# Constant-utility paths in a resource-based economy

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**Abstract**

This paper analyzes a social planner's solution in a resource-based economy under a constant-utility criterion. The utility function includes social progress in a multiplicative form. The resulting paths of consumption include the patterns of growth that are conventionally used in the literature. This approach extends conventional link between the utilitarian criterion and the maximin for the cases with finite elasticity of marginal utility. The closed form solutions, derived for the Dasgupta-Heal-Solow-Stiglitz (DHSS) model, include the result of Solow (1974) and Hartwick (1977) as a specific case. The approach is applied to an example of a distorted resource-extracting economy under the requirement for smoothness of the paths with respect to historical data.

*Key words:*
essential nonrenewable resource; sustainable growth; geometrically weighted percent; distorted economy
JEL : O13; O47; Q32; Q38

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## 1. Introduction

This paper introduces a modification of the Rawls's (1971) difference principle (maximin), and analyzes a social planner's solution under this modification in a resource-based economy. There is a vast literature devoted to the construction of the criteria of economic growth that do not use the discounting procedure.[1] The essential part of this literature is based on the maximin.

The conventional approach of using the maximin in the problems of intergenerational resource allocation is to maximize the level of per capita consumption $c$ or utility $u(c)$ of the least advantageous generation.[2] A negative consequence of this approach, which is referred to as "perpetuating poverty," attracts a major criticism of the maximin. There are studies that address this shortcoming by introducing a plausible generalization of the utility function. This generalization is based on the assumption – offered by Rawls – that the measure of utility should take into account not only the current level of consumption but also the social progress in the form of sympathy for future generations (Arrow, 1973; Dasgupta, 1974; Calvo, 1978; Leininger, 1985; Asheim, 1988; Long, 2007).

The idea of using sympathy for the future can be extended by introducing the consumption prehistory into the utility function. This extension is intuitive since the same person estimates the same level of current consumption in different ways, depending on whether this level resulted from gains or from losses.[3] A resulting model with the consumption prehistory can yield "Rawlsian growth,"

[1] The list of references and a review can be found, e.g., in Fleurbaey (2007).

[2] See, e.g., Solow (1974), Hartwick (1977), Leininger (1985), Asheim et al (2007), Alvarez–Cuadrado and Long (2009).

[3] There are findings supporting the idea that for estimating utility it is not enough to calculate a vector of measurable static indicators. Lecomber (1979) noted that "people become accustomed to rising living standards and are dissatisfied with static ones" (p. 33). Scanlon (1991) further mentioned that "we can ask ... how well a person's life is going and whether that person is ... better off than he or she was a year ago" (p. 18). There is also evidence that has "documented the claim that people are relatively insensitive to steady states, but highly sensitive to changes" and that "the main carriers of value are gains and losses rather than overall wealth" (Kahneman and Varey, 1991, p. 148).

even in a purely egoistic framework.[4]

The authors of the approach that introduses social progress into the utility function used an additively separable form of this function, justifying this form only by technical simplicity (Arrow, 1973, p. 326; Dasgupta, 1974, p. 409). However, it is interesting to analyze the properties of the constant-utility paths (a particular case of the maximin) under a multiplicative (Cobb-Douglas) form of the utility function. This analysis is interesting because the multiplicative form of utility includes commonly used utility measures as specific cases, and also because the resulting patterns of growth belong to the family of paths usually considered in the literature. Therefore, the problem with the constant-utility criterion can be an interesting theoretical tool since all the problems of growth theory that yield the "regular" patterns of growth (Groth et al., 2006) are equivalent (in the sense of the resulting paths) to this simple problem.

This paper offers the patterns of optimal investment and the resulting paths of nonrenewable resource extraction, capital, output, and consumption under the constant-utility criterion. The closed form solutions are derived for the Dasgupta-Heal-Solow-Stiglitz (DHSS) model (Dasgupta and Heal, 1974; Solow, 1974; Stiglitz, 1974).[5] The solution includes the Solow-Hartwick result (stagnation)[6] as a specific case and establishes the dependence between the value of

[4] See, e.g., Phelps and Riley (1978).

[5] There is mixed evidence about the elasticity of factor substitution between capital and resource including the results showing that this value is close to unity (Griffin and Gregory, 1976; Pindyck, 1979), which means that the use of the Cobb-Douglas technology is not implausible in this framework. However, plausibility is not the main reason for its use in this paper. As Asheim (2005) put it, "I do not claim that this model describes accurately ... production possibilities in the real world ... however, it is well-suited to illustrate how a small variation in the parameters ... may lead to very different consequences when combined with criteria for intergenerational justice" (p. 316).

[6] Solow (1974) showed that per capita consumption can be maintained constant over time in an economy with a limited nonrenewable resource, which is an input in the Cobb-Douglas production function. Hartwick (1977) showed that constant consumption in this model results from investing the resource rent into man-made capital.

the constant investment rate and the pattern of growth.

This approach is applied to a distorted economy under the requirement of the smoothness of paths with respect to historical data. The distortion results, for example, from the instantaneous increment in the resource reserve. The smoothness of paths results from endogenization of a preference parameter depending on the reserve and the economy's current state. These smooth paths can be used either as an independent solution or as transition paths to the new paths that are optimal with respect to the original preferences.

The paper is structured as follows: Section 2 introduces a version of the modified maximin; Section 3 derives an optimal investment rule in a resource-based economy and specifies it for the DHSS model; Section 4 analyzes the closed form solutions for the DHSS model; Sections 5 and 6 offer a smooth solution for a distorted economy. The conclusions are presented in Section 7.

## 2. A modified maximin: care for social progress

Assume that utility depends on social progress expressed both in the form of the sympathy to the future generations and in consumption prehistory. Then, the Arrow - Dasgupta approach, in a discrete setting, implies that the utility function takes the form

$$\widetilde{u}(c(t)) = \sum_{i\geqslant 1} \sigma^i (c_t - c_{t-i}) + c_t + \sum_{i\geqslant 1} \sigma^i c_{t+i} = c_t \sum_{i\geqslant 0} \sigma^i + \sum_{i\geqslant 0} \sigma^i (c_{t+i} - c_{t-i}),$$

where $\sigma \in (0,1)$ is the discount factor, and the term $\sum_{i\geqslant 0} \sigma^i (c_{t+i} - c_{t-i})$ is a weighted average of the slopes of the consumption path. Then, there exists such a value of $\gamma$ that $\widetilde{u}(c(t)) = \widetilde{C} u(c_t, \dot{c}_t)$, where $\widetilde{C}$ is a constant and $u(c_t, \dot{c}_t) = c_t + \gamma \dot{c}_t$,[7] where $\dot{c} = dc/dt$.

Since the additively separable form was introduced only for simplicity, this paper uses below a multiplicative utility, which, in the general case, takes the form: $u(c, \dot{c}) = sgn(\dot{c}) \cdot |\dot{c}|^\gamma c^\mu$. Following Solow (1974), the maximin applied to

[7] This form of utility function was used by Long (2007).

$u(c,\dot{c})$ implies that already this combination, not consumption per se, should be kept constant over time.[8] Assume for simplicity that $\mu = 1-\gamma$ and $\dot{c} > 0$. Then, the constant-utility criterion with *the growth weight* $\gamma$ is

$$\dot{c}^{\gamma} c^{1-\gamma} = \overline{u} = const, \tag{1}$$

yielding the pattern of "regular growth" (Groth et al., 2006, p. 4):

$$c(t) = c_0 (1 + \varphi t)^{\gamma}, \tag{2}$$

where $\varphi := (\dot{c}_0/c_0)/\gamma$. The pattern (2) is stagnation when $\gamma = 0$ or one of the following forms of growth: quasi-arithmetic (or sub-arithmetic) when $\gamma \in (0,1)$, linear when $\gamma = 1$, super-arithmetic when $\gamma > 1$, or exponential when $\gamma$ goes to infinity. This relationship between the form of the criterion and the pattern of growth can be formulated as follows.

**Proposition 1.** *The problem of the construction of the regular sustainable pattern of growth* (2) *is equivalent to the solution of the social planner's problem with the constant-utility criterion* (1).

One of the main approaches to fair allocation of limited resources is the *no-envy principle* (Foley, 1967; Kolm, 1997). When there is no strict equality in distribution, the principle is usually combined with a compensation procedure. The form (1) of no-envy, which can be rewritten as follows: $(\dot{c}/c)^{\gamma} c = \overline{u}$, means that the decline in the *rate* of growth $\dot{c}/c$ should be compensated by the growing *level* of consumption $c$. The multiplicative form $\dot{c}^{\gamma} c^{\mu}$ includes as specific cases:

(a) the conventional function for measuring the utility of the *level* of growing with no limit consumption $c^{1-\eta}/(1-\eta)$ for $\gamma = 0$, $\mu = 1-\eta$, and $\overline{u} = \widetilde{u}(1-\eta)$;

[8] Although the criterion $\max_c \min_t sgn(\dot{c}) \cdot |\dot{c}|^{\gamma} c^{\mu}$ is the "dictatorship of the least advantaged" (Alvarez–Cuadrado and Long, 2009), it does not imply that the generation in crisis ($\dot{c} < 0$) should increase its current consumption by decreasing saving and undermining the consumption of the future generations. In a crisis, the combination $sgn(\dot{c}) \cdot |\dot{c}|^{\gamma} c^{\mu} = \overline{u}$ can be maximized by decreasing the current level of consumption and increasing investment (increasing $\dot{c}$) until $\overline{u}$ reaches its maximum sustainable level. Hence, the current generation, maximizing its own utility, can maximize the utility of future generations, and this "care about the future" can originate from purely egoistic incentives.

(b) percent change as a conventional measure of the *growth* of consumption for $\gamma = 1$ and $\mu = -1$;

(c) a sample value function that relates value to an initial consumption $c$ and to a change of consumption $\dot{c}$ (Kahneman and Varey, 1991, p. 157): $V(\dot{c}, c) = b\dot{c}^a/c$ for $\dot{c} > 0$, where $a < 1$ and $b > 0; V(0, c) = 0; V(\dot{c}, c) = -Kb(-\dot{c})^a/c$ for $\dot{c} < 0$, where $K > 1$.

