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Carbon, Michel and Francq, Christian

University Rennes 2 and ENSAI, CREST (CNRS) and University Lille 3 (EQUIPPE)

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Michel Carbon[1] and Christian Francq[2]
[1]University Rennes 2 and ENSAI, France
[2]CREST (CNRS) and University Lille 3 (EQUIPPE), France

Abstract: The asymptotic distribution of a vector of autocorrelations of squared residuals is derived for a wide class of asymmetric GARCH models. Portmanteau adequacy tests are deduced. These results are obtained under moment assumptions on the iid process, but fat tails are allowed for the observed process, which is particularly relevant for series of financial returns. A Monte Carlo experiment and an illustration to financial series are also presented.

Keywords: ARCH models, Leverage effect, Portmanteau test, Goodness-of-fit test, Diagnostic checking.

1 Introduction

There exists a huge number of extensions to the initial autoregressive conditional heteroscedastic (ARCH) model introduced by Engle (1982) (see Bollerslev (2009) for an impressive list of more than one hundred of these models, and Francq and Zakoïan (2010) for a recent book on the ARCH models). The univariate ARCH-type models are generally written in the multiplicative form

$$\epsilon_t = \sigma_t \eta_t,$$

where the sequence (η_t) is independent and identically distributed (iid), η_t being independent to the σ-field $\mathcal{F}_{t-1}$ generated by $\{\epsilon_u, u < t\}$, and the so-called volatility σ_t is positive and measurable with respect to $\mathcal{F}_{t-1}$. The different ARCH specifications differ by the parametrization of the volatility $\sigma_t = \sigma(\theta_0; \epsilon_u, u < t)$. The GARCH model introduced by Bollerslev (1986) is the leading specification, but it has the important drawback of being insensible to the sign of the past returns. Black (1976) first noted that the signs of the returns are relevant because past negative returns tend to have more impact on the current volatility than past positive returns of the same magnitude. This stylized fact, known as the leverage effect, is present in many financial series. A large class of models allowing for the leverage effect is the asymmetric power GARCH model of order (p, q) (denoted as APARCH(p, q)) of Ding, Granger, and Engle (1993), defined by

$$\begin{cases} \epsilon_t = \sigma_t \eta_t \\ \sigma_t^\delta = \omega_0 + \sum_{i=1}^q \left\{ \alpha_{0i+}(\epsilon_{t-i}^+)^\delta + \alpha_{0i-}(-\epsilon_{t-i}^-)^\delta \right\} + \sum_{j=1}^p \beta_{0j}\sigma_{t-j}^\delta \end{cases} \tag{1}$$

where δ is a positive constant and, using the notation $x^+ = \max(x, 0), x^- = \min(x, 0)$, the parameter $\theta_0 = (\omega_0, \alpha_{01+}, \dots, \alpha_{0q+}, \alpha_{01-}, \dots, \alpha_{0q-}, \beta_{01}, \dots, \beta_{0p})'$ satisfies the positivity constraints $\theta_0 \in (0, \infty) \times [0, \infty)^{2q+p}$. This formulation contains the standard

GARCH and also two widely used asymmetric models: the GJR model of Glosten, Jaganathan, and Runkle (1993) for $\delta = 2$ and the Threshold ARCH (TARCH) model of Rabemananjara and Zakoïan (1993) for $\delta = 1$.

Since the seminal works of Box and Pierce (1970), Ljung and Box (1978) and McLeod (1978), portmanteau tests have been important tools in time series analysis, in particular for testing the adequacy of an estimated ARMA(p, q) model (see Section 9.4 in Brockwell and Davis (1991), and Li (2004) for an entire book devoted to the portmanteau tests). Under the null assumption that a model with iid innovations η_t is appropriate for the data at hand, the autocorrelations of the residuals $\hat{\eta}_t$ should be close to zero, which is the theoretical value of the autocorrelations of η_t. The standard portmanteau tests thus consist in rejecting the adequacy of the model for large values of some quadratic form of the residual autocorrelations.

