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Abstract

The Nash equilibrium concept combines two fundamental ideas. First, rational players choose the most preferred strategy given their beliefs about what other players will do. Second, it imposes the consistency condition that all players' beliefs are correct. This consistency condition has often been considered too strong and different solution concepts have been introduced in the literature in order to take into account ambiguous beliefs. In this paper, we show, by means of examples, that in some situation beliefs might be dependent on the strategy profile and that this kind of contingent ambiguity affects equilibrium behavior differently with respect to the existing models of ambiguous games. Hence we consider a multiple prior approach and subjective beliefs correspondences which depend on the strategy profile; we investigate existence of the equilibrium concepts corresponding to different attitudes towards ambiguity (namely optimism and pessimism). Finally we analyze particular beliefs correspondences: beliefs given by correlated equilibria and by ambiguity levels on events.

Keywords: Noncooperative games, ambiguity, beliefs correspondence, equilibrium

1 Introduction

In the theory of decision making under uncertainty actions of decision makers are usually assumed to lead to well-defined probability distributions over outcomes, meaning that choices of actions could be identified with choices of probability distributions. The subjective expected utility theory (Savage (1954)) provides a strongly founded approach for ranking probability distributions over outcomes for decision-makers endowed with subjective risk preferences. While this approach has led to many theoretical achievements in economics over the past decades, the evidence from Ellsberg (1961) suggests that beliefs cannot always be represented by conventional probabilities. Empirical research seems to confirm Ellsberg's conjecture on the inconsistencies between Savage's theory and empirically observed behavior. In order to fit these discrepancies, alternative theories have been proposed. The most known theories are the Choquet expected utility theory (henceforth CEU, see Schmeidler (1989)), which involves individuals maximizing the expected value of an utility function with respect to non-additive beliefs (capacities) by means of Choquet integrals (Choquet(1953)) and the maxmin expected utility theory (henceforth MEU, see Gilboa and Schmeidler (1989)) in which beliefs are represented by a convex set of probability distributions over outcomes (multiple priors) and individuals maximize the minimum over the set of beliefs of the corresponding expected utilities. These theories and their extensions are known as *ambiguity theory*.

Part of the growing literature on ambiguity has focused in the last years on games with players having ambiguous beliefs; in particular, the analysis has been directed to the concept of Nash equilibrium. In fact, the Nash equilibrium concept combines two fundamental ideas: First, rational players choose one of their most preferred strategies given their beliefs about what other players will do. Second, it imposes the consistency condition that all players' beliefs are correct. One of the major criticisms to the Nash equilibrium concept has always been the strength of the consistency condition. In fact, in many settings it is not clear why players should have exactly correct beliefs about each other. Therefore, different solution concepts have been introduced in order to weaken such consistency condition by taking into account ambiguous beliefs; most of such solution concept are founded on the MEU approach (see for instance, Dow and Werlang (1994), Eichberger and Kelsey (2000) , Lo (1996) and Klibanoff (1993)). In Eichberger, Kelsey and Schipper (2008) and in Marinacci (2000) instead, the CEU approach has been taken into account. However, in all these papers the set of beliefs of each player is fixed, while it can be exogenous or endogenously given by the solution concept; moreover, ambiguity concerns only beliefs on opponents' strategies. Examples suggest, instead, that ambiguity may concern also the rules of the game and that it may vary with the strategy profile; this is the case, for instance, of the models of coalition formation investigated in De Marco and Romaniello (2010;a,b) in which ambiguity concerns also the *rules* of coalition formation. More precisely, those papers extend previous literature in which stability of coalition structures has been analyzed by using concepts of equilibrium in associated strategic form games (see Hart and Kurz (1983)). In this class of games the strategy set of each player i is the set of all subgroups of players containing i and his choice represents the coalition he wishes to join. It is well known that, given a strategy profile (i.e. a coalition for each player), the coalition structure formed is not unequivocally determined since it depends on the so-called *rules of coalition structure formation* which are functions associating to every strategy profile a coalition structure. The usual assumption in this literature is that each player i makes his choice having correct beliefs about the strategies of every other player and about the formation rule of coalitions in which i is not involved. However, other literature argues that the formation of a coalition is the outcome of private communication within the members of the coalition (see Moreno and Wooders (1996) and references therein). Hence, differently from the previous literature, in De Marco and Romaniello (2010;a,b) it has been considered the case in which each player has vague expectations about the choices of his opponents corresponding to the coalitions in which is not involved and about the formation rule of these coalitions. It is shown that the join of those two different sources of ambiguity implies multiplicity of additive beliefs over outcomes, which turn also to be strategy profile dependent.

Therefore, in this paper we take into account a general setting in which the set of beliefs over outcomes varies with the strategy profile. In particular, this model embodies ambiguity about beliefs over opponents' strategies. We show, by means of examples, that slight variations to already existing models give rise quite naturally to the *contingent ambiguity* we investigate in this work. We follow the *multiple prior* approach: for each player, beliefs are given by a set-valued map (correspondence) which provides a set of subjective additive beliefs (probability distribution) over outcomes for every strategy profile. In line with the work of Marinacci (2000), ambiguity is solved by considering two different kind of (extreme) attitudes towards ambiguity: pessimism and optimism. Players that, in the presence of ambiguity, emphasize the lower payoffs are called *pessimistic* and players that instead emphasize the higher ones are called *optimistic*. In fact, the emphasis on higher and lower payoffs may be thought of as dependent on whether or not the player expects that ambiguity will be resolved in his favor. We provide existence results for the equilibria in games in which every player is optimistic or pessimistic. These results are based on the topological properties of the belief correspondences. The last section of the paper analyzes whether two specific kind of belief correspondence satisfy the requirements of the existence results. In the first model, beliefs to a player over his opponents' strategy profiles depend only on his strategies: beliefs are given by the correlated equilibria of the game between the opponents once they have observed players' action. In the second model, beliefs are determined by contingent assignment of ambiguity levels on a family of disjoint events in the set of outcomes.

2 Illustrative Examples

As already mentioned in the Introduction and as it will be formally stated in the next section, aim of this work is an equilibrium analysis in games in which players have ambiguous beliefs over the outcomes of the game and ambiguous beliefs depend on the strategy profile. In this section we give two examples showing that this kind of ambiguous beliefs may arise quite naturally in simple models and affect the equilibrium behavior differently with respect to the already existing models of ambiguous beliefs.

In the first example, we revise the arguments contained in Marinacci (2000) regarding the effects of ambiguity on equilibrium behavior in a variation of the *stung hunt game*. The second example is devoted to the equilibrium analysis in the *noisy leader game* (Bagwell (1982)).

The stung hunt game

The stung hunt game consists in the following 2 player game:

	c	d
c	$9, 9$	$0, 8$
d	$8, 0$	$7, 7$

Call Alice the row player and Bob the column player. This game has two equilibria in pure strategies: (c, c) which Pareto dominates the other equilibrium (d, d). However the strategy (d, d) risk dominates (c, c)[1].

Aumann (1990) argues that whenever players are prudent the equilibrium (c, c) cannot be obtained even in case of pre-play communication:
"Let us now change the scenario by permitting pre-play communication. On the face of it, it seems that the players can then agree to play (c, c); though the agreement is not enforceable, it removes each player's doubt about the other one playing c. But does it indeed remove this doubt? Suppose that Alice (player 1) is a careful, prudent person, and in the absence of an agreement, would play d. Suppose now that the players agree on (c, c), and each retires to his corner in order actually to make a choice. Alice is about to choose c, when she says to herself: Wait; I have a few minutes; let me think this over. Suppose that Bob doesn't trust me, and so will play d in spite of our agreement. Then he would still want me to play c, because that way he will get 8 rather than 7. And of course, also if he does play c, it is better for him that I play c. Thus he wants me to play c no matter what ... it is as if there were no agreement. So I will choose now what I would have chosen without an agreement, namely d".