## 3. Investment in resource-based economy

In the general case, a resource-based economy produces output $q$ with

$$\text{the technology:} \qquad f(k, r) = q, \tag{3}$$

$$\text{the investment rule:} \qquad \dot{k} = wq, \tag{4}$$

$$\text{the initial stocks:} \qquad k(0) = k_0, \quad s(0) = s_0, \tag{5}$$

where $k$ is man-made capital and $r$ is the rate of the resource extraction.[9] The variables are in per capita units, time-dependent, and smooth enough.

Lemma 1 below provides a known necessary condition for optimal prices in the problem of finding

$$u^* = const\,[c(t), r(t)] = \max_{c(t), r(t)} \dot{c}(t)^\gamma c(t)^{1-\gamma}, \tag{6}$$

where $r$ is a nonrenewable resource, $c = q - \dot{k}$, and $\dot{s} = -r$.

**Lemma 1.** *The optimal resource price $f_r$ in economy (3) – (5) under criterion (6) satisfies the Hotelling rule $\dot{f}_r/f_r = f_k + \tau$ with $\tau \equiv 0$.*

**Proof.** The approach of Leonard and Long (1992, pp. 300-304) reformulates problem (6) into the following equivalent form:

$$\text{maximize} \quad V(t) \equiv \int_t^\infty u^* \delta e^{-\delta\xi} d\xi \text{ for } t = 0 \ (V(0) = u^* = const) \tag{7}$$

[9] Economy (3) – (5) represents the conventional approach, which defines the optimal (equilibrium) initial value of the rate of extraction $r_0$ and all other initial values (e.g., $q_0$, $c_0$) that depend on $r_0$. This approach provides a discontinuous solution with respect to economy's prehistory (Bazhanov, 2010) and can be used, e.g., for a resource-extractive firm that has just obtained the stock of a resource $s_0$.

by choosing $c(t)$ and $r(t)$ for an arbitrary constant $\delta$ subject to (omitting the dependence on time) $\dot{k} = q - c, \quad \dot{s} = -r$, and $u(\dot{c}, c) = u^*$. The Hamiltonian of this problem is $H = u^* \delta e^{-\delta t} + \mu_k (q - c) - \mu_s r$. The utility constraint yields the Lagrangian to be maximized: $L = H + \lambda (u - u^*)$. Then, the Pontryagin-type necessary conditions for the state variables $k$ and $s$ are[10]

$$L_c = \lambda u_c - \mu_k = 0, \tag{8}$$

$$L_r = \mu_k f_r - \mu_s = 0, \tag{9}$$

$$\dot{\mu}_k = -\frac{\partial L}{\partial k} = -\mu_k f_k, \tag{10}$$

$$\dot{\mu}_s = -\frac{\partial L}{\partial s} = 0, \tag{11}$$

$$\int_0^\infty L_{u^*} dt = \int_0^\infty \left(\delta e^{-\delta t} - \lambda\right) dt = 1 - \int_0^\infty \lambda dt = 0. \tag{12}$$

The time derivative of Eq. (9) is $\dot{\mu}_k f_r + \mu_k \dot{f}_r - \dot{\mu}_s = 0$, which, combined with Eq. (10) and divided by $\mu_k f_r$, yields the result of the Lemma■

In the conventional approach, where $c_0$ is not fixed, the solution to problem (6) is not unique. For simplicity, the optimal paths can be found for a constant optimal investment rate if this rate exists. The optimal investment rate can be derived, at first, for the optimal path of output by reformulating problem (6).[11] Then, if there exists a constant optimal investment rate for the problem of finding $v^* = const\,[q(t), r(t)] = \max_{q(t),r(t)} \dot{q}(t)^\gamma q(t)^{1-\gamma}$, the same investment rate will be optimal for the initial problem (6).

Criterion (1) implies the specific patterns of growth, therefore, Proposition 2 below provides a general formula for the investment rate $w(t)$ that guarantees the given growth rate when the investment rate is feasible, for example, $w \in (0, 1)$ for a closed economy. The application of this result is illustrated below for the DHSS economy.

[10] Here $\mu_k$ and $\mu_s$ are indexed dual variables unlike $u_c$, $f_k$, and $f_r$, which are the partial derivatives of $u$ and $f$.

[11] The substitution of $\dot{q}$ and $q$ for $\dot{c}$ and $c$ in (6) does not change the result of Lemma 1.

**Proposition 2.** *The economy's output $q = f(k, r)$ grows with the rate $\dot{q}/q$ under the investment rule $\dot{k} = wq$ iff $w$ is feasible and*

$$w = \left(\frac{\dot{q}}{q} - \frac{f_r \dot{f}_r}{q f_{rr}}\right) / \left(f_k - \frac{f_r f_{kr}}{f_{rr}}\right), \tag{13}$$

*where $f_x = \partial f / \partial x$ and $f$ is smooth enough.*

**Proof.** The growth rate is $\dot{q}/q = f_k \dot{k}/q + f_r \dot{r}/q = f_k w + f_r \dot{r}/q$, yielding $w = (\dot{q}/q - f_r \dot{r}/q) / f_k$. Substitutions for $\dot{r}$ from the equation $\dot{f}_r = f_{kr}\dot{k} + f_{rr}\dot{r}$ and then for $\dot{k} = wq$ result in equation (13)■

This result can be specified for various criteria, kinds of the resource, and technologies $f(k, r)$. A classical benchmark in resource economics, the DHSS model, specifies the production technology as the Cobb-Douglas function: $q = f(k, r) = k^\alpha r^\beta$, where $\alpha$, $\beta \in (0, 1)$, $\alpha + \beta < 1$ are constants ($1 - \alpha - \beta$ is the share of labor in this economy). Assume that there is no population growth,[12] extraction cost is zero, and the TFP (Total Factor Productivity) exactly compensates for capital depreciation.[13] Then, the following result holds.

**Corollary 1.** *The economy's output $q = k^\alpha r^\beta$ grows with the rate $\dot{q}/q$ under the investment rule $\dot{k} = wq$ iff $w$ is feasible and*

$$w = \frac{\dot{q}}{q} \cdot \frac{1 - \beta}{f_k} + \beta \left(1 + \tau / f_k\right), \tag{14}$$

*where $\tau$ is the deviation from the standard Hotelling rule: $\tau := \dot{f}_r / f_r - f_k$.*[14]

[12] The United Nations estimates that the world's population growth is going to flatten out at a level around 9 billion (UN, 2004). Stabilization has already happened in developed countries, which are the main users of nonrenewable resources.

[13] This assumption allows for considering the basic DHSS model with no capital depreciation and no TFP. At the same time, this approach makes it possible to examine correctly various patterns of growth in the economy. This TFP is somewhere between optimistic and pessimistic assumptions about technical change: it is asymptotically linear with a small slope.

[14] An example of the modified rule was provided, e.g., by Stollery (1998) ($\tau(t) = (f_T + u_T/u_c)T_{s_0 - s(t)}/f_r$) in the problem, where utility $u(c, T)$ and production are negatively affected ($u_T < 0$, $f_T < 0$) by growing damage $T(s_0 - s)$, and the damage is rising due to oil use in the economy. A review of the literature and the reasons, distorting the standard Hotelling rule, can be found, e.g., in Gaudet (2007).

**Proof.** In the DHSS case, the expressions for the derivatives in equation (13) are: $f_r = \beta q/r$, $f_k = \alpha q/k$, $f_{kr} = \alpha\beta q/(rk)$, $f_{rr} = \beta q(\beta-1)/r^2$, and the generalized Hotelling rule gives $\dot{f}_r = (f_k + \tau)\,\beta q/r$. Direct substitution of these formulas into equation (13) results in equation (14)■

In the Solow (1974) - Hartwick (1977) case, namely, when $\tau \equiv 0$ and $\dot{q} \equiv 0$ ($\gamma = 0$), Corollary 1 implies that the Hartwick rule ($w(t) \equiv \beta$) is a necessary and sufficient condition for constant per capita consumption in this economy, which coincides with known results (Dixit et al., 1980).

Another interesting illustration of Corollary 1 is Stollery's (1998) problem, for example, with $\tau = u_T T_{s_0-s}/\left(u_c f_r\right)$, when utility $u$ alone is affected by damage $T$. In the DHSS case, $\tau = -\dot{q}(1-\beta)/(q\beta)$, which yields $w(t) \equiv \beta$, coinciding with Stollery's conclusion.[15]

The next result extends the Solow - Hartwick case by defining the optimal investment rule depending on the pattern of growth, determined by $\gamma$.

**Corollary 2.** *Let the economy $q = k^\alpha r^\beta$ follow the investment rule $\dot{k} = wq$, and $\tau \equiv 0$. Then $q(t) = q_0(1+\varphi t)^\gamma$ iff $w$ is feasible and satisfies the equation:*

$$w(t) = w^* - (w^* - w_0)(1+\varphi t)^a, \tag{15}$$

*where $w_0 = w(0)$,*

$$w^* = \beta/\left[1 - \frac{\gamma(1-\beta)}{\alpha(1+\gamma)}\right] = \beta\left[1 + \frac{\gamma(1-\beta)}{\alpha - \gamma(1-\alpha-\beta)}\right], \tag{16}$$

$a = \gamma(1-\beta)/(\alpha q_0) - \gamma - 1$, $q_0 = k_0^\alpha r_0^\beta(s_0)$, $\varphi := \left(\dot{c}_0/c_0\right)/\gamma$, $c_0 = (1-w_0)q_0$, $\dot{c}_0 = (1-w_0)\dot{q}_0 - \dot{w}_0 q_0$, $\dot{q}_0 = \alpha w_0 k_0^{2\alpha-1} r_0^{2\beta}(s_0)\left[w_0 - \beta(1-w_0)/(1-\beta)\right]$, $\dot{w}_0 = -a(w^* - w_0)$.

