \Rightarrow \inf\{x^k\}_k \succ x^0. \tag{3b}$$

Clearly, both conditions (3) hold if $\succ$ is *upper semicontinuous*, i.e., has open lower contours, or just satisfies Milgrom and Shannon's (1994) "upper semicontinuity on chains" condition. On the other hand, a mono-ω-transitive ordering need not be upper semicontinuous on chains (e.g., the lexicographic order on $\mathbb{R}^m$).

Remark. There is an obvious similarity with the (topological) notion of "ω-transitivity" (Gillies, 1959; Smith, 1974; Kukushkin, 2008a).

Theorem 2. *A regular ordering $\succ$ on a chain-complete poset A has the property that $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$ if and only if $\succ$ is mono-ω-transitive.*

Proof. The sufficiency proof is deferred to after Theorem 5. Assuming (3a) violated, we define $X := \{x^k\}_{k\in\mathbb{N}} \cup \{\sup_k x^k\}$; clearly, $X \in \mathfrak{C}_A$. The assumption $x^0 \succeq \sup_k x^k$ implies $x^1 \succ \sup_k x^k$ since $\succ$ is an ordering; therefore, $M(X, \succ) = \emptyset$. A violation of (3b) is treated dually. □

Milgrom and Shannon's (1994) definition of a *quasisupermodular* function on a lattice remains meaningful for a mapping to an arbitrary chain:

$$\forall x, y \in A\,\big[u(x) > u(y \wedge x) \Rightarrow u(y \vee x) > u(y)\big]; \tag{4a}$$

$$\forall x, y \in A\,\big[u(y) > u(y \vee x) \Rightarrow u(y \wedge x) > u(x)\big]. \tag{4b}$$

We partition quasisupermodularity (4) into four independent conditions, cf. Eq. (1) and (2) of Li Calzi and Veinott (1992):

$$\forall x, y \in A \left[u(y) \vee u(x) > u(y \wedge x) \Rightarrow u(y \vee x) > u(y) \wedge u(x)\right]; \tag{5a}$$
$$\forall x, y \in A \left[u(y) \wedge u(x) > u(y \wedge x) \Rightarrow u(y \vee x) > u(y) \vee u(x)\right]; \tag{5b}$$
$$\forall x, y \in A \left[u(y) \wedge u(x) > u(y \vee x) \Rightarrow u(y \wedge x) > u(y) \vee u(x)\right]; \tag{5c}$$
$$\forall x, y \in A \left[u(y) \vee u(x) > u(y \vee x) \Rightarrow u(y \wedge x) > u(y) \wedge u(x)\right]. \tag{5d}$$

Proposition 3. *Let A be a lattice and $\mathcal{C}$ be a chain. Then a mapping $u\colon A \to \mathcal{C}$ satisfies condition* (4a) *if and only if it satisfies* (5a) *and* (5b); *u satisfies* (4b) *if and only if it satisfies* (5c) *and* (5d).

Proof. The necessity is obvious. To prove the sufficiency, we suppose the contrary. Let $u(x) > u(y \wedge x)$, but $u(y) \geq u(y \vee x)$; then $u(y \vee x) > u(x)$ by (5a), hence $u(y) > u(y \wedge x)$ by transitivity, which contradicts (5b). The proof of the equivalence (4b) $\equiv$ [(5c) & (5d)] is dual. □

Remark. When x and y are comparable in the basic order, each condition (5) holds trivially.

Example 4. Let $A := \{(0,0), (0,1), (1,0), (1,1)\} \subset \mathbb{R}^2$; we consider four mappings $A \to \mathbb{R}$ depicted in these matrices (the axes are directed upwards and rightwards):

$$\mathbf{a}.\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{b}.\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{c}.\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \qquad \mathbf{d}.\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}.$$

The mapping depicted in the matrix "a" satisfies all conditions (5) except (5a), and similarly with other matrices.

Remark. Agliardi (2000) called a function u on a lattice A *pseudosupermodular* if it satisfies (5a) and (5c).

Clearly, the conditions (5), as well as (4), are invariant to monotone transformations of u, i.e., they only depend on $\succ$. In particular, if an ordering $\succ$ admits a representation satisfying (5a), then *every* representation of $\succ$ satisfies (5a). Henceforth, we just say "$\succ$ satisfies (5a)."