In the GARCH framework, the portmanteau tests based on residual autocorrelations are irrelevant because the process $\hat{\eta}_t = \epsilon_t/\hat{\sigma}_t$ is always a white noise (and even a martingale difference) even when the volatility is misspecified, *i.e.* when $\epsilon_t = \sigma_t^* \eta_t$ with $\sigma_t^* \neq \sigma_t$. Li and Mak (1994) and Ling and Li (1997) proposed and studied a portmanteau test based on the autocovariances of the *squared* residuals. Their results apply to a large class of heteroscedastic time series, but they assume conditional normality and finite fourth-order moments for the observations. These assumptions are often considered as too strong for the financial series, which typically exhibit heavy tailed marginal distributions. Berkes, Horváth, and Kokoszka (2003) developed an asymptotic theory of portmanteau tests allowing for heavy tails in the standard GARCH framework (see also Theorem 8.2 in Francq & Zakoïan, 2010).

Our main aim in this paper is to extend the asymptotic theory developed by the above-mentioned authors to the wide class of the APARCH models (1). To obtain our results under weak assumptions, we exploit the recent results obtained by Hamadeh and Zakoïan (2011) on the estimation of this class of models.

2 Asymptotic distribution for quadratic forms of autocovariances of squared residuals

Let the parameter space $\Theta \subset (0, \infty) \times [0, \infty)^{2q+p}$. For all $\theta = (\theta_1, \dots, \theta_{2q+p+1})' = (\omega, \alpha_{1+}, \dots, \alpha_{q+}, \alpha_{1-}, \dots, \alpha_{q-}, \beta_1, \dots, \beta_p)'$, and for given initial values $\epsilon_0, \dots, \epsilon_{1-q}, \tilde{\sigma}_0 \geq 0, \dots, \tilde{\sigma}_{1-p} \geq 0$, we defined recursively on $t \geq 1$

$$\tilde{\sigma}_t^\delta(\theta) = \omega + \sum_{i=1}^{q} \left\{ \alpha_{i+}(\epsilon_{t-i}^+)^\delta + \alpha_{i-}(-\epsilon_{t-i}^-)^\delta \right\} + \sum_{j=1}^{p} \beta_j \tilde{\sigma}_{t-j}^\delta(\theta).$$

For ARCH-type models, the gaussian quasi-maximum likelihood estimator (QMLE) is the usual estimation procedure. Based on observations $(\epsilon_1, \dots, \epsilon_n)$ of Model (1), this estimator is solution of

$$\hat{\theta}_n = \arg\min_{\theta \in \Theta} n^{-1} \sum_{t=1}^{n} \tilde{\ell}_t(\theta), \quad \tilde{\ell}_t(\theta) = \frac{\epsilon_t^2}{\tilde{\sigma}_t^2} + \ln \tilde{\sigma}_t^2. \tag{2}$$

Hamadeh and Zakoïan (2011) showed the consistency and asymptotic normality of the QMLE under the assumption:

Assumption A: *θ_0 belongs to the interior of the compact set Θ; $E\eta_t^2 = 1$ and $\kappa_\eta = E\eta_t^4 < \infty$; $P[\eta_t > 0] \in (0,1)$ and the support of η_t has cardinality > 2; the top-Lyapounov exponent associated to Model (1) is strictly negative*[1]*; $\forall\theta \in \Theta, \quad \sum_{j=1}^{p} \beta_j < 1$; if $p > 0$, $\mathcal{B}_{\theta_0}(z) = 1 - \sum_{j=1}^{p} \beta_{0j} z^j$ has no common root with $\mathcal{A}_{\theta_0+}(z) = \sum_{i=1}^{q} \alpha_{0i+} z^i$ and $\mathcal{A}_{\theta_0-}(z) = \sum_{i=1}^{q} \alpha_{0i-} z^i$; $\mathcal{A}_{\theta_0+}(1) + \mathcal{A}_{\theta_0-}(1) \neq 0$ and $\alpha_{0q,+} + \alpha_{0q,-} + \beta_{0p} \neq 0$.*

For technical reasons, we will need to slightly reinforce the assumption on the distribution of η_t as follows:

Assumption B: *η_t takes more than 3 positive values and more than 3 negative values.*