**Proof.** Equation (14) implies, for $\tau \equiv 0$ and $q(t) = q_0(1+\varphi t)^\gamma$, that $w = \left[\dot{q}_0/(q_0^2(1+\varphi t)^{\gamma+1})\right] k(1-\beta)/\alpha + \beta$, which, using the investment rule, can be

[15] The growth of consumption in the Stollery's case is associated with $\tau < 0$, which is caused by the externality. The criterion $u(cT^{-1}) = const$ requires less initial rate of extraction $r_0$ and more gradual decline in $r(t)$, which, in combination with the same Hartwick rule as in the constant-consumption case, gives a richer flow of inputs, causing growth of consumption, starting from a lower level (Bazhanov, 2011).

rewritten as follows: $\int w(t)(1+\varphi t)^{\gamma}dt = [w(t)-\beta](1+\varphi t)^{\gamma+1}A$, where $A := \alpha q_0^2/[\dot{q}_0(1-\beta)]$. The last formula, after differentiating and dividing by $(1+\varphi t)^{\gamma+1}$, becomes a separable differential equation $\dot{w} = [wp_1 + p_0]/(1+\varphi t)$ with the solution $w(t) = [C(1+\varphi t)^a - p_0]/p_1$, where $a := p_1/\varphi$, $p_1 := 1/A - \varphi(\gamma+1)$, $p_0 := \beta\varphi(\gamma+1)$. The constant of integration $C$, defined from the initial condition $w(0) = w_0$, is $C = w_0 p_1 + p_0$. Then, the formula for $w(t)$ takes the form of (15) with $w^* := -p_0/p_1$, which after substitution of $p_0$ and $p_1$ yields formula (16), and the expression for $a$, using $\varphi = \dot{q}_0/(q_0\gamma)$, becomes $\gamma(1-\beta)/(\alpha q_0) - \gamma - 1$. Then $\dot{w}(t) = -a(w^* - w_0)(1+\varphi t)^{a-1}$, defining $\dot{w}_0$■

Corollary 2 provides the unique constant investment rate $w^*$, which maintains the specific pattern of growth, implied by criterion (1) for a given $\gamma$. When $w_0$ deviates from $w^*$, the path $q(t) = q_0(1+\varphi t)^{\gamma}$ can be sustained under a variable $w(t)$ that asymptotes to $w^*$ for $a < 0$[16] in accord with Eq. (15).

The result is intuitive since the faster growth requires more investment $(\partial w^*/\partial\gamma > 0)$ and less consumption. The optimal trade-off is defined here by the preference parameter $\gamma$. The same qualitative result for this economy was obtained by Hamilton et al. (2006, Proposition 1), showing that consumption grows when the investment rate is more than $\beta$.[17] A similar result was reported by Asheim et al. (2007) for the maximin with $\gamma = 0$ : an "additional" investment allows for quasi-arithmetic population growth and/or for quasi-arithmetic growth of per capita consumption.[18] Corollary 2 specifies the general result of Hamilton and Hartwick (2005, Proposition 1),[19] by providing the link between

[16] The definition of $a$ implies that $a < 0$ iff $q_0 > (1-\beta)/[\alpha(1+1/\gamma)]$. This condition, e.g., takes the form $q_0 > 0$ when $\gamma = 0$ or, when $\gamma = 1$, $q_0 > (1-\beta)/(2\alpha)$, which can be satisfied by the choice of units of measure for capital and extraction.

[17] The difference is that Hamilton et al. (2006) considered constant returns to scale with respect to capital and the resource $(\alpha + \beta = 1)$, which resulted in logarithmic growth for $w > \beta$, whereas here, following Solow (1974, p. 35), returns to scale are constant with respect to capital, resource, and labor.

[18] Similar to Corollary 2, Asheim et al (2007, Theorem 13) showed that the saving rate asymptotically converges to a constant $w^* \geqslant \beta$.

[19] The result implies, in particular, that a positive constant genuine saving $(w > \beta)$ with

the value of the investment rate and the pattern of growth.

Note also, that according to (16), the larger share of capital in production implies less effort in investment for the same rate of growth ($\partial w^*/\partial\alpha < 0$). Formula (16) establishes a strict relationship between the "desirable" rate of growth, expressed in $\gamma$, the optimal investment rate $w^*$, and the technological abilities of the economy ($\alpha$ and $\beta$). Then, the feasibility of the investment rate alone put the restriction on the pattern of growth that could be maintained forever. This result about the limitation of the rates of growth in a resource-based economy is specified in the following Corollary.

**Corollary 3.** *Under the conditions of Corollary 2, the optimal path exists if* $\gamma < \alpha/(1-\alpha)$.

**Proof** follows directly from the feasibility condition $w^* < 1$ after substitution for $w^*$ from formula (16).

In theory, the constraint $\gamma < \alpha/(1-\alpha)$ is not binding since $\gamma \to \infty$ with $\alpha \to 1$; however, empirical estimates of $\alpha$, which are around 0.3 (e.g., Nordhaus and Boyer, 2000), restrict the value of $\gamma$ by 0.43. Further restriction on the rate of growth, imposed by limitedness of the resource, is considered below.

## 4. Optimal paths in the DHSS economy

The following Proposition extends the Solow - Hartwick case by providing the social planner's optimal paths under the generalized criterion (6) with the optimal investment rate $w^*$, defined by formula (16).

**Proposition 3.** *The optimal with respect to criterion (6) paths in economy (3) – (5) with* $f(k,r) = k^\alpha r^\beta$ *and* $w = w^*$ *are:*

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$\tau \equiv 0$ yields the growth of consumption.

$$\begin{aligned} c(t) &= c_0(1+\varphi t)^{\gamma}, \\ q(t) &= q_0(1+\varphi t)^{\gamma}, \\ k(t) &= k_0 + \frac{w^* q_0}{(\gamma+1)\varphi}\left[(1+\varphi t)^{\gamma+1} - 1\right], \\ r(t) &= \left[q(t)k(t)^{-\alpha}\right]^{1/\beta}, \end{aligned}$$

*where*

$$\varphi := (\dot{c}_0/c_0)/\gamma = \beta\frac{f_k(0)}{\alpha - \gamma(1-\alpha-\beta)} = \frac{\alpha\beta r_0^{\beta}}{k_0^{1-\alpha}\left[\alpha - \gamma(1-\alpha-\beta)\right]}, \tag{17}$$

$\dot{c}_0 = (1-w^*)\left[\alpha k_0^{2\alpha-1} r_0^{2\beta}\left(w^* - \beta(1-w^*)/(1-\beta)\right)\right]$, $c_0 = (1-w^*)q_0$, $q_0 = k_0^{\alpha} r_0^{\beta}$, *and the relationship between* $k_0, s_0$ *and* $r_0$ *is:*

$$s_0 = \Phi_{\gamma}(k_0, r_0), \tag{18}$$

*where*

$$\Phi_{\gamma}(k_0, r_0) := \frac{k_0^{1-\alpha} r_0^{1-\beta}\left[1 - \gamma(1-\alpha-\beta)/\alpha\right]}{\left[\alpha - \beta - \gamma(1-\alpha)\right]} \times {}_2F_1(1, a_2; a_3; \beta) \tag{19}$$

*and* ${}_2F_1(\cdot)$ *is the Gauss hypergeometric function with the parameters* $a_2 := -\gamma(1-\beta)/\left[\beta(1+\gamma)\right]$, $a_3 := \alpha/\beta + a_2$.

*The optimal value of utility is*

$$u^* = \left\{\frac{\alpha\beta\gamma k_0^{\alpha} r_0^{\beta}}{k_0\left[\alpha - \gamma(1-\alpha-\beta)\right]}\right\}^{\gamma}(1-w^*)k_0^{\alpha} r_0^{\beta}. \tag{20}$$

**Proof** is in Appendix 1.

Formula (18) provides an explicit expression for $r_0(s_0, k_0)$ :

$$r_0 = \left\{\frac{s_0\left[\alpha - \beta - \gamma(1-\alpha)\right]}{k_0^{1-\alpha}\left[1 - \gamma(1-\alpha-\beta)/\alpha\right]{}_2F_1(1, a_2; a_3; \beta)}\right\}^{1/(1-\beta)}. \tag{21}$$

This expression can be used when the planner solves a discontinuous problem with respect to the initial rate of extraction, e.g., the stock $s_0$ has just been discovered or obtained at an auction.

The Solow - Hartwick case emerges here with $\gamma$ going to zero: the paths $c$ and $q$ are constant over time ($\varphi > 0$), capital is linear with $k(0) = k_0$, and the relationship between $k_0, s_0$, and $r_0$ becomes

$$s_0 = k_0^{1-\alpha} r_0^{1-\beta} / (\alpha - \beta) \tag{22}$$

(or $r_0 = \left\{ s_0(\alpha - \beta)/k_0^{1-\alpha} \right\}^{1/(1-\beta)}$) because all the terms in the series ${}_2F_1(\cdot)$ go to zero except the first one, which equals unity.

Quasi-arithmetic paths were derived in the literature from the different frameworks, namely, under the assumptions of quasi-arithmetic population growth (Asheim et al., 2007) or quasi-arithmetic technical change and discount factor (Pezzey, 2004), whereas, here, this pattern follows directly from the criterion.

Formulae (10.30) and (10.32) in Dasgupta and Heal (1979, p. 305) also yield quasi-arithmetic growth of consumption for the DHSS economy under the utilitarian criterion $\int_0^\infty e^{-\delta t} u(c(t))dt$ with $\delta = 0$ and $u(c) = -c^{-(\eta-1)}$, where $\eta > 1$. Proposition 3 implies that this problem is equivalent to the maximin applied to $u(\dot{c}, c) = \dot{c}^\gamma c^{1-\gamma}$ with

$$\gamma = \frac{\alpha\eta}{\beta(\eta-1)^2 + (\eta-1)(1+\beta-\alpha) + (1-\alpha)}. \tag{23}$$

Formula (23) extends the conventional link between the utilitarian criterion and the maximin for the cases with $\eta < \infty$.

Formula (18) allows to continue the analysis of existence of the sustainable optimal paths, which was started in Corollary 3. Note that the denominator of the fraction in formula (19) goes to zero when $\gamma$ approaches the value $\gamma_{\max} = (\alpha - \beta)/(1-\alpha)$, while ${}_2F_1(\cdot)$ monotonically declines, remaining positive when $\gamma$ increases from 0 to $\gamma_{\max}$.[20] Then, given $k_0$ and $s_0$, the initial rate of extraction strictly monotonically goes to zero when $\gamma$ approaches $\gamma_{\max}$.