Aumann (1990) points out that it is in a players interest to always signal c, regardless of whatever strategy he actually intends to use, since each strictly prefers that the other play c. He concludes that an agreement to play (c, c) conveys no information about what the players will do, and cannot be considered self-enforcing.

Marinacci (2000) argues that the attitudes towards ambiguity of the player determine whether (c, c) or (d, d) will be reached. More precisely, since there may be no obvious way to play, agents might well have low confidence in their own beliefs about their opponent's behavior and the way they react to this kind of ambiguity plays a significative role. With pessimistic players, (that is players that in the presence of ambiguity emphasize the lower payoffs), and if ambiguity is sufficiently high then only (d, d) is an equilibrium. This latter prediction accords with Aumann's arguments. When players are optimistic, (that is, they emphasize higher payoffs), and ambiguity is sufficiently high then the Pareto efficient Nash equilibrium (c, c) can be implemented as the unique equilibrium in the ambiguous game.

[1]The strategy pair (d,d) risk dominates (c,c) if the product of the deviation losses is highest for (d,d) (Harsanyi and Selten, 1988, Lemma 5.4.4). In other words, if the following inequality holds: $(f_A(c, d) - f_A(d, d))(f_B(d, c) - f_B(d, d)) > (f_A(d, c) - f_A(c, c))(f_B(c, d) - f_B(c, c))$. Since the game is symmetric, the inequality allows for a simple interpretation: assume the players are unsure about which strategy the opponent will pick and assign probabilities 1/2 to c and d each. Since (d, d) risk dominates (c, c), then the expected payoff from playing d exceeds the expected payoff from playing c, in fact $E_i\left(d, \frac{1}{2}c + \frac{1}{2}d\right) = \frac{15}{2} > \frac{9}{2} = E_i\left(c, \frac{1}{2}c + \frac{1}{2}d\right)$.

To understand better these arguments, consider the game in which players have multiple priors[2]. For every $i \in \{A, B\}$, denote with B_i the set of beliefs of player i over the strategies of his opponent, that is $b_i \in B_i$ denotes the probability of c and $1 - b_i$ the probability of d. For the sake of simplicity assume that $B_i = [\bar{b}, \bar{\bar{b}}]$ for $i = 1, 2$ and, for every $b_i \in B_i$, the expect payoffs from playing c or d are respectively $E_i(c, b_i) = 9b_i$ and $E_i(d, b_i) = 8b_i + 7(1 - b_i)$. Hence for an optimistic player i, the expect payoffs from playing c or d are respectively

$$\max_{b_i \in B_i} E_i(c, b_i) = 9\bar{\bar{b}} \quad \max_{b_i \in B_i} E_i(d, b_i) = 8\bar{\bar{b}} + 7(1 - \bar{\bar{b}})$$

while, for a pessimistic player i, the expect payoffs from playing c or d are respectively

$$\min_{b_i \in B_i} E_i(c, b_i) = 9\bar{b} \quad \min_{b_i \in B_i} E_i(d, b_i) = 8\bar{b} + 7(1 - \bar{b}).$$

Therefore, it follows that, for an optimistic player i, c is a best reply to B_i if and only if

$$9\bar{\bar{b}} \geq 8\bar{\bar{b}} + 7(1 - \bar{\bar{b}}) \iff \bar{\bar{b}} \geq \frac{7}{8}$$

and that, for a pessimistic player i, d is a best reply to B_i if and only if and only if

$$9\bar{b} \leq 8\bar{b} + 7(1 - \bar{b}) \iff \bar{b} \leq \frac{7}{8}.$$

The key point of this analysis is that the the set of multiple priors is fixed (that is, it does not depend on the strategy of each player). However, in such a context of vagueness, it is possible that a player (say Alice) has so vague expectations that she believes that with probability ε Bob will observe her action before choosing his strategy. Assuming that Alice believes that Bob will react optimally once observed her action then, the beliefs of Alice over Bob's strategies become

$$\mathcal{B}_A(c) = (1 - \varepsilon)B_A + \varepsilon(b_i = 1) = [(1 - \varepsilon)\bar{b} + \varepsilon, (1 - \varepsilon)\bar{\bar{b}} + \varepsilon];$$

$$\mathcal{B}_A(d) = (1 - \varepsilon)B_A + \varepsilon(b_A = 0) = [(1 - \varepsilon)\bar{b}, (1 - \varepsilon)\bar{\bar{b}}]$$

hence the expected payoffs of an optimistic Alice are

$$\max_{b_A \in \mathcal{B}_A(c)} E_A(c, b_A) = 9[(1 - \varepsilon)\bar{\bar{b}} + \varepsilon]; \quad \max_{b_A \in \mathcal{B}_A(d)} E_A(d, b_A) = 8[(1 - \varepsilon)\bar{\bar{b}}] + 7[1 - [(1 - \varepsilon)\bar{\bar{b}}]]$$

while, for a pessimistic Alice we get

$$\min_{b_A \in \mathcal{B}_A(c)} E_A(c, b_A) = 9[(1 - \varepsilon)\bar{b} + \varepsilon]; \quad \min_{b_A \in \mathcal{B}_A(d)} E_A(d, b_A) = 8[(1 - \varepsilon)\bar{b}] + 7[1 - [(1 - \varepsilon)\bar{b}]]$$

[2]Marinacci (2000) considers instead capacities and the Choquet expected utility approach. However in this example it is possible to obtain similar insights by considering the multiple priors approach.

Analogous arguments can hold for Bob. Therefore, it can be checked that c is a best reply if and only if $\bar{\bar{b}} \geq \frac{7-8\varepsilon}{8(1-\varepsilon)}$ and d is a best reply if and only if $\bar{b} \leq \frac{7-8\varepsilon}{8(1-\varepsilon)}$. Since $\frac{7-8\varepsilon}{8(1-\varepsilon)} < \frac{7}{8}$. Hence, these new beliefs imply that (c, c) requires a lower level of ambiguity to be implemented as an equilibrium for optimistic players while the implementation of (d, d) requires more vagueness. However, in this case there are no substantial differences with the analysis involving a fixed set of beliefs. To better understand the impact of variable beliefs we need to consider a variation of the stung hunt game:

	c	d
c	$8, 8$	$0, 8$
d	$8, 0$	$7, 7$

This game has yet two equilibria (c, c) (which remains the unique strong nash equilibrium) and (d, d) which not only is risk dominant but it is also in weakly dominant strategies. Obviously, (c, c) can be implemented by optimistic players in the game with fixed set of beliefs only if such set is of the following kind $B_i = [\bar{b}, 1]$. However, for the set of beliefs also the Pareto dominated profile (d, d) is an equilibrium for optimistic players. However, the introduction of variable beliefs allows for a resolution of this drawback. In fact, in this case the beliefs correspondence are

$$\mathcal{B}_A(c) = (1-\varepsilon)B_A + \varepsilon[0, 1] = [(1-\varepsilon)\bar{b} + \varepsilon, 1];$$

$$\mathcal{B}_A(d) = (1-\varepsilon)B_A + \varepsilon(b_A = 0) = [(1-\varepsilon)\bar{b}, (1-\varepsilon)]$$

hence

$$\max_{b_A \in \mathcal{B}_A(c)} E_A(c, b_A) = 8; \quad \max_{b_A \in \mathcal{B}_A(d)} E_A(d, b_A) = 8[(1-\varepsilon)] + 7[1 - [(1-\varepsilon)]] = 8 - \varepsilon$$

which implies that c is always the unique best reply and (c, c) the unique equilibrium for optimistic players.