The autocovariances of the squared residuals are defined by

$$\hat{r}_h = \frac{1}{n} \sum_{t=|h|+1}^{n} (\hat{\eta}_t^2 - 1)(\hat{\eta}_{t-|h|}^2 - 1), \qquad \hat{\eta}_t^2 = \frac{\epsilon_t^2}{\hat{\sigma}_t^2}$$

where $|h| < n$ and $\hat{\sigma}_t = \tilde{\sigma}_t(\hat{\theta}_n)$. For any fixed integer m, $1 \leq m < n$, consider the statistic $\hat{\boldsymbol{r}}_m = (\hat{r}_1, \ldots, \hat{r}_m)'$. Let $\hat{\kappa}_\eta$ and $\hat{J}$ be weakly consistent estimators of κ_η and J. For instance, one can take

$$\hat{\kappa}_\eta = \frac{1}{n} \sum_{t=1}^{n} \frac{\epsilon_t^4}{\tilde{\sigma}_t^4(\hat{\theta}_n)}, \qquad \hat{J} = \frac{4}{\delta^4} \frac{1}{n} \sum_{t=1}^{n} \frac{1}{\tilde{\sigma}_t^{2\delta}(\hat{\theta}_n)} \frac{\partial \tilde{\sigma}_t^\delta(\hat{\theta}_n)}{\partial \theta} \frac{\partial \tilde{\sigma}_t^\delta(\hat{\theta}_n)}{\partial \theta'}.$$

In the previous expression the derivatives can be recursively computed on $t > 0$ by

$$\frac{\partial \tilde{\sigma}_t^\delta(\theta)}{\partial \theta} = \underline{c}_t(\theta) + \sum_{j=1}^{p} \beta_j \frac{\partial \tilde{\sigma}_{t-j}^\delta(\theta)}{\partial \theta}, \tag{3}$$

with the additional initial values $\partial \tilde{\sigma}_t^2(\theta)/\partial\theta = 0$ for $t = 0, \ldots, 1-p$, and

$$\underline{c}_t'(\theta) = \left(1, (\epsilon_{t-1}^+)^\delta, \cdots, (\epsilon_{t-q}^+)^\delta, (-\epsilon_{t-1}^-)^\delta, \cdots, (-\epsilon_{t-q}^-)^\delta, \beta_1, \cdots, \beta_p\right). \tag{4}$$

Define also the $m \times (2q+p+1)$ matrix $\hat{C}_m$ whose element (h,k), for $1 \leq h \leq m$ and $1 \leq k \leq 2q+p+1$, is given by

$$\hat{C}_m(h,k) = -\frac{2}{\delta} \frac{1}{n} \sum_{t=h+1}^{n} (\hat{\eta}_{t-h}^2 - 1) \frac{1}{\tilde{\sigma}_t^\delta(\hat{\theta}_n)} \frac{\partial \tilde{\sigma}_t^\delta(\hat{\theta}_n)}{\partial \theta_k}.$$

Theorem 2.1 *Under Assumptions* **A** *and* **B**,

$$n \hat{\boldsymbol{r}}_m' \hat{D}^{-1} \hat{\boldsymbol{r}}_m \xrightarrow{\mathcal{L}} \chi_m^2,$$

with $\hat{D} = (\hat{\kappa}_\eta - 1)^2 I_m - (\hat{\kappa}_\eta - 1)\hat{C}_m \hat{J}^{-1} \hat{C}_m'$.

The adequacy of the APARCH(p,q) model (1) is then rejected at the asymptotic level α when

$$\left\{ n \hat{\boldsymbol{r}}_m' \hat{D}^{-1} \hat{\boldsymbol{r}}_m > \chi_m^2(1-\alpha) \right\}. \tag{5}$$

[1]This condition is necessary and sufficient for the existence of a strictly stationary solution to (1). The reader is referred to Appendix A in Hamadeh and Zakoïan (2011) for a precise definition of that top-Lyapounov exponent.