[20] ${}_2F_1$ has the points of discontinuity when $a_3$ is negative integer; $a_3$ is positive when $\gamma < \alpha/(1-\alpha-\beta)$. Here, $a_3$ is always positive since $\gamma_{\max} < \alpha/(1-\alpha-\beta)$ for the feasible values of $\alpha, \beta$, and $\gamma$. The behavior of ${}_2F_1(\cdot)$ in the range $\gamma \in [0, \gamma_{\max}]$ was examined numerically for the whole range of parameters $0 < \beta < \alpha < 1$ s.t. $\alpha + \beta < 1$.

Another interpretation of this outcome is that, given $k_0$ and $r_0$, the higher rates of *sustainable* growth of consumption require larger reserve $s_0$, which strictly monotonically goes to infinity with $\gamma \to \gamma_{\max}$. The result can be formulated as follows.

**Corollary 4.** *Under the conditions of Proposition 3, the optimal paths exist if* $\gamma < (\alpha - \beta)/(1 - \alpha)$.

This restriction, imposed by the finiteness of the resource, is more binding than the one, placed by the feasibility of the saving rate (Corollary 3).

Comparison of this result with the results in the literature on the limit to population growth shows that the resource restriction binds the growth of consumption under the assumption of constant population more, than it binds the growth of population under the constant per capita consumption[21] since $\gamma_{\max} = (\alpha - \beta)/(1 - \alpha) < (\alpha - \beta)/\beta$. The latter limit was obtained for the quasi-arithmetic population growth by Mitra (1983) and Asheim et al. (2007, Theorem 12). Another comparison shows that the value of $\gamma_{\max}$ corresponds to $\eta_{\min} = (1 - \beta)/(\alpha - \beta)$ in Dasgupta and Heal (1979), which can be shown by direct substitution of $\eta_{\min}$ for $\eta$ in formula (23).

Following Groth et al. (2006), denote $g_1(t) := \dot{q}(t)/q(t)$ – the first order growth rate. For the constant investment rate, $g_1(t) = \dot{c}(t)/c(t)$. Then the limit on the rate of growth implied by Corollary 4 can be formulated as follows.

**Corollary 5.** *In the economy* $q = k^\alpha r^\beta$ *under the conditions of Proposition 3, the optimal rate of the sustainable growth of consumption (output) is restricted by the technology in the following way:*

$$g_1(t) < 1/\left(g_1^{-1}(0) + \omega t\right),$$

*where* $\omega := (1 - \alpha)/(\alpha - \beta)$.

**Proof** follows from formula (2): $g_1(t) = \dot{c}/c = g_1(0)/(1 + \varphi t)$. Substitution for $\varphi = g_1(0)/\gamma$ gives $g_1(t) = 1/(g_1^{-1}(0) + t/\gamma)$, which, after applying Corollary

[21] This result can be explained by the fact that population, unlike consumption, is an input in the production function.

4, yields the result$\blacksquare$

Regular growth, by the definition of Groth et al. (2006), satisfies the condition $g_2 = -(1/\gamma)g_1$, where $g_2(t) := \dot{g}_1(t)/g_1(t)$ is the second order growth rate, and $1/\gamma$ is the damping coefficient. The growth approaches exponential when $1/\gamma$ goes to zero ($\gamma \to \infty$), which is possible in this framework only when $\alpha \to 1$. The last condition means that the shares of the resource ($\beta$) and labor ($1-\alpha-\beta$) go to zero (complete automatization of the production with complete recycling and/or regeneration of the resource). Note also, that when the resource share is close to the one of capital ($\alpha - \beta$ close to zero), then, given other parameters fixed, the damping coefficient goes to infinity, resulting in stagnation.[22]

There is a conventional practice of formulating the goals of economic programs in the fixed values of the percent change of some indicators. This practice was questioned more than three decades ago, for example, by Dasgupta and Heal (1979): "The rate of growth of GNP cannot function well as a primitive ethical norm. And yet it is very often so used" (p. 311). This measure of progress is still commonly used because of its convenience, especially in the formulation of the programs of sustainable development,[23] where the measures of progress are presumed to be sustained for a long time. These practical needs and the fact that growth can be less than exponential imply an important application of the measure $\dot{c}^{\gamma}c^{1-\gamma}$ since it can be constant along a path, even if the path is not a stagnation and not an exponential growth. This expression can be called *geometrically weighted percent,* and it can be used as an alternative measure of sustainable growth instead of regular percent.

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[22] The conventional estimate of $\alpha = 0.3$ yields $\gamma_{\max} = 0.357$ for $\beta = 0.05$ and $\gamma_{\max} = 0.071$ for $\beta = 0.25$. The patterns of growth with these values of $\gamma_{\max}$ are closer to stagnation than to a linear function.

[23] For example, the Brundtland Report (World Commission, 1987) claimed that "the key elements of sustainability are: a minimum of 3 percent per capita income growth in developing countries" (p. 169). Further, the Report suggested that "annual global per capita GDP growth rates of around 3 percent can be achieved. This growth is at least as great as that regarded in this report as a minimum for reasonable development" (p. 173).

## 5. Smooth paths under distortions

In the conventional approach, the path of extraction $r(t)$ results from the optimal paths of output and capital or from an optimality condition in the form of a first-order differential equation[24] with the constant of integration derived from the efficiency condition $s_0 = \int_0^\infty r dt$. This $r(t)$, including $r_0$, is completely determined by $s_0, k_0$, and other parameters of the model.

The conventional approach proved a convenient tool for qualitative analyses of changes in an economy using, e.g., comparative statics. However, the resulting discontinuity of the paths at $t = 0$ can be inadequate with the goals of some studies when historical data at $t = 0$ do not satisfy the "perfection" condition (18) implied by the criterion. For example, according to the estimate published in *Oil & Gas Journal*, the world's oil reserve on January 1, 2010 was $s_0 = 185.5$ bln t.[25] The production function $q = k^\alpha r^\beta$ with $k_0 = 6.246$ yields from Eq. (22) for $\gamma = 0$ the socially optimal value of $r_0 = 3.525$ bln t/year, which is the rate of world oil extraction on January 1, 2010 (World Oil, 2009). At the same time, Cambridge Energy Research Associates claimed that the actual world's reserve is around 512.33 bln t (CERA, 2006). An approach requiring immediate satisfaction of the efficiency condition by a discontinuous shift in the rates of extraction would, in this case, result in the jump to $r_0 = 13.66$ bln t/year, which is unacceptable in the real economy.

Hence, when a planner is recalculating smooth optimal paths under some changes in the formulation of the problem, there are two general options:

(I) to adhere to the past preferences and to solve a transition problem in order to adjust the paths of $s(t), k(t)$, and $r(t)$ to condition (18), and to enter smoothly new optimal paths in finite time;

(II) to adjust a preference parameter in accord with the updates in order to satisfy condition (18) and enter smoothly new optimal paths at $t = 0$.

The first option was considered in Bazhanov (2010), where a distorted econ-

[24] For example, $\dot{r}/r = -f_k$ for $\gamma = 0$.

[25] Ton of crude oil equals here 7.3 barrels.

omy switched to a new level of constant consumption ($\gamma = 0$) in finite time. The second problem can be either an independent option or a partial solution to the first problem. The following section provides a solution to problem II.

## 6. Smooth second-best paths in a distorted DHSS economy

Formulae (18) and (21) mean that the socially optimal extraction starts with $r_0$, defined by the given parameter $\gamma$ and the initial stocks $k_0$ and $s_0$. This $r_0$ can take any feasible value since it is assumed that the stock $s_0$ has just been discovered or the transition from the historical $r_0$ to the optimal one is not relevant, and so $r_0$ is treated as "the future."

This section examines another problem, in which a social planner constructs smooth constant-utility paths starting from $t = 0$ in a distorted economy that has *already been extracting the resource* for a period of time. The paths are the smooth continuations of the economy's current state, including the short-run trend of extraction (growing or declining), so the values of $r_0$ and $\dot{r}_0$ coincide with the last available estimates – on January 1 of the current year, for example – implying zero adjustment costs at $t = 0$.

In this case, $r_0$ is treated as "the past," and condition (18) shows how much reserve $s_0$ the economy needs to maintain constant utility in the infinite horizon problem. If the actual reserve is larger or smaller than $s_0$, the economy is either inefficient or unsustainable. In this sense, the discrepancy in equation (18) can be used as a measure of distortion in the economy. The other indicators of distortion are connected with the deviations from the optimal investment rule and from the specific formulation of the Hotelling rule, when the model does not include all the phenomena that can modify the rule in the real economy. Hence, a distorted economy is defined here as follows.

**Definition 1.** A resource-extracting economy with the initial state $(k_0, s_0, r_0, \dot{r}_0)$ is *distorted with respect to a criterion* at $t = 0$ if either of the following holds:

(1) the relationship $s_0 - \Phi(k_0, r_0) = 0$, implied by the criterion, is violated;

(2) the economy does not follow the optimal investment rule;

(3) the path of the resource price is not optimal (a specific formulation of the Hotelling rule does not hold at $t = 0$).

A distortion can result either from "positive" or from "negative" effects. For example, the condition $s_0 - \Phi(k_0, r_0) = 0$ can be violated due to an instant increment in reserve (positive distortion) or due to overextraction in the case of insecure property rights (negative distortion).

**Definition 2.** A distorted economy is *imperfect* if the distortion negatively affects the sustainability of the economy.[26]

Let us assume for definiteness that the reasons distorting the Hotelling rule can be expressed in terms of effective tax,[27] and consider the following example of an economy distorted with respect to a benchmark (Solow-Hartwick) case under criterion (1) with $\gamma = \gamma^0 = 0$ :

(i) condition (18) is violated: $s_0 > \Phi_{\gamma^0}(k_0, r_0)$; [28]

(ii) the investment rule is optimal for $\gamma = 0$, namely, $w \equiv \beta$;

(iii) the Hotelling rule is distorted at $t = 0$, namely, $\dot{f}_r(0)/f_r(0) = f_k(0) + \tau_0$, where $\tau_0 = \tau(0) < 0$.