The noisy leader game

Consider a simple 2 x 2 setting in which there are two players who choose one of two actions, C and S. If the game is played simultaneously, the payoff matrix is

	S	C
S	$5, 2$	$3, 1$
C	$6, 3$	$4, 4$

The (S, S) outcome is the "Stackelberg outcome" since this is the unique subgame perfect equilibrium outcome for the game in which player 1 moves first in a perfectly observable fashion. The unique Nash equilibrium outcome of the simultaneous-move game is (C, C), and this corresponds to the "Cournot outcome". Bagwell (1982) then considers the noisy-leader game. In this game, a pure strategy for player 1 is simply an action in the set

$\{C, S\}$. Let the signal received by player 2 be denoted by ϕ, and assume for simplicity that ϕ either takes value C or S. The signal technology works as follows:

$$Prob(\phi = S|S) = 1 - \varepsilon = Prob(\phi = C|C),$$

where $\varepsilon \in]0, 1[$. In other words, when player 1 chooses a particular action, the probability that player 2 will observe a signal specifying that same action is $1 - \varepsilon$. If $a_2 \in \{C, S\}$ represents an action for player 2, then a pure strategy for player 2 is a function, $a_2 = \omega(\phi)$, where $\omega(\phi) \in \{C, S\}$ for all ϕ. The noisy-leader game admits no off-equilibrium-path information sets, since, for any given action by Player 1, each signal is realized with positive probability. Hence, backward-induction-based refinements of Nash equilibrium are not effective (helpful) in this game and the equilibria can be found from the following strategic form:

	SS	SC	CS	CC
S	$5, 2$	$5 - 2\varepsilon, 2 - \varepsilon$	$3 + 2\varepsilon, 1 + \varepsilon$	$3, 1$
C	$6, 3$	$4 + 2\varepsilon, 4 - \varepsilon$	$6 - 2\varepsilon, 3 - \varepsilon$	$4, 4$

The Stackelberg outcome therefore fails to emerge as a Nash equilibrium outcome for the noisy leader game, and this is true no matter how precise the signal may be (i.e., no matter how small is ε). In fact, the unique pure-strategy Nash equilibrium of the noisy-leader game occurs when player 1 selects C and player 2 also selects C for all signal values.

Now we look at the effects of ambiguity: suppose that the probability ε is vague and in particular assume that it can be any probability in an interval $[\overline{\varepsilon}, \overline{\overline{\varepsilon}}]$. It is easy to check that if $\overline{\varepsilon} > 0$ then only $(C, (C, C))$ is an equilibrium, independently from the attitudes of the players towards ambiguity. Therefore, assume that $\overline{\varepsilon} = 0$ implying that the signal might be precise. If player 2 is pessimistic, then, again $(C, (C, C))$ is the unique equilibrium independently from the attitudes of player 1 towards ambiguity. Suppose now that player 2 is optimistic, then the payoff matrix becomes

	SS	SC	CS	CC
S	$5, 2$	$5 - 2\varepsilon, 2$	$3 + 2\varepsilon, 1 + \overline{\overline{\varepsilon}}$	$3, 1$
C	$6, 3$	$4 + 2\varepsilon, 4$	$6 - 2\varepsilon, 3$	$4, 4$

This game has at least another equilibrium in which player 2 chooses his strategy according to the signal he receives, that is the equilibrium strategy of player 2 is (S, C). The best reply of player 1 to player 2's strategy (S, C) depends on the level of ambiguity of the signal (ambiguity is larger as $\overline{\varepsilon}$ increases) and not on the attitude of player 1 towards ambiguity. In fact, independently of his attitudes, when $\overline{\overline{\varepsilon}} < 1/2$ player 1 prefers S to C. Whenever $\overline{\overline{\varepsilon}} > 1/2$ things are exactly the opposite since player 1 prefers C to S.

Therefore, vagueness on the technology of the signal and the presence of an optimistic player 2 alters the scenario since it allows for the existence of other equilibria in which player 2 plays the strategy (S, C). Moreover, one can check that in those equilibria an optimistic player 1 gets an expected payoff which is greater or equal than 5 which is the Stackelberg outcome. This means that an optimistic attitude towards ambiguity *restores* the first mover advantage.

3 Games, contingent beliefs and equilibria

3.1 The model

We consider a finite set on players $I = \{1, \dots n\}$; for every player i, $\Psi_i = \{\psi_i^1, \dots, \psi_i^{k(i)}\}$ is the (finite) pure strategy set of player i, $\Psi = \prod_{i \in I} \Psi_i$ and $\Psi_{-i} = \prod_{j \neq i} \Psi_j$. Denote with X_i the set of mixed strategies of player i and each strategy $x_i \in X_i$ is a vector $x_i = (x_i(\psi_i))_{\psi_i \in \Psi_i} \in \mathbb{R}_+^{k(i)}$ such that $\sum_{\psi_i \in \Psi_i} x_i(\psi_i) = 1$. Denote also with $X = \prod_{j=1}^n X_j$ and with $X_{-i} = \prod_{j \neq i} X_j$.

Differently from the classical literature on games, in this work we do not assume the existence of a one to one correspondence between strategies and outcomes of a game. Instead, we denote with $\Omega \subseteq \mathbb{R}^n$ the set of outcomes of the game, where ω_i represents the payoff to player i when outcome ω is realized. Let $\mathcal{P}$ be the set of all probability distributions on Ω, we consider the general situation in which each player is endowed with a set-valued map $\mathcal{B}_i : X \rightsquigarrow \mathcal{P}$, called *beliefs correspondence*, which gives to player i the set $\mathcal{B}_i(x)$ of subjective beliefs over outcomes, for every strategy profile $x \in X$. We consider the (extreme) situation in which players are either *pessimistic* or *optimistic* where a player is pessimistic if, in the presence of ambiguity, emphasizes the lower payoffs while he is optimistic if he emphasizes the higher ones instead.

More precisely a pessimistic player has the pessimistic payoff $F_i^P : X \to \mathbb{R}$ defined by

$$F_i^P(x) = \min_{\varrho \in \mathcal{B}_i(x)} \sum_{\omega \in \Omega} \varrho(\omega)\omega_i \quad \forall x \in X, \tag{1}$$

while an optimistic player has the optimistic payoff $F_i^O : X \to \mathbb{R}$ defined by

$$F_i^O(x) = \max_{\varrho \in \mathcal{B}_i(x)} \sum_{\omega \in \Omega} \varrho(\omega)\omega_i \quad \forall x \in X. \tag{2}$$

Assuming that players are partitioned in optimistic and pessimistic ones, that is, $I = I^O \cup I^P$ with $I^O \cap I^P = \emptyset$; we consider the game

$$\Gamma^{O,P} = \{I; X_1, \dots, X_n; (F_i^O)_{i \in I^O}, (F_i^P)_{i \in I^P}\}.$$

Remark 3.1: In the usual interpretation of a game, each agent is endowed with a payoff function $f_i : \Psi \to \mathbb{R}$; when the pure strategy profile ψ is played then every player knows that the outcome will be the payoff vector $(f_1(\psi), \dots, f_n(\psi))$. Being $E_i : X \to \mathbb{R}$ defined by

$$E_i(x) = \sum_{\psi \in \Psi} \left[\prod_{i \in I} x_i(\psi_i)\right] f_i(\psi) \quad \text{for all } x \in X$$

the expected payoff of player i, the choice of a mixed strategy profile x implies that each player has the same expectation on the outcomes of the game given by the expected payoff vector $(E_1(x), \dots, E_n(x))$ since, for each player, beliefs about what other players will do are correct.