Theorem 5. *Let A be a complete lattice and $\succ$ be a regular ordering on A satisfying* (5a). *Then $\succ$ has the property that $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{L}_A$ if and only if it is mono-ω-transitive.*

Proof. The necessity follows from Theorem 2 because $\mathfrak{C}_A \subseteq \mathfrak{L}_A$ by Theorem B above. We start the sufficiency proof with a couple of auxiliary statements.

Claim 5.1. *If X is a sublattice of A, $x \in M(X, \succsim)$, and $X \ni y \succ x$, then $y \wedge x \succeq y$.*

Proof. If $y \succ y \wedge x$, then condition (5a) would imply that $u(y \vee x) > u(y) \wedge u(x) = u(x)$, which contradicts the assumption $x \in M(X, \mathrel{\overset{\succ}{>}})$ since X is a sublattice. $\square$

Claim 5.2. *If $x \in X \in \mathfrak{L}_A$, then either $x \in M(X, \mathrel{\overset{\succ}{>}})$ or there is $y \in M(X, \mathrel{\overset{\succ}{>}})$ such that $y \mathrel{\overset{\succ}{>}} x$.*

Proof. First, we set $X^* := \{y \in X \mid y \mathrel{\overset{\succ}{>}} x\}$. It is enough to show that $M(X^*, \mathrel{\overset{\succ}{>}}) \neq \emptyset$ if $X^* \neq \emptyset$. We do that invoking Zorn's Lemma (Theorem A above). Let $Y \subseteq X^*$ be a chain w.r.t. $\mathrel{\overset{\succ}{>}}$, i.e., Y is a chain such that $y \succ x$ whenever $y, x \in Y$ and $y > x$. If there exists $\max Y$, it is an upper bound of Y. Otherwise, let $\langle y^k \rangle_{k \in \mathbb{N}}$ be a maximizing sequence in Y and $y^+ := \sup_k y^k$; y^+ exists and belongs to X because the latter is subcomplete. For every $x \in Y$, there is $m \in \mathbb{N}$ such that $y^m \mathrel{\overset{\succ}{>}} x$. Since $y^+ = \sup_{k \geq m} y^k$, we have $y^+ \mathrel{\overset{\succ}{>}} y^m \mathrel{\overset{\succ}{>}} x$ by (3a). Therefore, $y^+ \in X^*$ and is an upper bound of Y in X^*. $\square$

Let $X \in \mathfrak{L}_A$ and $\langle y^h \rangle_{h \in \mathbb{N}}$ be a maximizing sequence in X. (If there is no such sequence, then $M(X, \succ) \neq \emptyset$, and we are already home.) We recursively define a sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ such that $x^k \in M(X, \mathrel{\overset{\succ}{>}})$, $x^{k+1} \mathrel{\overset{\succ}{<}} x^k$, and $x^{k+1} \succ y^{k+1}$ for all k. First, $x^0 := \bigvee X$.

Having $x^k \in M(X, \mathrel{\overset{\succ}{>}})$ defined, we first check whether $x^k \in M(X, \succ)$; if the answer is "yes," we are home again. Otherwise, we pick $h \in \mathbb{N}$ such that $y^h \succ x^k$ and $h > k+1$, hence $y^h \succ y^{k+1}$. Denoting $x^* := y^h \wedge x^k$, we have $x^* \succeq y^h \succ x^k$ by Claim 5.1, hence $x^* \mathrel{\overset{\succ}{<}} x^k$. Now we define $Y := X \cap [x^*, x^k]$ and, applying Claim 5.2, obtain $x^{k+1} \in M(Y, \mathrel{\overset{\succ}{>}})$ such that $x^{k+1} \succeq x^*$.

Let us show that $x^{k+1} \in M(X, \mathrel{\overset{\succ}{>}})$. Otherwise, there is $y \in X$ such that $y \mathrel{\overset{\succ}{>}} x^{k+1}$, hence $y \succ x^k$. We define $y^* := y \wedge x^k$ and apply Claim 5.1, obtaining $y^* \succeq y \succ x^{k+1}$. Besides, $x^k \geq y^* \geq x^{k+1} \geq x^*$, hence $y^* \in Y$ and $y^* \mathrel{\overset{\succ}{>}} x^{k+1}$, which contradicts $x^{k+1} \in M(Y, \mathrel{\overset{\succ}{>}})$. Thus, $x^{k+1} \in M(X, \mathrel{\overset{\succ}{>}})$ indeed.