The motivation for choosing the example is twofold: first, to show how the constant-utility criterion can work in a distorted economy, and second, to provide an illustration of Proposition 1 in Bazhanov (2008), which claims that a resource-based economy can grow even with underinvestment. The growth can be sustainable if the reserve is large enough and the resource is optimally allocated among generations in the sense of a constant-utility criterion.

Hence, the problem of a planner is: to construct a sustainable path of con-

[26] Arrow, Dasgupta and Mäler (2003) define imperfect economies as the "economies suffering from weak, or even bad, governance" (p. 648). Imperfection can also result from imperfect knowledge, e.g., in justice theory or in estimate of the path of technical change, even when the decisions of a planner are "perfect."

[27] For example, insecure property rights lead to shifting extraction from the future towards the present (Long, 1975). The same effect can be obtained by subsidizing the resource extracting industry.

[28] I consider large $s_0$ since the paper is devoted to the analysis of the patterns of *growth*. When $s_0 < \Phi_{\gamma^0}(k_0, r_0)$, the economy needs a transition period with declining consumption.

sumption growth (2) with the $\gamma$ satisfying condition (18) (if this path exists) subject to the condition that the paths in the economy are the smooth continuations of the given initial state. The planner imposes a tax that, for simplicity, is only extraction-distorting, while the pattern of investment remains unchanged.[29]

It follows from the inequality $s_0 > \Phi_{\gamma^0}(k_0, r_0)$ and from the strict monotonicity of the dependence between $s_0$ and $\gamma$ that there exists a unique $\gamma^*(s_0) > \gamma^0$ such that $s_0 = \Phi_{\gamma^*}(k_0, r_0)$, satisfying the efficiency condition. The strict monotonicity of $w^*(\gamma)$ and the optimality of $w^*$ imply that $\gamma^*(\beta) < \gamma^*(w^*)$ for the same reserve $s_0 = \Phi_{\gamma^*(\beta)}(k_0, r_0, \beta) = \Phi_{\gamma^*(w^*)}(k_0, r_0, w^*)$, which is intuitive since the optimal investment rate gives a higher rate of growth for the same $s_0$.

The existence of the growth path in this example, despite the underinvestment, follows from Proposition 1 in Bazhanov (2008), which states that $\dot{q} > 0$ in the DHSS economy iff $\tau < f_k(w/\beta - 1)$. Hence, any negative deviations from the standard Hotelling rule ($\tau < 0$) result in output growth for $w \equiv \beta$.

Endogenization of the preference parameter is a well-known approach in justice theory and in human practice.[30] In the current case, this approach solves the following problems: (a) the optimal paths are smooth despite the changes in the parameters (consistent with the given initial state), (b) the path of extraction satisfies the efficiency condition $s_0 = \int_0^\infty r(t, \gamma^*)dt$, and (c) consumption grows with the maximum $\gamma$ among the sustainable paths.

Technically, the approach introduces the two new fixed parameters: $r_0$ and $\dot{r}_0$, which are used to find the two new unknowns: the parameter $\gamma$, solving dis-

[29] The change in $\gamma$ means that $w \equiv \beta$ becomes non-optimal (the preference of population does not coincide with the preference of the planner), providing only the second-best optimum.

[30] Pezzey (2004, formula (15), p. 477) endogenized preference parameter by specifying the discount factor in utilitarian criterion for given technological parameters and the current state of economy in order to solve the problem of dynamic inconsistency. The approach is consistent with Koopmans' (1964, p. 253) idea about adjusting preferences to economic opportunities, "viewing physical assets as opportunities;" with Hadamard's (1902) principle of a well-posed mathematical problem, and with Bellman's Principle of Optimality. The review of studies in justice theory can be found in Elster (1989).

tortion (i), and the initial value of the distortion $\tau_0$, which includes the influence of known and unknown effects of imperfect institutions, government policies, and externalities.[31] The given initial state $(k_0, s_0, r_0, \dot{r}_0)$ and investment rule imply that $c_0$ and $\dot{c}_0$ are known, which results in the unique sustainable optimal path determined by $\gamma$. A way of constructing this path is shown below.

**Lemma 2.** *Let a distorted economy $q = k^\alpha r^\beta$ with the initial state $(k_0, s_0, r_0, \dot{r}_0)$ follows the investment rule $\dot{k} = \beta q$, and the Hotelling rule at $t = 0$ is $\dot{f}_r(0)/f_r(0) = f_k(0) + \tau_0$, where $\tau_0$ is determined by the initial state: $\tau_0 = \tau(0) = -(1 - \beta)\left[\dot{r}_0/r_0 + \alpha k_0^{\alpha-1} r_0^\beta\right]$. Then the unique path of the Hotelling rule distortion*

$$\tau(t) = -\frac{1-\beta}{\beta}\frac{\dot{q}_0/q_0}{1+\varphi t} = -\frac{1-\beta}{\beta}\frac{\dot{q}}{q}, \tag{24}$$

*where $\varphi := (\dot{q}_0/q_0)/\gamma$, is socially optimal with respect to criterion (1).*

**Proof.** The general investment rule $\dot{k} = wq$ implies that $\dot{f}_r/f_r = wf_k - (1-\beta)(\dot{r}/r)$, which, according to the Hotelling rule, equals $f_k + \tau$, or $f_k(w-1) - (1-\beta)(\dot{r}/r) = \tau$. The last equation yields $\dot{r}/r = -\left[(1-w)/(1-\beta)\right]\left[f_k + \tau/(1-w)\right]$, which for $w \equiv \beta$ becomes $\dot{r}/r = -f_k - \tau/(1-\beta)$. Then, $\dot{q}/q = \alpha\dot{k}/k + \beta\dot{r}/r = \beta\left(f_k + \dot{r}/r\right) = -\beta\tau/(1-\beta)$.

From the criterion, $\dot{c}^\gamma c^{1-\gamma} = (1-\beta)\dot{q}^\gamma q^{1-\gamma} = \overline{u}$ or $\dot{q}^\gamma q^{1-\gamma} = \overline{u}/(1 - \beta)$. Substitutions for $q = c_0(1+\varphi t)^\gamma/(1-\beta)$ and $\dot{q} = q\beta\tau/(\beta-1)$ give: $(q\tau\beta/(\beta-1))^\gamma q^{1-\gamma} = (\tau\beta/(\beta-1))^\gamma q = \overline{u}/(1-\beta)$. Substitution for $q$ yields $\left[\tau\beta/(\beta-1)(1+\varphi t)\right]^\gamma = \overline{u}/c_0$ or $\tau = (\overline{u}/c_0)^{\frac{1}{\gamma}}(\beta-1)/\left[\beta(1+\varphi t)\right] = -(1 - \beta)(\dot{q}_0/q_0)/\left[\beta(1+\varphi t)\right]$. The value of $\tau_0$ can be derived from the Hotelling rule. For the general investment rate $w$, it is $\tau_0 = -(1-\beta)\dot{r}_0/r_0 - (1-w)\alpha k_0^{\alpha-1} r_0^\beta$ ■

The following Lemma provides the path of an effective extraction tax $T$ that is equivalent to the distortion $\tau(t)$ shifting the economy from the Solow-Hartwick case. The tax includes all the existing at $t = 0$ taxes/subsidies[32] and the tax

[31] This approach, as any approach with an aggregate model, does not pretend on high quantitative accuracy; therefore the path of $\tau(t)$ includes also the inaccuracy of the model.

[32] This tax, however, does not include the tax that brings the economy from the laissez-faire state to the Solow-Hartwick case.

imposed by the planner at $t = 0$ to adjust the path of extraction in accord with the criterion. The Solow-Hartwick case, therefore, corresponds here to $T \equiv 0$.

**Lemma 3.** *Under the conditions of Lemma 2, the effect of the tax*

$$T(t) = e^{\int f_k(t)dt} \left[ \widehat{T} + \int \tau f_r e^{-\int f_k(t)dt} dt \right], \tag{25}$$

*with* $\widehat{T} = \widehat{T}(T_0)$ *and* $T_0 = T(0) = f_r(0) - \beta k_0 / [s_0 (\alpha - \beta)]$, *on the distortion in the Hotelling rule is equivalent to the effect of* $\tau(t)$.

**Proof.** Since $\tau(t)$ can be expressed in terms of tax, there exists an effective tax $T(t)$ such that the equation $\dot{f}_r/f_r = f_k + \tau$ takes the form:[33] $(\dot{f}_r - \dot{T})/(f_r - T) = \dot{f}_r/f_r - \tau = f_k$. This equation can be rewritten as follows: $\dot{T} - \dot{f}_r + f_k(f_r - T) = 0$ or $\dot{T} - Tf_k - \dot{f}_r + f_k f_r = 0$, which is equivalent to the following dynamic condition for tax

$$\dot{T} - Tf_k - \tau f_r = 0 \tag{26}$$

with the general solution in the form of (25). The initial condition $T(0)$ can be found from the fact that, for $\gamma = 0$ (Solow-Hartwick case), the condition $s_0 - \Phi(k_0, r_0) = 0$ takes the form (22). Then, $f_r^*(0)$ with no distortions equals

$$f_r(0) - T(0) = \beta q_0^* / r_0^*, \tag{27}$$

where $q_0^* = k_0^\alpha (r_0^*)^\beta$ and $r_0^*$ satisfies "perfection" condition (22): $(r_0^*)^{\beta-1} = k_0^{1-\alpha} / [s_0 (\alpha - \beta)]$. Substitution of this expression into (27) yields $T(0)$, and equation (26) gives the initial tax change: $\dot{T}(0) = T(0) f_k + \tau(0) f_r(0)$[34] ■

Lemmas 3 and 4 have established the link between $\tau$, the planner's tax, and the rate of growth, providing the way to construct the paths with desirable properties. The following Proposition uses this link for deriving the smooth closed form solutions by using the given $r_0, \dot{r}_0$ and redetermining $\gamma^* > \gamma^0$ from the efficiency condition.