3.2 Equilibria

Aim of this subsection is to provide an existence result for the equilibria of the strategic form game $\Gamma^{O,P}$. This result depend on the properties of the beliefs correspondences, so the analysis starts by recalling well known definitions and results on set-valued maps which we use below.

Preliminaries on set-valued maps

Following Aubin and Frankowska (1989)[3], recall that if Z and Y are two metric spaces and $F : Z \rightsquigarrow Y$ a set-valued map, then

$$\begin{array}{rc} i) & \operatorname*{Lim\,inf}_{z\to z'} F(z) = \{y \in Y \mid \lim_{z\to z'} d(y, F(z)) = 0\} \\ ii) & \operatorname*{Lim\,sup}_{z\to z'} F(z) = \left\{y \in Y \mid \liminf_{z\to z'} d(y, F(z)) = 0\right\} \\ iii) & \operatorname*{Lim\,inf}_{z\to z'} F(z) \subseteq F(z') \subseteq \operatorname*{Lim\,sup}_{z\to z'} F(z). \end{array}$$

Moreover

Definition 3.2: Given the set valued map $F : Z \rightsquigarrow Y$, then

i) F is *lower semicontinuous* in z' if $F(z') \subseteq \operatorname*{Lim\,inf}_{z\to z'} F(z)$; that is, F is lower semicontinuous in z' if for every $y \in F(z')$ and every sequence $(z_\nu)_{\nu\in\mathbb{N}}$ converging to z' there exists a sequence $(y_\nu)_{\nu\in\mathbb{N}}$ converging to y such that $y_\nu \in F(z_\nu)$ for every $\nu \in \mathbb{N}$. Moreover, F is lower semicontinuous in Z if it is lower semicontinuous for all z' in Z.

ii) F is *closed* in z' if $\operatorname*{Lim\,sup}_{z\to z'} F(z) \subseteq F(z')$; that is, F is closed in z' if for every sequence $(z_\nu)_{\nu\in\mathbb{N}}$ converging to z' and every sequence $(y_\nu)_{\nu\in\mathbb{N}}$ converging to y such that $y_\nu \in F(z_\nu)$ for every $\nu \in \mathbb{N}$, it follows that $y \in F(z')$. Moreover, F is closed in Z if it is closed for all z' in Z.

iii) F is *upper semicontinuous* in z' if for every open set U such that $F(z') \subseteq U$ there exists $\eta > 0$ such that $F(z) \subseteq U$ for all $z \in B_Z(z', \eta) = \{\zeta \in Z \mid ; ||\zeta - z'|| < \eta\}$. Moreover, F is upper semicontinuous in Z if it is upper semicontinuous for all z' in Z.

iv) F is *continuous* (in the sense of Painlevé-Kuratowski) in z' if it is lower semicontinuous and upper semicontinuous in z'.

The following proposition is very useful in this work.

[3] All the definitions and the propositions we use, together with the proofs can be found in this book.

Proposition 3.3: *Assume that Z is closed, Y is compact and the set-valued map $F : Z \rightsquigarrow Y$ has closed values, i.e. $F(z)$ is closed for all $z \in Z$. Then, F is upper semicontinuous in $z \in Z$ if and only if F is closed in z*[4].

Recall also that

Definition 3.4: Let Z a convex set, then the set valued map $F : Z \rightsquigarrow Y$ is a said to be *concave* if

$$tF(\overline{z}) + (1-t)F(\widehat{z}) \subseteq F(t\overline{z} + (1-t)\widehat{z}) \quad \forall\, \overline{z}, \widehat{z} \in Z,\ \forall\, t \in [0,1] \tag{3}$$

while it is *convex* if

$$F(t\overline{z} + (1-t)\widehat{z}) \subseteq tF(\overline{z}) + (1-t)F(\widehat{z}) \quad \forall\, \overline{z}, \widehat{z} \in Z,\ \forall\, t \in [0,1] \tag{4}$$

Existence theorems

Denote with $BR_i^O : X_{-i} \rightsquigarrow X_i$ and with $BR_i^P : X_{-i} \rightsquigarrow X_i$ the set valued maps defined by

$$BR_i^O(x_{-i}) = \{\overline{x}_i \in X_i \mid F_i^O(\overline{x}_i, x_{-i}) = \max_{x_i \in X_i} F_i^O(x_i, x_{-i})\} \quad \forall x_{-i} \in X_{-i} \tag{5}$$

$$BR_i^P(x_{-i}) = \{\overline{x}_i \in X_i \mid F_i^P(\overline{x}_i, x_{-i}) = \max_{x_i \in X_i} F_i^P(x_i, x_{-i})\} \quad \forall x_{-i} \in X_{-i} \tag{6}$$

and recall that

Definition 3.5: If $D \subseteq \mathbb{R}^n$ is a convex set and $g : D \to \mathbb{R}$ then g is said to be quasi concave if for every $x, y \in D$ and $\alpha \in]0,1[$ it results that $f(\alpha x + (1-\alpha)y) \geq min\{f(x), f(y)\}$.

Then

Proposition 3.6: *Assume that $\mathcal{B}_i$ is continuous with not empty compact and convex images for every $x \in X$ and $\mathcal{B}_i(\cdot, x_{-i})$ is concave in X_i for every $x_{-i} \in X_{-i}$, that is*

$$t\mathcal{B}_i(\overline{x}_i, x_{-i}) + (1-t)\mathcal{B}_i(\widehat{x}_i, x_{-i}) \subseteq \mathcal{B}_i(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i}) \quad \forall x_{-i} \in X_{-i}. \tag{7}$$

Then F_i^O is continuous in X and $F_i^O(\cdot, x_{-i})$ is quasi concave in X_i for every $x_{-i} \in X_{-i}$. Therefore, the set valued map BR_i^O is upper semicontinuous with not empty, closed and convex images for every $x_{-i} \in X_{-i}$.

Proof. From the assumptions each function F_i^O is well defined and by applying the Berge maximum theorem (see also Aubin and Frankowska (1990), Border (1985)), F_i^O is continuous on the compact set X and BR_i^O is upper semicontinuous with not empty and closed images for every $x_{-i} \in X_{-i}$.

Now we prove that $F_i^O(\cdot, x_{-i})$ is quasi concave for all $x_{-i} \in X_{-i}$. Let $\overline{x}_i$ and $\widehat{x}_h$ in X_i, and consider $\overline{\varrho} \in \mathcal{B}_i(\overline{x}_i, x_{-i})$ and $\widehat{\varrho} \in \mathcal{B}_i(\widehat{x}_i, x_{-i})$ such that $F_i^O(\overline{x}_i, x_{-i}) = E(\overline{\varrho})$ and

[4] Every set valued map in this paper satisfies the assumptions of this proposition. Hence upper semicontinuity and closeness coincide in this work.