Finally, we set $x^+ := \inf_k x^k$. By (3b), we have $x^+ \succ x^k$ for each $k \in \mathbb{N}$. Since $x^{k+1} \succ y^{k+1}$ for each k, we see that $x^+ \in M(X, \succ)$. $\square$

Corollary. *A quasisupermodular function on a complete lattice attains its maximum if it attains a maximum on every subcomplete chain.*

Proof of sufficiency in Theorem 2. Let $X \in \mathfrak{C}_A$. The restriction of u to X satisfies (5a), hence the sufficiency part of Theorem 5 applies. $\square$

Theorem 6. *Let A be a complete lattice and $\succ$ be a regular ordering on A satisfying* (5d). *Then $\succ$ has the property that $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{L}_A$ if and only if it is mono-ω-transitive.*

The proof is dual to that of Theorem 5.

Remark. The very possibility to replace one condition with another shows that there is no clear prospect for a necessity result, in the style of Theorem 2 above or Theorem 4.1 of Smith (1974), about a complete lattice A and $X \in \mathfrak{L}_A$.

Conditions (5a) or (5d) cannot be replaced with (5b) or (5c), or even with their conjunction.

Example 7. Let $A := \big(\{n/(n+1)\}_{n\in\mathbb{N}} \cup \{1\}\big) \times \big(\{0\} \cup \{1/(n+1)\}_{n\in\mathbb{N}}\big) \subset \mathbb{R}^2$ and $u\colon A \to \mathbb{R}$ be as follows: $u(1, x_2) = u(x_1, 0) := 0$; $u(n_1/(n_1+1), 1/(n_2+1)) := \min\{n_1, n_2\}$. It is easy to check that A with the order induced from $\mathbb{R}^2$ is a complete lattice and u is mono-ω-transitive; by Theorem 2, $M(X, \succ) \neq \emptyset$ for every $X \in \mathfrak{C}_A$. Moreover, u satisfies (5b) and (5c). Nonetheless, $\sup_{x\in A} u(x) = +\infty$, hence there is no maximizer.

4 Further results

We introduce a pair of conditions stronger than (3), but still weaker than upper semicontinuity on chains.

$$\forall X \in \mathfrak{C}_A\, \forall Y \in \mathfrak{B}_X \,\big[\forall y, x \in Y\, [y > x \Rightarrow y \succeq x] \Rightarrow \forall x \in Y\, [\sup Y \succeq x]\big]; \tag{6a}$$

$$\forall X \in \mathfrak{C}_A\, \forall Y \in \mathfrak{B}_X \,\big[\forall y, x \in Y\, [x > y \Rightarrow y \succeq x] \Rightarrow \forall x \in Y\, [\inf Y \succeq x]\big]. \tag{6b}$$

It is easy to see that (6a)⇒(3a) while (6b)⇒(3b).

Theorem 8. *Let A be a complete lattice and $\succ$ be a regular ordering on A satisfying* (3b)*,* (6a)*, and* (5c)*. Then $M(A, \succ) \neq \emptyset$.*

Proof. The basic construction is virtually the same as in the proof of Theorem 5. Again, we start with a couple of auxiliary statements.

Claim 8.1. *If X is a sublattice of A, $x \in M(X, \underset{>}{\succeq})$, and $X \ni y \succ x$, then $y \wedge x \succ y$.*

Proof. The maximality of x implies that $u(y) \wedge u(x) = u(x) > u(y \vee x)$, hence $u(y \wedge x) > u(x) \vee u(y) = u(y)$ by (5c). ◻

Claim 8.2. *If $x \in X \in \mathfrak{L}_A$, then either $x \in M(X, \underset{>}{\succeq})$ or there is $y \in M(X, \underset{>}{\succeq})$ such that $y \underset{>}{\succeq} x$.*

Proof. Setting $X^* := \{y \in X \mid y \underset{>}{\succeq} x\}$, we show that $M(X^*, \underset{>}{\succeq}) \neq \emptyset$, invoking Zorn's Lemma (Theorem A above). If $Y \subseteq X^*$ is a chain w.r.t. $\underset{>}{\succeq}$, then the "left hand side" of (6a) applies to Y, hence $\sup Y$ is an upper bound of Y (w.r.t. $\underset{>}{\succeq}$) in X^*. ◻

Let $\langle y^h \rangle_{h\in\mathbb{N}}$ be a maximizing sequence in A. Exactly in the same way as in the proof of Theorem 5, we recursively construct a sequence $\langle x^k \rangle_{k\in\mathbb{N}}$ such that $x^k \in M(A, \underset{>}{\succeq})$, $x^{k+1} \underset{<}{\succ} x^k$, and $x^{k+1} \succ y^{k+1}$ for all k. We again set $x^0 := \bigvee A$ and then rely on Claims 8.1 and 8.2 instead of Claims 5.1 and 5.2. Finally, (3b) gives us $\inf_k x^k \in M(A, \succ)$. ◻

Theorem 9. *Let A be a complete lattice and $\succ$ be a regular ordering on A satisfying* (3a)*,* (6b)*, and* (5b)*. Then $M(A, \succ) \neq \emptyset$.*

The proof is dual to that of Theorem 8.