[33] This dynamic efficiency condition was used by Hamilton (1994) in the form $\dot{n}/n = f_k$ for the net rent per unit of resource $n = f_r - C - T$, where $C$ is the marginal cost of extraction.

[34] In the Solow-Hartwick case, $f_r \equiv \beta q^* / r^*$ implying $T \equiv 0$. When, e.g., the initial extraction is small ($r_0 < r_0^*$) and growing ($\dot{r}_0 > 0$), the tax $T$ is positive and declining.

**Proposition 4.** *Let a distorted economy* $q = k^{\alpha} r^{\beta}$ *with the initial state* $(k_0, s_0, r_0, \dot{r}_0)$ *satisfy conditions (i)-(iii). Then the effective tax*

$$T(t) = T_0 (k/k_0)^{\alpha/\beta} - f_r \left[ (q/q_0)^{1/\beta - 1} - 1 \right]$$

*is socially optimal with respect to criterion (1). This tax implies the following paths of capital and the resource use:*

$$\begin{aligned} k(t) &= k_0 + \frac{\beta q_0}{(\gamma^* + 1)\varphi} \left[ (1 + \varphi t)^{\gamma^* + 1} - 1 \right], \\ r(t) &= q_0^{1/\beta} (1 + \varphi t)^{\gamma^*/\beta} k(t)^{-\alpha/\beta}, \end{aligned}$$

*where* $\varphi := (\dot{q}_0 / q_0) / \gamma^*$, $q_0 = k_0^{\alpha} r_0^{\beta}$, $\dot{q}_0 = \beta k_0^{\alpha} r_0^{\beta} (\alpha k_0^{\alpha - 1} r_0^{\beta} + \dot{r}_0 / r_0)$, *and* $\gamma^* = \gamma^*(s_0)$ *is a unique solution of the equation*

$$s_0 = \frac{1 + \gamma^*}{\alpha - \beta - \gamma^*(1 - \alpha)} \cdot k_0^{1-\alpha} r_0^{1-\beta} \cdot {}_2F_1 (1, a_2; a_3; z), \tag{28}$$

*where* $a_2 := -\frac{\gamma^*(1-\beta)}{\beta(1+\gamma^*)}$, $a_3 := \alpha/\beta + a_2$, *and* $z := 1 - k_0 \varphi (1 + \gamma^*) / (\beta q_0)$.

**Proof** is in Appendix 2.

The paths, offered in Proposition 4, are the smooth continuations of the initial conditions (Fig. 1). Indeed, the initial value of the effective tax coincides with the historical value $T_0$, which means that the "additional" tax, introduced at $t = 0$, is zero at this moment, regardless of the shocks in the parameters at $t = 0$ including the shock in $s_0$. Unlike the conventional approach, the claim of CERA (2006) about larger reserve results here only in changes in the plans for the paths of the tax ($\dot{T} < 0$), extraction ($\ddot{r} > 0$), and consumption ($\ddot{c} > 0$) (dotted lines in Fig. 1). This sustainable economy is asymptotically efficient because $\tau \to 0$ with $t \to \infty$, and $\gamma^*$ is specified by the necessary efficiency condition[35] $\int_0^{\infty} r(t, \gamma^*) dt = s_0$.

Another interesting property of this solution is that the path of extraction $r$ includes the multiplier $(1 + \varphi t)^{\gamma^*/\beta}$ implying that the "second-best" initial extraction can be growing (Fig. 1a). Indeed, the distorted Hotelling rule with

[35] I mean here the conventional notion of efficiency in terms of consumption (e.g., Malinvaud (1953), Mitra (1978), Dasgupta and Heal (1979, p. 216)) without $\dot{c}$ in utility ($\gamma = 0$).

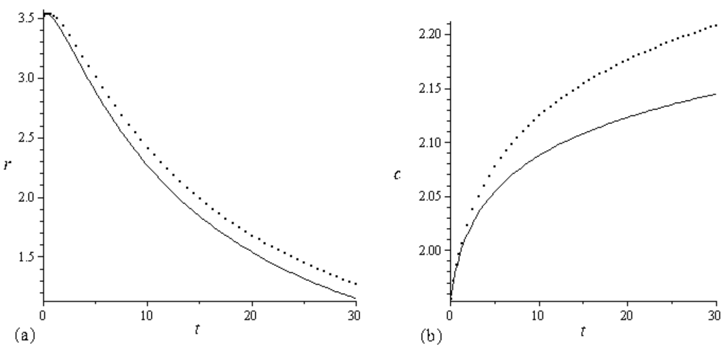


Figure 1: The second-best paths of (a) extraction [bln t/year] and (b) consumption in a distorted economy for the world's oil reserve estimated by: *Oil & Gas Journal* - as a solid line; CERA (2006) - as a dotted line; time is in years starting from 2010.

the initial investment $\dot{k}_0 = w_0 q_0$ yields $\dot{r}_0 = -r_0 \left[\alpha k_0^{\alpha-1} r_0^{\beta}(1-w_0) + \tau_0\right]/(1-\beta)$, which is positive when $\tau_0 < -f_k(0)(1-w_0)$.

It is natural to expect that sustainable growth is not affordable for any initial states. Formulas (18) and (28) show that, for overconsuming economies ($s_0 < \Phi_{\gamma^0}(k_0, r_0)$), sustainable growth paths, including stagnation, do not exist. The condition $s_0 < \Phi_{\gamma^0}(k_0, r_0)$ implies that the current level of consumption $c_0$ is higher than the maximum sustainable level of consumption available for the economy by a discontinuous jump at $t = 0$.[36]

In a smooth economy, however, the notion of "the maximum sustainable level of consumption" is undefined because, for example for $s_0 > \Phi_{\gamma^0}(k_0, r_0)$, the economy's consumption can grow quasi-arithmetically, and, at any $\bar{t} > 0$, the economy can switch to a sustainable constant consumption path with the level of consumption higher than $c(\bar{t})$ (Bazhanov, 2010). Hence, the longer is the "transition period" along the quasi-arithmetic path the higher is the maximum sustainable level of consumption with $\lim_{\bar{t}\to\infty} c(\bar{t}) = \infty$ due to the unboundedness of quasi-arithmetic growth.

[36] The latter level is defined in Martinet (2007) as a Sustainable Consumption Indicator.

In an overconsuming economy, the maximin applied to the expression $sgn(\dot{c}) \cdot |\dot{c}|^{\gamma} c^{1-\gamma}$ do not imply that this expression is constant along the optimal path. In this case, a simplified formulation of the criterion, for example, in the form of the fixed percent change or the constant-utility criterion, is not applicable to the formulation of a long-run program.

## 7. Concluding remarks

This paper has examined the social planner's solutions in a resource-based economy under the constant-utility criterion. The utility function depends on social progress $\dot{c}$ in the multiplicative form $u(c, \dot{c}) = \dot{c}^{\gamma} c^{1-\gamma} = c(\dot{c}/c)^{\gamma}$, realizing a form of the no-envy principle where the lower rate of growth is compensated by the higher level of consumption. This criterion implies the "regular" (Groth et al., 2006) paths of consumption growth, which include conventional patterns such as stagnation ($\gamma = 0$), quasi-arithmetic ($0 < \gamma < 1$), linear ($\gamma = 1$), super-arithmetic ($\gamma > 1$), and exponential ($\gamma \to \infty$). This link renders the problem with the constant-utility criterion an interesting theoretical tool since this problem is equivalent – in the sense of resulting growth – to any problem in growth theory resulting in a path from this family. For example, this tool extends the conventional link between the utilitarian criterion and the maximin for the cases with finite values of the elasticity of marginal utility $\eta$ by providing the dependence between $\gamma$ and $\eta$ in the form of (23).

The optimal investment rule was obtained for a general resource-based economy and specified for the DHSS model. The optimal constant investment rate depends on the shares of capital ($\alpha$), the resource ($\beta$), and labor ($1 - \alpha - \beta$) in the following way: $w^* = \beta \left\{1 + \gamma(1-\beta)/\left[\alpha - \gamma(1-\alpha-\beta)\right]\right\}$. This formula includes the Hartwick rule ($w^* \equiv \beta$) as a particular case for $\gamma = 0$. The closed form solutions showed in particular that $\gamma$, determining the rate of growth, is limited from above: $\gamma < (\alpha - \beta)/(1 - \alpha)$. This restriction implies that growth can be exponential only when $\alpha \to 1$, which is possible when the shares of the resource and labor go to zero (complete automatization of the production with complete recycling and/or regeneration of the resource).

Since economic growth can be less then exponential, the measure $\dot{c}^\gamma c^{1-\gamma}$ or geometrically weighted percent can be used as an alternative measure of sustainable growth instead of regular percent. This combination can be constant along the path with declining rates, which is convenient for formulating long-run programs of sustainable development.

A modification of this problem was considered for a distorted (underextracting) resource economy under the constant-consumption criterion ($\gamma = 0$). The requirement for the paths to be smooth continuations of the given initial state combined with the endogenization of $\gamma$ and a monotonically declining tax result in the smooth, asymptotically efficient paths with the monotonic (quasi-arithmetic) growth of per capita consumption. Using these paths for transition to a new constant level of consumption can result in unrestrictedly high new levels of consumption depending on the duration of the transition period.

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## References

[1] Alvarez-Cuadrado, F., Long, N.V., 2009. A mixed Bentham-Rawls criterion for intergenerational equity: theory and implications. Journal of Environmental Economics and Management 58(2), 154–168.

[2] Arrow, K.J., 1973. Rawls's principle of just saving. Swedish Journal of Economics 75(4), 323–335.

[3] Arrow, K.J., Dasgupta, P., Mäler, K.G., 2003. Evaluating projects and assessing sustainable development in imperfect economies. Environmental and Resource Economics 26, 647–685.

[4] Asheim, G.B., 1988. Rawlsian intergenerational justice as a Markov-perfect equilibrium in a resource technology. Review of Economic Studies, 55(3), 469–483.

[5] Asheim, G.B., 2005. Intergenerational ethics under resource constraints. Swiss Journal of Economics and Statistics 141(3), 313–330.