$F_i^O(\widehat{x}_h, x_{-i}) = E(\widehat{\varrho})$. If $t \in]0,1[$ then from the assumptions it follows that $t\overline{\varrho} + (1-t)\widehat{\varrho} \in \mathcal{B}_i(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i})$. Since

$$E(t\overline{\varrho} + (1-t)\widehat{\varrho}) = tE(\overline{\varrho}) + (1-t)E(\widehat{\varrho}) \geq \min\{E(\overline{\varrho}), E((\widehat{\varrho})\}$$

then

$$F_i^O(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i}) = \max_{\varrho \in \mathcal{B}_i(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i})} E(\varrho) \geq \min\{E(\overline{\varrho}), E(\widehat{\varrho})\} \tag{8}$$

$$= \min\{F_i^O(\overline{x}_i, x_{-i}), F_i^O(\widehat{x}_i, x_{-i})\}. \tag{9}$$

Therefore $F_i^O(\cdot, x_{-i})$ is quasi concave for every $x_{-i} \in X_{-i}$. Then it follows that BR_i^O has convex images for every $x_{-i} \in X_{-i}$. □

PROPOSITION 3.7: *Assume that $\mathcal{B}_i$ is continuous with not empty compact and convex images for every $x \in X$ and $\mathcal{B}_i(\cdot, x_{-i})$ is convex in X_i for every $x_{-i} \in X_{-i}$, that is*

$$t\mathcal{B}_i(\overline{x}_i, x_{-i}) + (1-t)\mathcal{B}_i(\widehat{x}_i, x_{-i}) \supseteq \mathcal{B}_i(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i}) \quad \forall x_{-i} \in X_{-i}. \tag{10}$$

Then F_i^P is continuous in X and $F_i^P(\cdot, x_{-i})$ is quasi concave in X_i for every $x_{-i} \in X_{-i}$. Therefore, the set valued map BR_i^P is upper semicontinuous with not empty, closed and convex images for every $x_{-i} \in X_{-i}$.

Proof. We follow the same steps of the proof of the previous Proposition. From the assumptions each function F_i^P is well defined and by applying the Berge maximum theorem (see also Aubin and Frankowska (1990), Border (1985)), F_i^P is continuous on the compact set X and BR_i^P is upper semicontinuous with not empty and closed images for every $x_{-i} \in X_{-i}$.

Now we prove that $F_i^P(\cdot, x_{-i})$ is quasi concave for all $x_{-i} \in \prod_{j \neq i} X_j$. Let $\overline{x}_i$ and $\widehat{x}_h$ be in X_i and $t \in [0,1]$. Let $F_i^P(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i}) = E(\varrho^*)$ with $\varrho^* \in \mathcal{B}_i(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i})$. Then, in light of the assumptions, there exist $\overline{\varrho} \in \mathcal{B}_i(\overline{x}_i, x_{-i})$ and $\widehat{\varrho} \in \mathcal{B}_i(\widehat{x}_i, x_{-i})$ such that $\varrho^* = t\overline{\varrho} + (1-t)\widehat{\varrho}$; therefore

$$E(\varrho^*) = tE(\overline{\varrho}) + (1-t)E(\widehat{\varrho}) \geq$$

$$t\left[\min_{\varrho \in \mathcal{B}_i(\overline{x}_i, x_{-i})} E(\varrho)\right] + (1-t)\left[\min_{\varrho \in \mathcal{B}_i(\widetilde{x}_i, x_{-i})} E(\varrho)\right] = tF_i^P(\widetilde{x}_i, x_{-i}) + (1-t)F_i^P(\widetilde{x}_i, x_{-i})$$

and $F_i^P(\cdot, x_{-i})$ is quasi concave for all x_{-i}. Then it follows that BR_i^P has convex images for every $x_{-i} \in X_{-i}$. □

From the Nash equilibrium existence theorems (see for instance Rosen (1965), it immediately follows that

THEOREM 3.8: *Assume that for every player i, $\mathcal{B}_i$ is continuous with not empty compact and convex images for every $x \in X$. If, for every player $i \in I^O$, $\mathcal{B}_i(\cdot, x_{-i})$ is concave in X_i for every $x_{-i} \in X_{-i}$ and, for every player $i \in I^P$, $\mathcal{B}_i(\cdot, x_{-i})$ is convex in X_i for every $x_{-i} \in X_{-i}$, then, the game $\Gamma^{O,P}$ has at least an equilibrium.*

4 Examples of Beliefs Correspondences

In this section we propose two different kind of beliefs correspondence and investigate whether they satisfy the requirements of the existence results. In the first model, beliefs to a player over his opponents' strategy profiles are given by the correlated equilibria of the game between the opponents once they have observed player's action. In the second model, beliefs are determined by contingent assignment of ambiguity levels on a family of disjoint events on the set of outcomes.

4.1 Beliefs given by correlated equilibria

The idea underlying the kind of beliefs correspondence investigated in this subsection is that a player believes his opponents will observe his action before choosing their strategies. Assuming that the player believes that his opponents will react optimally and in a correlated way once observed his action then his beliefs are given by the correlated equilibria of the game between the opponents given the player's action.

For a given player i, denote with $J_i = I \setminus \{i\}$, then, for every pure strategy $\psi_i \in \Psi_i$, consider the game

$$G(\psi_i) = \{J_i; (\Psi_j)_{j\in J_i};\ (g_j^{\psi_i})_{j\in J_i}\}$$

where Ψ_j is the pure strategy set of player j and the payoff function $g_j^{\psi_i} : \Psi_{-i} \to \mathbb{R}$ is the payoff function of player j which corresponds to the payoff of player j in the game Γ when player i chooses ψ_i, i.e., $g_j^{\psi_i}((\widehat{\psi}_h)_{h\in J_i}) = f_j(\widehat{\psi}_1, .., \widehat{\psi}_{i-1}, \psi_i, \widehat{\psi}_{i+1}, .., \widehat{\psi}_n)$ for every $\widehat{\psi}_{-i} \in \Psi_{-i}$. Now we recall the definition of correlated equilibrium (Aumann (1974, 1987)) for the game $G(\psi_i)$. To this purpose we denote with $\psi_{-(i,j)} = (\psi_h)_{h\in I\setminus\{i,j\}}$ and with $\Psi_{-(i,j)} = (\Psi_h)_{h\in I\setminus\{i,j\}}$

Definition 4.1 (Aumann): A probability distribution μ on Ψ_{-i} is a correlated equilibrium for the game $G(\psi_i)$ if for every player $j \in J_i$ and every pure strategy $\bar{\psi}_j \in \Psi_j$,

$$\sum_{\psi_{-(i,j)}\in\Psi_{-(i,j)}} \mu\left(\psi_{-(i,j)}|\bar{\psi}_j\right) g_j^{\psi_i}(\bar{\psi}_j, \psi_{-(i,j)}) \geq \sum_{\psi_{-(i,j)}\in\Psi_{-(i,j)}} \mu\left(\psi_{-(i,j)}|\bar{\psi}_j\right) g_j^{\psi_i}(\psi_j, \psi_{-(i,j)}) \quad \forall\psi_j \in \Psi_j. \tag{11}$$

where

$$\mu\left(\psi_{-(i,j)}|\bar{\psi}_j\right) = \frac{\mu\left(\bar{\psi}_j, \psi_{-(i,j)}\right)}{\sum_{\widetilde{\psi}_{-(i,j)}\in\Psi_{-(i,j)}} \mu\left(\bar{\psi}_j, \widetilde{\psi}_{-(i,j)}\right)}$$

if $\sum_{\widetilde{\psi}_{-(i,j)}\in\Psi_{-(i,j)}} \mu\left(\bar{\psi}_j, \widetilde{\psi}_{-(i,j)}\right) \neq 0$ and $\mu\left(\psi_{-(i,j)}|\bar{\psi}_j\right) = 0$ otherwise. Therefore, $\mu\left(\psi_{-(i,j)}|\bar{\psi}_j\right)$ is player j's conditional probability of $\psi_{-(i,j)}$ given $\bar{\psi}_j$; that is, the probability that player j assigns to the strategy profile $\psi_{-(i,j)}$ of his opponents in J_i once the mediator has communicated player j to play $\bar{\psi}_j$. In other words, μ is a correlated equilibrium if the expected payoff from playing the recommended strategy is no worse than playing any other strategy.