Remark. In Example 7, conditions (6a) and (6b) are violated by chains $\{n_1/(n_1+1)\}_{n_1 \geq n_2} \times \{1/(n_2+1)\}$ ($n_2 \in \mathbb{N}$) and $\{n_1/(n_1+1)\} \times \{1/(n_2+1)\}_{n_2 \geq n_1}$ ($n_1 \in \mathbb{N}$), respectively.

An ordering $\succ$ on a lattice A is *weakly quasisupermodular* (Shannon, 1995) if

$$\forall x, y \in A \left[x \succ y \vee x \Rightarrow y \wedge x \succeq y \right]. \tag{7}$$

It is easy to see that (7) is "self-dual" and follows from either condition (4). Each ordering in Example 4 is weakly quasisupermodular, hence (7) does not imply any condition (5). Similarly, none of conditions (5) implies (7) by itself.

Example 10. Let $A := \{(0,0), (0,1), (1,0), (1,1)\} \subset \mathbb{R}^2$; we consider two functions $A \to \mathbb{R}$ depicted in these matrices (the axes are directed upwards and rightwards):

$$\mathbf{a.} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad \mathbf{b.} \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}.$$

Neither function represents a weakly quasisupermodular ordering; that depicted in the matrix "a" satisfies (5a) and (5c); that depicted in the matrix "b," (5b) and (5d).

Proposition 11. *An ordering $\succ$ on a lattice A satisfies* (7) *if it satisfies* (5a) *or* (5c)*, and satisfies* (5b) *or* (5d).

Proof. Suppose the contrary: $x \succ y \vee x$, but $y \succ y \wedge x$. Then $y \vee x \succeq y$ by (5a) or (5c), hence $x \succ y \wedge x$ by transitivity, which is incompatible with either (5b) or (5d). □

Theorem 12. *Let A be a complete lattice and $\succ$ be a regular ordering on A satisfying* (3b)*,* (6a)*, and* (7)*. Then $M(A, \succ) \neq \emptyset$.*

Proof. The basic scheme remains the same.

Claim 12.1. *If X is a sublattice of A, $x \in M(X, \mathrel{\substack{\succeq \\ >}})$, and $X \ni y \succ x$, then $y \wedge x \succeq y$.*

Proof. The maximality of x implies that $x \succ y \vee x$, hence $y \wedge x \succeq y$ by (7). □

Claim 12.2. *If $x \in X \in \mathfrak{L}_A$, then either $x \in M(X, \mathrel{\substack{\succeq \\ >}})$ or there is $y \in M(X, \mathrel{\substack{\succeq \\ >}})$ such that $y \mathrel{\substack{\succeq \\ >}} x$.*

The proof is exactly the same as in Claim 8.2; it only needs (6a).

The end of the proof is again standard. Given a maximizing sequence $\langle y^h \rangle_{h \in \mathbb{N}}$ in A, we, relying on Claims 12.1 and 12.2, recursively construct a sequence $\langle x^k \rangle_{k \in \mathbb{N}}$ such that $x^k \in M(A, \mathrel{\substack{\succeq \\ >}})$, $x^{k+1} \mathrel{\substack{\succ \\ <}} x^k$, and $x^{k+1} \succ y^{k+1}$ for all k. Again, (3b) gives us $\inf_k x^k \in M(A, \succ)$. □

Theorem 13. *Let A be a complete lattice and $\succ$ be a regular ordering on A satisfying* (3a), (6b), *and* (7). *Then $M(A, \succ) \neq \emptyset$.*

The proof is dual to that of Theorem 8.

Remark. The preference ordering in Example 7 is weakly quasisupermodular; therefore, *both* conditions (6) cannot be replaced with (3).