[6] Asheim, G.B., Buchholz, W., Hartwick, J.M., Mitra, T., Withagen, C., 2007. Constant savings rates and quasi-arithmetic population growth under exhaustible resource constraints. Journal of Environmental Economics and Management 53, 213–229.

[7] Bazhanov, A.V., 2008. Sustainable growth in a resource-based economy: the extraction-saving relationship. MPRA Paper No. 12350.

[8] Bazhanov, A.V., 2010. Sustainable growth: Compatibility between a plausible growth criterion and the initial state. Resources Policy 35(2), 116–125.

[9] Bazhanov, A.V., 2011. A closed-form solution to Stollery's problem with damage in utility. Computational Economics (in print), DOI: 10.1007/s10614-010-9249-4.

[10] Becken, W., Schmelcher, P., 2000. The analytic continuation of the Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$ for arbitrary parameters. Journal of Computational and Applied Mathematics 126, 449–478.

[11] Calvo, G., 1978. Some notes on time inconsistency and Rawls' maximin criterion. Review of Economic Studies, 45(1), 97–102.

[12] CERA, 2006. Why the Peak Oil Theory Falls Down: Myths, legends, and the future of oil resources. (November 10, 2006), http://www.cera.com/aspx/cda/client/report/reportpreview.aspx? CID=8437&KID= . Accessed 25 December 2007.

[13] Dasgupta, P., 1974. On some alternative criteria for justice between generations. Journal of Public Economics 3, 405–423.

[14] Dasgupta, P., Heal, G., 1974. The optimal depletion of exhaustible resources. Review of Economic Studies 41, 3–28.

[15] Dasgupta, P., Heal, G., 1979. Economic Theory and Exhaustible Resources. Cambridge University Press, Cambridge, England.

[16] Elster, J., 1989. Solomonic judgements: studies in the limitations of rationality. Cambridge University Press, Cambridge, England.

[17] Fleurbaey, M., 2007. Intergenerational fairness, in: Roemer, J.E., Suzumura K. (Eds.), Intergenerational equity and sustainability. Palgrave Macmillan, New York, pp. 155–175.

[18] Foley, D., 1967. Resource allocation and the public sector. Yale Economic Essays 7, 45–98.

[19] Gaudet, G., 2007. Natural resource economics under the rule of Hotelling. Canadian Journal of Economics 40(4), 1033–1059.

[20] Griffin, J.M., Gregory, P.R., 1976. An intercountry translog model of energy substitution responses. American Economic Review 66, 845–857.

[21] Groth, C., Koch, K.J., Steger, T.M., 2006. Rethinking the concept of long-run economic growth. CESifo Working Paper N 1701.

[22] Hadamard, J., 1902. Sur les problemes aux derivees partielles et leur signification physique. Princeton University Bulletin 13, 49–52.

[23] Hamilton, K., 1994. The Hartwick rule in a greenhouse world. Ch. 4, in: Sustainable development and green national accounts. Ph.D. Thesis, University College, London.

[24] Hamilton, K., Ruta, G., Tajibaeva, L., 2006. Capital accumulation and resource depletion: a Hartwick Rule counterfactual. Environmental and Resource Economics 34, 517–533.

[25] Hartwick, J.M., 1977. Intergenerational equity and the investing of rents from exhaustible resources. American Economic Review 67, 972–974.

[26] Hotelling, H., 1931. The economics of exhaustible resources. The Journal of Political Economy 39(2), 137–175.

[27] Hubbert, M.K., 1956. Nuclear energy and the fossil fuels. American Petroleum Institute, Drilling and Production Practice. Proceedings of Spring Meeting, San Antonio, Texas, pp. 7–25.

[28] Kahneman, D., Varey, C., 1991. Notes on the psychology of utility, in: Elster, J., Roemer J.E. (Eds.), Interpersonal comparisons of well-being. Cambridge University Press, New York, pp. 127–163.

[29] Kolm, S.C., 1997. Justice and Equity. MIT Press, Cambridge, MA.

[30] Koopmans, T.C., 1964. On flexibility of future preference, in: Bryan, G. L., Shelly, M. W. II (Eds.), Human judgments and optimality. John Wiley, New York, pp. 243–254.

[31] Lecomber, R., 1979. The Economics of Natural Resources. John Wiley & Sons, New York.

[32] Leininger, W., 1985. Rawls' maximin criterion and time-consistency: further results. Review of Economic Studies 52(3), 505–513.

[33] Levhari, D., Liviatan, N., 1977. Notes on Hotelling's economics of exhaustible resources. Canadian Journal of Economics 10, 177–192.

[34] Long, N. V., 1975. Resource extraction under the uncertainty about possible nationalization. Journal of Economic Theory 10, 42–53.

[35] Long, N.V., 2007. Toward a just savings principle, in: Roemer, J.E., Suzumura K. (Eds.), Intergenerational equity and sustainability. Palgrave Macmillan, New York, pp. 291–319.

[36] Luke, Y.L., 1969. The Special Functions and their Approximations, Vol. 1, Academic Press, New York, 1969.

[37] Malinvaud, E., 1953. Capital accumulation and efficient allocation of resources. Econometrica 21 (2), 233–268.

[38] Martinet, V., 2007. A step beside the maximin path: Can we sustain the economy by following Hartwick's investment rule? Ecological Economics 64, 103–108.

[39] Mitra, T., 1978. Efficient growth with exhaustible resources in a neoclassical model. Journal of Economic Theory 17, 114–129.

[40] Mitra, T., 1983. Limits on population growth under exhaustible resource constraints. International Economic Review 24, 155–168.

[41] Nordhaus, W.D., Boyer, J., 2000. Warming the World: Economic Models of Global Warming. MIT Press, Cambridge.

[42] Pezzey, J.C.V., 2004. Exact measures of income in a hyperbolic economy. Environment and Development Economics 9, 473–484.

[43] Phelps, E. S., Riley, J. G., 1978. Rawlsian growth: dynamic programming of capital and wealth for intergeneration "maximin" justice. Review of Economic Studies, 45(1), 103–120.

[44] Pindyck, R.S., 1979. Interfuel substitution and the demand for energy: an international comparison. Review of Economics and Statistics 61, 169–179.

[45] Radler, M., 2006. Oil production, reserves increase slightly in 2006. Oil and Gas Journal 104(47), 20–23.

[46] Rawls, J., 1971. A Theory of Justice. Belknap Press of Harvard University Press, Cambridge.

[47] Scanlon, T.M., 1991. The moral basis of interpersonal comparisons, in: Elster, J., Roemer, J.E. (Eds.), Interpersonal comparisons of well-being. Cambridge University Press, New York, pp. 17–44.

[48] Solow, R.M., 1974. Intergenerational equity and exhaustible resources. Review of Economic Studies 41, 29–45.

[49] Stiglitz, J., 1974. Growth with exhaustible natural resources: efficient and optimal growth paths. Review of Economic Studies 41, 123–137.

[50] Stollery, K.R., 1998. Constant utility paths and irreversible global warming. Canadian Journal of Economics 31(3), 730–742.

[51] UN, 2004. World Population to 2300. United Nations, New York.

[52] World Commission, 1987. World Commission on Environment and Development. Our Common Future. Oxford University Press, Oxford/New York.

[53] World Oil, 2009. Worldwide look at reserves and production. Oil & Gas Journal 107(47), 20–21.

## 9. Appendix 1 (Proof of Proposition 3)

The paths of consumption and output follow directly from criterion (1) and from the investment rule. Then the investment rule $\dot{k} = w^* q$ gives the path of capital, implying the path of extraction $r$ from the production function, given $q$ and $k$. Then, the initial rate of extraction $r_0$ can be expressed via the initial stock $s_0$ from the necessary efficiency condition $s_0 = \int_0^\infty r(r_0, t)dt$ in the following way.

The production function and the investment rule imply $r = q^{1/\beta} k^{-\alpha/\beta} = (1/w)^{1/\beta} \dot{k}^{1/\beta - 1} k^{-\alpha/\beta} \dot{k}$. Integration by parts with $p := \dot{k}^{1/\beta-1}$ and $dv := k^{-\alpha/\beta}\dot{k}dt$ yields $s_0 = (1/w)^{1/\beta} \int_0^\infty pdv = (1/w)^{1/\beta} \left[\dot{k}_0^{1/\beta-1} k_0^{1-\alpha/\beta} / (\alpha/\beta - 1) + I_1 (1-\beta) / (\alpha - \beta)\right]$, where $I_1 := \int_0^\infty k^{1-\alpha/\beta} \dot{k}^{1/\beta-2} \ddot{k} dt$. Note that criterion (1) implies that $\ddot{k} = d\left[w k_0^\alpha r_0^\beta (1+\varphi t)^\gamma\right]/dt = \gamma\varphi k_0^{1/\gamma} \dot{k}^{1-1/\gamma}$. Let $\widetilde{u} := \gamma\varphi = \alpha w k_0^{\alpha-1} r_0^\beta + \beta r_0^{-1} \dot{r}_0$ and $\widehat{k} := k_0 - \dot{k}_0 / [\widetilde{u} + \varphi]$. Then $I_1 = \widetilde{u} \dot{k}_0^{1/\gamma} I_2$, where $I_2 = \int_0^\infty k^{1-\alpha/\beta} \dot{k}^{1/\beta-1-1/\gamma} dt$, and $k\dot{k}^{-1-1/\gamma} = \widehat{k}\dot{k}^{-1-1/\gamma} + \dot{k}_0^{-1/\gamma}$, which gives $k^{1-\alpha/\beta} \dot{k}^{1/\beta-1-1/\gamma} = k^{-\alpha/\beta} \dot{k}^{1/\beta} \left[\widehat{k}\dot{k}^{-1-1/\gamma} + \dot{k}_0^{-1/\gamma} / (\widetilde{u} + \varphi)\right]$. Then $I_2 = \widehat{k} I_3 + \left[\dot{k}_0^{-1/\gamma} / (\widetilde{u} + \varphi)\right] \int_0^\infty k^{-\alpha/\beta} \dot{k}^{1/\beta} dt$, where $I_3 := \int_0^\infty k^{-\alpha/\beta} \dot{k}^{1/\beta - (1+1/\gamma)} dt$. The second integral in the formula for $I_2$ equals $w^{1/\beta} s_0$ and then the original integral can be expressed from the equation

$$\begin{aligned} \int_0^\infty r dt &= (1/w)^{1/\beta} \left\{ \beta \dot{k}_0^{1/\beta-1} k_0^{1-\alpha/\beta} / (\alpha - \beta) + \widetilde{u} \dot{k}_0^{-1/\gamma} (1-\beta) / (\alpha - \beta) \right. \\ &\quad \left. \times \left[ \dot{k}_0^{-1/\gamma} / (\widetilde{u} + \varphi) \int_0^\infty r dt + \widehat{k} I_3 \right] \right\} \end{aligned}$$

as follows:

$$\int_0^\infty r dt = (1/w)^{1/\beta} \frac{\widetilde{u} + \varphi}{\widetilde{u}(\alpha - 1) + \varphi(\alpha - \beta)} \left[ \beta \dot{k}_0^{1/\beta-1} k_0^{1-\alpha/\beta} + (1-\beta) \widetilde{u} \dot{k}_0^{1/\gamma} \widehat{k} I_3 \right].$$