Denote with $\mathcal{C}_i(\psi_i)$ the set of correlated equilibria of the game $G(\psi_i)$. Then, assume that $\Omega = \{f(\psi) \mid \psi \in \Psi\}$ and let $\mathcal{B}_i : X \rightsquigarrow \mathcal{P}$ be the set-valued map defined by

$$\mathcal{B}_i(x) = \sum_{\psi_i \in \Psi_i} x_i(\psi_i)\mathcal{C}_i(\psi_i) \quad \forall x \in X \tag{12}$$

that is, for every $x \in X$,

$$\varrho \in \mathcal{B}_i(x) \iff \forall \psi_i \in \Psi_i \;\; \exists \mu_{\psi_i} \in \mathcal{C}_i(\psi_i) \text{ such that } \varrho = \sum_{\psi_i \in \Psi_i} x_i(\psi_i)\mu_{\psi_i}.$$

We emphasize that this set valued map means that player i believes that the other players will observe his play and then they will react by choosing a correlated equilibrium.

Lemma 4.2: *The set valued map $\mathcal{B}_i$ defined in (12) is continuous with not empty convex and closed values for every $x \in X$. Moreover $\mathcal{B}_i(\cdot, x_{-i})$ is concave and convex for every $x_{-i} \in X_{-i}$.*

Proof. For every ψ_i the set $\mathcal{C}_i(\psi_i)$ of correlated equilibria of the game $G(\psi_i)$ is not empty closed and convex (see Aumann (1974, 1987)). Let ϱ' and ϱ'' in $\mathcal{B}_i(x)$. Hence, for every strategy ψ_i there exist correlated equilibria μ'_{ψ_i} and μ''_{ψ_i} of the game $G(\psi_i)$ such that

$$\varrho' = \sum_{\psi_i \in \Psi_i} x_i(\psi_i)\mu'_{\psi_i}, \; \varrho'' = \sum_{\psi_i \in \Psi_i} x_i(\psi_i)\mu''_{\psi_i}$$

hence

$$\alpha\varrho' + (1-\alpha)\varrho'' = \sum_{\psi_i \in \Psi_i} x_i(\psi_i)[\alpha\mu'_{\psi_i} + (1-\alpha)\mu''_{\psi_i}]$$

for every $\alpha \in]0,1[$ which implies that $\mathcal{B}_i(x)$ is convex for every $x \in X$.

Now, we show that the set valued map $\mathcal{B}_i$ is closed for every $x \in X$. In fact, given a point $x \in X$, let $(x_\nu)_{\nu\in\mathbb{N}}$ be a sequence in X converging to x and $(\varrho_\nu)_{\nu\in\mathbb{N}}$ be a sequence converging to ϱ with in $\varrho_\nu \in \mathcal{B}_i(x_\nu)$ for every $\nu \in \mathbb{N}$. Denote with $x_\nu = (x_{1,\nu}, \dots, x_{n,\nu})$, then $\varrho_\nu = \sum_{\psi_i \in \Psi_i} x_{i,\nu}(\psi_i)\mu^\nu_{\psi_i}$ with $\mu^\nu_{\psi_i} \in \mathcal{C}_i(\psi_i)$ for every $\psi_i \in \Psi_i$ and every $\nu \in \mathbb{N}$. Since $\mu^\nu_{\psi_i} \to \mu_{\psi_i}$ and $\mathcal{C}_i(\psi_i)$ is closed then $\mu_{\psi_i} \in \mathcal{C}_i(\psi_i)$ for every ψ_i and $\varrho \in \mathcal{B}_i(x)$. Therefore $\mathcal{B}_i$ is closed in x. Applying the previous arguments at the constant sequence $(x_\nu)_{\nu\in\mathbb{N}}$ with $x_\nu = x$ for every $\nu \in \mathbb{N}$, it follows that $\mathcal{B}_i(x)$ is also closed for every $x \in X$. Being $\mathcal{P}$ compact and X closed it follows that $\mathcal{B}_i$ is upper semicontinuous in X.

$\mathcal{B}_i$ is also lower semicontinuous in every $x \in X$. In fact, given a point $x \in X$, consider $\varrho \in \mathcal{B}_i(x)$ and a sequence $(x_\nu)_{\nu\in\mathbb{N}}$ in X converging to x. Since $\varrho = \sum_{\psi_i \in \Psi_i} x_i(\psi_i)\mu_{\psi_i}$ with $\mu_{\psi_i} \in \mathcal{C}_i(\psi_i)$ for every $\psi_i \in \Psi_i$, consider $\varrho_\nu = \sum_{\psi_i \in \Psi_i} x_{i,\nu}(\psi_i)\mu_{\psi_i}$ with $\mu_{\psi_i} \in \mathcal{C}_i(\psi_i)$ for every $\psi_i \in \Psi_i$ and every $\nu \in \mathbb{N}$. It immediately follows that $\varrho_\nu \to \varrho$ as $\nu \to \infty$ which implies that $\mathcal{B}_i$ is lower semicontinuous in x.

Finally, since

$$\mathcal{B}_i(x'_i, x_{-i}) = \sum_{\psi_i \in \Psi_i} x'_i(\psi_i)\mathcal{C}_i(\psi_i), \quad \text{and} \quad \mathcal{B}_i(x''_i, x_{-i}) = \sum_{\psi_i \in \Psi_i} x''_i(\psi_i)\mathcal{C}_i(\psi_i)$$

then, for every $\alpha \in]0,1[$ it follows that

$$\mathcal{B}_i(\alpha x'_i + (1-\alpha)x''_i, x_{-i}) = \sum_{\psi_i \in \Psi_i} [\alpha x'_i + (1-\alpha)x''_i]\mathcal{C}_i(\psi_i) =$$

$$\alpha \left[\sum_{\psi_i \in \Psi_i} x'_i(\psi_i)\mathcal{C}_i(\psi_i) \right] + (1-\alpha) \left[\sum_{\psi_i \in \Psi_i} x''_i(\psi_i)\mathcal{C}_i(\psi_i) \right] = \alpha \mathcal{B}_i(x'_i, x_{-i}) + (1-\alpha)\mathcal{B}_i(x''_i, x_{-i})$$

which implies that $\mathcal{B}_i(\cdot, x_{-i})$ is concave and convex for every $x_{-i} \in X_{-i}$. □

Remark 4.3: Note that an analogous construction of a beliefs correspondence involving the set of Nash equilibria $N_i(\psi_i)$ of the game $G(\psi_i)$ instead of the set of correlated equilibria $\mathcal{C}_i(\psi_i)$ does not guarantee that all the properties required for the existence of the equilibria of the game $\Gamma^{O,P}$ are satisfied. More precisely, since the set of Nash equilibria is not always convex[5], each $N_i(\psi_i)$ is not necessarily convex and hence the set-valued map $\mathcal{N}_i : X \rightsquigarrow \mathcal{P}$ defined by

$$\mathcal{N}_i(x) = \sum_{\psi_i \in \Psi_i} x_i(\psi_i) N_i(\psi_i) \quad \forall x \in X$$

does not have convex values in general.