5 Concluding remarks

5.1. We may call a lattice A *relatively complete* if meet $\bigwedge X$ exists for all $X \in \mathfrak{B}_A$ that are bounded below, while join $\bigvee X$ exists for all $X \in \mathfrak{B}_A$ that are bounded above. $\mathbb{R}^m$ with the natural order is a relatively complete lattice which is not complete. If A is a relatively complete lattice, $X \in \mathfrak{B}_A$ is a *subcomplete sublattice* of A if $\bigwedge Y$ and $\bigvee Y$ exist in A and belong to X for all $Y \in \mathfrak{B}_X$. Theorems 5 and 6 admit straightforward generalizations to relatively complete lattices A. Theorem 2 admits a similar generalization to "relatively chain-complete posets."

5.2. Conditions (5) play crucial roles in the study of monotone comparative statics with constant feasible sets (Kukushkin, 2008b, 2009). If the feasible set is varied while the preference ordering remains fixed, then those roles are taken by conditions (4) as well as strict and weak quasisupermodularity (Li Calzi and Veinott, 1992; Milgrom and Shannon, 1994; Shannon, 1995).

5.3. An ordering that is not regular in our sense could hardly be relevant to any decision problem. Still, one may wonder whether the same results could be obtained without the assumption. The answer is negative, even concerning Theorem 2.

Example 14. Let A' be a well-ordered uncountable set. We define $A^* := \{a \in A' \mid \{x \in A' \mid x < a\}$ is countable$\}$ and $A := A^* \cup \{\sup A^*\}$. It is easy to see that A is a complete chain and $\sup A^* \notin A^*$. Then we define a preference ordering (actually, a linear order) on A:

$$y \succ x \rightleftharpoons \big[y \in A^* \;\&\; [y > x \;\text{ or }\; x = \sup A^*]\big].$$

Condition (3b) holds by default; (3a), because $\sup\{x^k\}_k \in A^*$ whenever $\{x^k\}_k \subseteq A^*$. However, $M(A, \succ) = \emptyset$.

Sufficient conditions for the existence of optima without the regularity assumptions can be obtained (Kukushkin, 2008b) by supplementing any one of conditions (5) with "chain-related" conditions in the style of (6); the proofs require transfinite recursion. An analog of Theorem 2 can be obtained, but there is no clear prospect for a necessity result concerning optimization on subcomplete sublattices, like in Theorems 5 and 6.

5.4. Similarly to Theorem 4 of Kukushkin (2008a), Theorem 2 remains valid if $\succ$ is a semiorder (e.g., ε-improvement). In principle, the sufficiency part can be extended far beyond semiorders, cf. Theorem 1 of Kukushkin (2008a), but the notion of a maximizing sequence has to be modified considerably. The necessity does not hold even for interval orders (Kukushkin, 2008a, Example 3). As to Theorems 5 and 6, it is unclear whether they remain valid even for semiorders.

Acknowledgments

Financial support from the Russian Foundation for Basic Research (project 08-07-00158), the Russian Foundation for Humanities (project 08-02-00347), and the Spanish Ministry of Education and Science (project SEJ 2007-67135) is acknowledged. I have benefitted from fruitful contacts with Francisco Marhuenda, Herve Moulin, and Kevin Reffett.

References

Agliardi, E., 2000. A generalization of supermodularity. Economics Letters 68, 251–254.

Gillies, D.B., 1959. Solutions to general non-zero-sum games. In: Tucker, A.W., and R.D. Luce (Eds.) Contributions to the Theory of Games 4 (Annals of Mathematical Studies 40). Princeton University Press, 47–85.

Kukushkin, N.S., 2008a. Maximizing an interval order on compact subsets of its domain. Mathematical Social Sciences 56, 195–206.

Kukushkin, N.S., 2008b. Monotonicity Conditions, Monotone Selections, and Equilibria. Russian Academy of Sciences, Dorodnicyn Computing Center, Moscow.
A version available at http://www.ccas.ru/mmes/mmeda/ququ/Mono.pdf

Kukushkin, N.S., 2009. Another characterization of quasisupermodularity. Munich Personal RePEc Archive Paper. Available at http://mpra.ub.uni-muenchen.de/18237/

Li Calzi, M., and A.F. Veinott, Jr., 1992. Subextremal functions and lattice programming. Unpublished manuscript. Available at
http://ideas.repec.org/p/wpa/wuwpge/0509001.html

Milgrom, P., and C. Shannon, 1994. Monotone comparative statics. Econometrica 62, 157–180.

Shannon, C., 1995. Weak and strong monotone comparative statics. Economic Theory 5, 209–227.

Smith, T.E., 1974. On the existence of most-preferred alternatives. International Economic Review 15, 184–194.

Veinott, A.F., Jr., 1989. Lattice Programming. Unpublished lectures.