Integration of $I_3$ by parts with $p := \dot{k}^{1/\beta - 1 - (1+1/\gamma)}$, $dv := k^{-\alpha/\beta}\dot{k}dt$, and with the same substitutions yields

$$\begin{aligned} \int_0^\infty r dt &= (1/w)^{1/\beta} \frac{\gamma + 1}{\alpha - \beta - \gamma(1-\alpha)} \left\{ \beta \dot{k}_0^{1/\beta-1} k_0^{1-\alpha/\beta} + \right. \\ &\quad \left. + \frac{\gamma(\gamma+1)(1-\beta)\varphi \dot{k}_0^{1/\gamma} \widehat{k}}{\alpha - \beta - \gamma(1-\alpha) + \beta(\gamma+1)} \left[ \beta \dot{k}_0^{1/\beta - 1 - (1+1/\gamma)} k_0^{1-\alpha/\beta} + (1 - \beta - \beta(1 + \frac{1}{\gamma})) \widetilde{u} \dot{k}_0^{1/\gamma} \widehat{k} I_6 \right] \right\}, \end{aligned}$$

where $I_6 := \int_0^\infty k^{-\alpha/\beta} \dot{k}^{1/\beta - 2(1+1/\gamma)} dt$. Consecutive application of this procedure with the substitution of the investment rule $\dot{k}_0 = w k_o^\alpha r_0^\beta$ and rearrangement of terms leads to the following formula for $s_0$ :

$$\int_0^\infty rdt = \frac{\beta}{w}\frac{1+\gamma}{\alpha-\beta-\gamma(1-\alpha)}k_0^{1-\alpha}r_0^{1-\beta}\{1+ \tag{29}$$
$$+\frac{\gamma(1-\beta)}{\alpha-\beta-\gamma(1-\alpha)+\beta(\gamma+1)}\left[\left(\frac{\varphi(\gamma+1)}{w}k_0^{1-\alpha}r_0^{-\beta}-1\right)+\dots\right.$$
$$\left.\left.+\frac{\gamma(1-\beta)-(i-1)\beta(\gamma+1))}{\alpha-\beta-\gamma(1-\alpha)+i\beta(\gamma+1)}\left\{\left(\frac{\varphi(\gamma+1)}{w}k_0^{1-\alpha}r_0^{-\beta}-\frac{1}{\gamma+1}\right)^i+\dots\right\}\right]\right\}.$$

Note that $\varphi := \dot{c}_0/(c_0\gamma) = \left[\alpha w k_0^{\alpha-1}r_0^{\beta}+\beta\dot{r}_0/r_0\right]/\gamma$. The value of $\dot{r}_0$ can be obtained from the expression $\left[\dot{f}_r(0)/f_r(0)\right] = \alpha\dot{k}_0/k_0-(1-\beta)\dot{r}_0/r_0$, which, according to the Hotelling rule, equals $\alpha q_0/k_0$. After substitution of $\dot{k}=wq$, the formula for $\dot{r}_0$ becomes $\dot{r}_0 = -\alpha k_0^{\alpha-1}r_0^{1+\beta}(1-w)/(1-\beta)$, which implies, first,

$$\varphi = \alpha r_0^{\beta}(w-\beta)/\left[\gamma k_0^{1-\alpha}(1-\beta)\right], \tag{30}$$

and, second, $\varphi(\gamma+1)k_0^{1-\alpha}r_0^{-\beta}/w = \alpha\left(1-\beta/w\right)(\gamma+1)/\gamma$. Denote $z := 1-\alpha\left(1-\beta/w\right)(\gamma+1)/\gamma$; then, after dividing the fractions before brackets by $\beta(\gamma+1)$, denoting $a_2 := -\gamma(1-\beta)/\left[\beta(\gamma+1)\right]$, $a_3 := \alpha/\beta-(\gamma+\beta)/\left[\beta(1+\gamma)\right]+1 = \alpha/\beta+a_2$, and opening the brackets, formula (29) takes the form:

$$s_0 = \int_0^\infty rdt = \frac{\beta}{w}\frac{1+\gamma}{\alpha-\beta-\gamma(1-\alpha)}k_0^{1-\alpha}r_0^{1-\beta}\sum_{i=0}^{\infty}\frac{(1,i)(a_2,i)}{(a_3,i)(1,i)}z^i, \tag{31}$$

where $(d,i)$ is the Pochhammer symbol: $(d,i) := d(d+1)\cdots(d+i-1)$ and $(d,0) := 1$. The sum $\sum_{i=0}^{\infty}(\cdot)$ in formula (31) coincides with the definition of the Gauss hypergeometric function ${}_2F_1(1,a_2;a_3;z)$ (Luke 1969, p. 39); therefore the connection between $s_0$ and $r_0$ is:

$$s_0 = \frac{\beta}{w}\frac{1+\gamma}{\alpha-\beta-\gamma(1-\alpha)}k_0^{1-\alpha}r_0^{1-\beta}{}_2F_1(1,a_2;a_3;z). \tag{32}$$

Function ${}_2F_1(\cdot)$ converges for $|z|<1$. The substitution $w=w^*$[37] results in $z=\beta<1$ and in formula (17) after substituting $w=w^*$ into equation (30). Then formula (32) becomes

$$s_0 = \frac{1-\gamma(1-\alpha-\beta)/\alpha}{\alpha-\beta-\gamma(1-\alpha)}k_0^{1-\alpha}r_0^{1-\beta}{}_2F_1(1,a_2;a_3;\beta),$$

[37] Formula (16).

which coincides with formula (18) in the Proposition, and substitution for $\varphi$ from formula (17) into the equation $u^* = (\gamma\varphi)^\gamma(1 - w^*)k_0^\alpha r_0^\beta$ yields formula (20).

## 10. Appendix 2 (Proof of Proposition 4)

Lemma 2 showed that the path $\tau(t) = -(1 - \beta)(\dot{q}_0/q_0)/\left[\beta(1 + \varphi t)\right]$ is bi-uniquely connected with the pattern of growth $q(t) = q_0(1 + \varphi t)^\gamma$ implied by criterion (1). For this $\tau(t)$, Proposition 3 provides the patterns of capital and extraction.

The path of effective tax can be obtained from formula (25) in Lemma 3:

$$T(t) = \exp\left[\int f_k(t)dt\right]\left\{\widehat{T} + \int \tau f_r \exp\left[-\int f_k(t)dt\right]dt\right\}.$$

Consider the following integral, given the investment rule: $\int f_k dt = \alpha \int (q/k)dt = (\alpha/\beta)\int(\dot{k}/k)dt = (\alpha/\beta)\ln k + C_1$. This expression implies $\exp\left[\int f_k dt\right] = C_2 k^{\alpha/\beta}$ and equation (24) gives $\tau f_r \exp\left[-\int f_k dt\right] = \left[\dot{q}/q\right]\left[(\beta-1)/\beta\right]\left[\beta q/r\right]k^{-\alpha/\beta}/C_2 = \left[(\beta-1)/C_2\right]q^{-1/\beta}\dot{q} = \left[\beta/C_2\right]\left[d(q^{1-1/\beta})/dt\right]$, which yields

$$T(t) = k^{\alpha/\beta}\left[\widehat{T} + \beta q^{1-1/\beta}\right], \tag{33}$$

where $\widehat{T}$ can be expressed via $T_0 = T(0)$: $\widehat{T} = T_0 k_0^{-\alpha/\beta} - \beta q_0^{1-1/\beta}$. Then, since $k^{\alpha/\beta}q^{-1/\beta} = r^{-1}$, formula (33) becomes

$$T(t) = T_0\,(k/k_0)^{\alpha/\beta} - f_r\left[(q/q_0)^{1/\beta-1} - 1\right],$$

which is the expression formulated in the proposition.

Formula (28) results from the same procedure as formula (18) derived in Appendix 1. A technical difference is that for a non-optimal investment rule ($w \equiv \beta < w^*$) the variable $z$ in ${}_2F_1$ depends on $\gamma$ and on the initial values $k_0, r_0, \dot{r}_0$, which can result in $|z| \geqslant 1$. However, there are formulas for the analytic continuation of ${}_2F_1$ for any parameters (Luke 1969, p. 69; Becken, Schmelcher 2000). These formulas are the part of major software like MAPLE, MATHEMATICA and MATLAB.

Note that the value of ${}_2F_1(\cdot)$ is 1.157 for the numerical example (Section 6) and so, taking into account the existing uncertainty in the reserve estimate, the following formula can be used in some cases as a “first-order approximation:” $s_0 = \int_0^\infty r dt = k_0^{1-\alpha} r_0^{1-\beta}(1+\gamma)/\left[\alpha-\beta-\gamma(1-\alpha)\right]$, which yields an explicit expression for $\gamma(s_0) : \gamma = \left[(\alpha-\beta)s_0 - k_0^{1-\alpha} r_0^{1-\beta}\right] / \left[(1-\alpha)s_0 + k_0^{1-\alpha} r_0^{1-\beta}\right]$. This formula captures the main qualitative properties of the behavior of the closed form solution (28). In particular, it has the same horizontal and vertical asymptotes■