Remark 4.4: Given the beliefs correspondence $\mathcal{B}_i(\cdot)$ defined by (12), we can consider a generalization of the beliefs correspondence considered in the stung hunt game: player i has a fixed set of ambiguous beliefs D_i but he believes that with probability ε his opponents will observe his action and will react optimally in a correlated way. In fact, let D_i be a convex and closed set of probability distributions over Ψ_{-i} and let $\varepsilon > 0$, it is possible to consider the beliefs correspondence $\mathcal{D}_i : X \rightsquigarrow \mathcal{P}$ defined by

$$\mathcal{D}_i(x) = \sum_{\psi_i \in \Psi_i} x_i(\psi_i)[(1-\varepsilon)D_i + \varepsilon \mathcal{C}_i(\psi_i)] \quad \forall x \in X.$$

Following the same steps in the proof of Lemma 4.2, it results that $\mathcal{D}_i$ is continuous with not empty convex and closed values for every $x \in X$ and $\mathcal{D}_i(\cdot, x_{-i})$ is concave and convex for every $x_{-i} \in X_{-i}$.

4.2 Ambiguity Levels

We consider now the case in which the set valued maps $\mathcal{B}_i$ are determined by an assignment of upper levels for the probabilities of a family of disjoint events in Ω[6].

In particular, given a player i, we assume there exist a family of subsets of Ω, denoted with $\mathscr{F}_i$ such that

$$\cup_{\mathcal{F} \in \mathscr{F}_i} \mathcal{F} = \Omega; \quad \mathcal{E}, \mathcal{F} \in \mathscr{F}_i \text{ and } \mathcal{E} \neq \mathcal{F} \implies \mathcal{E} \cap \mathcal{F} = \emptyset$$

[5]Indeed, examples show that usually the set of Nash equilibria is not convex.

[6]In De Marco and Romaniello (2010;a,b) ambiguity derives from assignments of probabilities on disjoint events

and a family of functions $(g_{i,\mathcal{F}})_{\mathcal{F}\in\mathscr{F}_i}$ such that $g_{i,\mathcal{F}} : X \to [0,1]$. Each $g_{i,\mathcal{F}}(x)$ gives the maximal probability of the event $\mathcal{F}$ to player i given the mixed strategy profile x. Hence, the following consistency condition should be satisfied

$$\sum_{\mathcal{F}\in\mathscr{F}_i} g_{i,\mathcal{F}}(x) \geq 1 \quad \forall x \in X.$$

The set-valued map of feasible beliefs to player i $\mathcal{B}_i$ is therefore given by

$$\varrho \in \mathcal{B}_i(x) \iff \begin{cases} \sum_{\xi\in\mathcal{F}} \varrho(\xi) \leq g_{i,\mathcal{F}}(x) & \forall \mathcal{F} \in \mathscr{F}_i \\ \text{and} \\ \varrho \in \mathcal{P} \end{cases} \tag{13}$$

So we have that

THEOREM 4.5: *If the function g_i is continuous on X then the set valued map $\mathcal{B}_i$ defined by (13) is upper and lower semicontinuous with not empty compact and closed images.*

Proof. By definition $\mathcal{B}_i(x)$ is compact and convex. Now we prove that the graph of $\mathcal{B}_i$ that is

$$Graph(\mathcal{B}_i) = \{(x,\varrho) \in X \times \mathcal{P} \mid \varrho \in \mathcal{B}_i(x)\} \tag{14}$$

is closed. In fact, let $\{(x_\nu, \varrho_\nu)\}_\nu$ be a sequence converging to (x,ϱ) with $(x_\nu, \varrho_\nu) \in Graph(\mathcal{B}_i)$ for all ν. Obviously, from compactness, $x \in X$ and $\varrho \in \mathcal{P}$. Being $\sum_{\xi\in\mathcal{F}} \varrho_\nu(\xi) \leq g_{i,\mathcal{F}}(x_\nu)$ for all $\nu \in \mathbb{N}$, from continuity of $g_{i,\mathcal{F}}$ for all $\mathcal{F} \in \mathscr{F}$ it follows that $\sum_{\xi\in\mathcal{F}} \varrho(\xi) \leq g_{i,\mathcal{F}}(x)$ for all $\mathcal{F} \in \mathscr{F}$ and hence $\varrho \in \mathcal{B}_i(x)$. Hence $\mathcal{B}_i$ has closed graph and compact images so it is upper semicontinuous.

Now we show that $\mathcal{B}_i$ is lower semicontinuous in X, that is, for every $x \in X$, $\varrho \in \mathcal{B}_i(x)$ and $x_\nu \to x$, there exists $\varrho_\nu \to \varrho$ with $\varrho_\nu \in \mathcal{B}_i(x_\nu)$ for all $\nu \in \mathbb{N}$. In fact, given the distribution ϱ and for every $\xi \in \Omega$ let $\mathcal{F}_\xi$ the unique element of $\mathscr{F}_i$ containing ξ. Denote with $\mathbb{P}_\varrho(\mathcal{F}_\xi)$ the probability of the event $\mathcal{F}_\xi$ given the distribution ϱ (i.e. $\mathbb{P}_\varrho(\mathcal{F}_\xi) = \sum_{\widehat{\xi}\in\mathcal{F}_\xi} \varrho(\widehat{\xi})$) and with $\mathbb{P}_\varrho(\xi|\mathcal{F}_\xi)$ the conditional probability of ξ given $\mathcal{F}_\xi$ (set $\mathbb{P}_\varrho(\xi|\mathcal{F}_\xi) = 0$ if $\mathbb{P}_\varrho(\mathcal{F}_\xi)$), then we have

$$\varrho(\xi) = \mathbb{P}_\varrho(\mathcal{F}_\xi)\left[\mathbb{P}_\varrho(\xi|\mathcal{F}_\xi)\right].$$

Denote with $\mathscr{F}_i^0 = \{\mathcal{F} \in \mathscr{F}_i \mid \mathbb{P}_\varrho(\mathcal{F}) = 0\}$, $\mathscr{F}_i^1 = \{\mathcal{F} \in \mathscr{F}_i \setminus \mathscr{F}_i^0 \mid \mathbb{P}_\varrho(\mathcal{F}) = g_{i,\mathcal{F}}(x)\}$ and with $\mathscr{F}_i^2 = \{\mathcal{F} \in \mathscr{F}_i \setminus \mathscr{F}_i^0 \mid \mathbb{P}_\varrho(\mathcal{F}) < g_{i,\mathcal{F}}(x)\}$. From the total probability theorem (i.e. $\sum_{\mathcal{F}\in\mathscr{F}} \mathbb{P}_\varrho(\mathcal{F}) = 1$) it follows that the vector $(\mathbb{P}_\varrho(\mathcal{F}))_{\mathcal{F}\in\mathscr{F}_i^2}$ is a solution of the following equation

$$\sum_{\mathcal{F}\in\mathscr{F}_i^2} Y_{\mathcal{F}} = H(x) \tag{15}$$

where $H(x) = 1 - \sum_{\mathcal{F}\in\mathscr{F}_i^1} g_{i,\mathcal{F}}(x)$. From the continuity of each function $g_{i,\mathcal{F}}$ it immediately follows that the set valued map of the solutions of the system (15), $x \rightsquigarrow S(x)$, is lower semicontinuous in X. This implies that, given the sequence $x_\nu \to x$, there exists a sequence $(Y^\nu)_\nu$ (where $Y^\nu = (Y^\nu_{\mathcal{F}})_{\mathcal{F}\in\mathscr{F}_i^2})$) which converges to $(\mathbb{P}_\varrho(\mathcal{F}))_{\mathcal{F}\in\mathscr{F}_i^2}$. Hence, define for every $\nu \in \mathbb{N}$

$$\mathbb{P}^\nu_\varrho(\mathcal{F}) = \begin{cases} 0 & \text{if } \mathcal{F}\in\mathscr{F}_i^0 \\ g_{i,\mathcal{F}}(x_\nu) & \text{if } \mathcal{F}\in\mathscr{F}_i^1 \\ Y^\nu_{\mathcal{F}} & \text{if } \mathcal{F}\in\mathscr{F}_i^2 \end{cases}$$

Note that, by construction, $\sum_{\mathcal{F}\in\mathscr{F}} \mathbb{P}^\nu_\varrho(\mathcal{F}) = 1$ for every $\nu\in\mathbb{N}$. Define also

$$\varrho_\nu(\xi) = \mathbb{P}^\nu_\varrho(\mathcal{F}_\xi)\left[\mathbb{P}_\varrho(\xi|\mathcal{F}_\xi)\right] \quad \forall\xi\in X,$$

where $\mathcal{F}_\xi$ is the unique set in $\mathscr{F}_i$ containing ξ; then, from the continuity of each $g_{i,\mathcal{F}}$, it follows that $\varrho_\nu(\xi)\to\varrho(\xi)$ as $x_\nu\to x$. Moreover, it also follows that $\sum_{\xi\in X}\varrho_\nu(\xi) = 1$; in fact

$$\sum_{\xi\in X}\varrho_\nu(\xi) = \sum_{\mathcal{F}\in\mathscr{F}_i\setminus\mathscr{F}_i^0}\left[\sum_{\xi\in\mathcal{F}}\varrho_\nu(\xi)\right] = \sum_{\mathcal{F}\in\mathscr{F}_i\setminus\mathscr{F}_i^0}\sum_{\xi\in\mathcal{F}}\mathbb{P}^\nu_\varrho(\mathcal{F})\left[\mathbb{P}_\varrho(\xi|\mathcal{F})\right] =$$

$$\sum_{\mathcal{F}\in\mathscr{F}_i\setminus\mathscr{F}_i^0}\mathbb{P}^\nu_\varrho(\mathcal{F})\left[\sum_{\xi\in\mathcal{F}}\mathbb{P}_\varrho(\xi|\mathcal{F})\right] = 1$$

since, for every $\mathcal{F}\in\mathscr{F}_i\setminus\mathscr{F}_i^0$, it results that $\sum_{\xi\in\mathcal{F}}\mathbb{P}_\varrho(\xi|\mathcal{F}) = 1$. So $(\varrho_\nu(\xi))_{\xi\in\Omega}$ is a probability distribution on Ω.

Finally, $(\varrho_\nu(\xi))_{\xi\in\Omega}$ satisfies the consistency constraints in (13). In fact it immediately follows that

$$\sum_{\xi\in\mathcal{F}}\varrho_\nu(\xi)\le g_{i,\mathcal{F}}(x_\nu) \quad \forall\mathcal{F}\in\mathscr{F}_i^1\cup\mathscr{F}_i^0$$

Now, let $\mathcal{F}\in\mathscr{F}_i^2$, then

$$\sum_{\xi\in\mathcal{F}}\varrho_\nu(\xi) = \sum_{\xi\in\mathcal{F}} Y^\nu_{\mathcal{F}}\left[\mathbb{P}_\varrho(\xi|\mathcal{F})\right] = Y^\nu_{\mathcal{F}}\left[\sum_{\xi\in\mathcal{F}}\mathbb{P}_\varrho(\xi|\mathcal{F})\right] = Y^\nu_{\mathcal{F}}.$$

Since

$$\lim_{\nu\to\infty} Y^\nu_{\mathcal{F}} = \mathbb{P}_\varrho(\mathcal{F}) < g_{i,\mathcal{F}}(x) = \lim_{\nu\to\infty} g_{i,\mathcal{F}}(x_\nu)$$

then there exist $\widehat{\nu}$ such that, for all $\nu\ge\widehat{\nu}$ it results that

$$\sum_{\xi\in\mathcal{F}}\varrho_\nu(\xi) = Y^\nu_{\mathcal{F}} < g_{i,\mathcal{F}}(x_\nu).$$

So, by redefining ϱ_ν for $\nu<\widehat{\nu}$, it follows that $\varrho_\nu =\in \mathcal{B}_i(x_\nu)$ for all $\nu\in\mathbb{N}$ and the assertion follows. □

THEOREM 4.6: *If the function $g_i(\cdot, x_{-i})$ is concave in X_i for every $x_{-i} \in X_{-i}$ then the set-valued map $\mathcal{B}_i(\cdot, x_{-i})$ is concave for every $x_{-i} \in X_{-i}$, that is*

$$\mathcal{B}_i(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i}) \supseteq \mathcal{B}_i(\overline{x}_i, x_{-i}) + (1-t)\mathcal{B}_i(\widehat{x}_i, x_{-i}) \quad \forall x_{-i} \in X_{-i} \tag{16}$$

Proof. Let $\overline{\varrho} \in \mathcal{B}_i(\overline{x}_i, x_{-i})$ and $\tilde{\varrho} \in \mathcal{B}_i(\tilde{x}_i, x_{-i})$ and $t \in]0,1[$. Obviously $t\overline{\varrho} + (1-t)\widehat{\varrho} \in \mathcal{P}$; moroever, by defition follows that

$$\sum_{\xi \in \mathcal{F}} \overline{\varrho}(\xi) \leq g_{i,\mathcal{F}}(\overline{x}_i, x_{-i}) \quad \text{and} \quad \sum_{\xi \in \mathcal{F}} \widehat{\varrho}(\xi) \leq g_{i,\mathcal{F}}(\widehat{x}_i, x_{-i}) \quad \forall \mathcal{F} \in \mathscr{F}_i.$$

Hence

$$\sum_{\xi \in \mathcal{F}} [t\overline{\varrho}(\xi) + (1-t)\widehat{\varrho}(\xi)] \leq t g_{i,\mathcal{F}}(\overline{x}_i, x_{-i}) + (1-t) g_{i,\mathcal{F}}(\widehat{x}_i, x_{-i}) \leq g_{i,\mathcal{F}}(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i}) \quad \forall \mathcal{F} \in \mathscr{F}_i,$$

which implies that $t\overline{\varrho} + (1-t)\widehat{\varrho} \in \mathcal{B}_i(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i})$ and

$$t\mathcal{B}_i(\overline{x}_i, x_{-i}) + (1-t)\mathcal{B}_i(\widehat{x}_i, x_{-i}) \subseteq \mathcal{B}_i(t\overline{x}_i + (1-t)\widehat{x}_i, x_{-i}) \quad \forall x_{-i} \in X_{-i}.$$

□

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