3 Monte Carlo results

We simulated $N = 1,000$ independent replications of an APARCH(1,1) for several powers δ, with parameters $\omega_0 = 0.04$, $\alpha_{01+} = 0.02$, $\alpha_{01-} = 0.13$, $\beta_{01} = 0.85$, and η_t distributed as a Student with $\nu = 9$ degrees of freedom, standardized in such a way that the variance be equal to 1. These parameters are close to those obtained when a TARCH(1,1) model (*i.e.* an APARCH(1,1) with $\delta = 1$) or a GJR(1,1) model (*i.e.* an APARCH(1,1) with $\delta = 2$) is fitted to daily stock indices (such as those considered in the next section). The length of the simulations is $n = 4,000$, which is also a current value for daily returns. Table 1 displays the empirical sizes of the portmanteau tests at the nominal level $\alpha = 5\%$. If the actual level coincides with the nominal level, the empirical size over the $N = 1,000$ independent replications should belong to the interval [3.6%, 6.4%] with probability 95%, and to the interval [3.2%, 6.9%] with probability 99%. Table 1 indicates that the error of first kind is well controlled (most of the rejection frequencies of the left array, and those of the line $\delta = 2$ in the right array, are within the 99% significant limits). In term of power performance, the portmanteau tests are more disappointing since they fail to detect alternatives of the form $\delta > 2$ when the null is $\delta = 2$ (see the right array in Table 1). Other Monte Carlo experiments, not reported here, reveal that the portmanteau tests are much more powerful to detect wrong values of the order (p, q).

Table 1: For the portmanteau tests (3), relative frequencies (in %) of rejection of an APARCH(1,1) model for several values of δ, when the DGP follows the same model (left array) and when the DGP is an APARCH(1,1) with $\delta = 2$ (right array). The nominal level is 5% and the number of replications is $N = 1,000$.

Empirical size							Empirical power (when $\delta \neq 2$)						
δ	m						δ	m					
	2	4	6	8	10	12		2	4	6	8	10	12
0.5	4.2	4.9	5.5	5.3	6.5	6.2	0.5	44.9	65.1	75.7	80.8	82.3	82.0
1	4.9	4.9	5.7	6.0	5.2	4.8	1	18.8	26.6	32.6	35.5	38.9	40.4
1.5	5.3	6.5	7.6	7.6	7.6	7.5	1.5	7.3	11.0	13.4	13.8	15.0	15.7
2	5.8	5.9	6.3	6.6	6.8	5.1	2	5.8	5.9	6.3	6.6	6.8	5.1
2.5	4.9	4.9	4.9	5.1	5.4	4.7	2.5	4.2	3.9	3.7	4.2	4.2	4.4
3	3.6	4.4	4.0	5.0	5.5	5.4	3	2.7	2.3	2.7	3.0	3.0	3.8

4 Checking the adequacy of APARCH models for stock market returns

We consider daily returns of eight major world stock indices: CAC (Paris), DAX (Frankfurt, FTSE (London), Nikkei (Tokyo), OMX (Copenhagen), SP500 (New York), SPTSX

(Toronto), and SPTSX (Shanghai). The observations cover the period from January, 2 1990 to November, 6 2010 (except for the OMX, SPTSX and SSE whose first observations are posterior to 1990). Table 2 shows that the TARCH$(0,5)$ and GJR$(0,5)$ models are generally rejected, whereas the TARCH(1,1) and GJR$(1,1)$ are only occasionally rejected.

From this empirical study and the simulation experiments made in the previous section, we draw the conclusion that the portmanteau tests based on squared TARCH(p,q) residuals constitute valuable tools to detect a misspecification of the order (p,q), but are not able to distinguish certain models with different parameters δ. In particular they do not seem to be able to make the difference between TARCH and GJR models. This is not very surprising because these two models are close and lead to similar volatility processes (see Section 10.6 in Francq and Zakoïan (2010)). Moreover, Hamadeh and Zakoïan (2011) showed that the likelihood of an APARCH model can be virtually identical at different values of the parameter δ.

5 Proofs

For all $\theta \in \Theta$, let $\tilde{\sigma}_t^\delta(\theta)$ be the strictly stationary and non-anticipative solution of

$$\sigma_t^\delta(\theta) = \omega + \sum_{i=1}^q \left\{\alpha_{i+}(\epsilon_{t-i}^+)^\delta + \alpha_{i-}(-\epsilon_{t-i}^-)^\delta\right\} + \sum_{j=1}^p \beta_j \sigma_{t-j}^\delta(\theta).$$

Note that $\sigma_t^\delta(\theta)$ and $\tilde{\sigma}_t^\delta(\theta)$ differ because of the initial values, and note that $\sigma_t^\delta = \sigma_t^\delta(\theta_0)$. Introduce also the matrix

$$J = \frac{4}{\delta^4} E\left(\frac{1}{\sigma_0^{2\delta}(\theta_0)} \frac{\partial \sigma_0^\delta(\theta_0)}{\partial \theta} \frac{\partial \sigma_0^\delta(\theta_0)}{\partial \theta'}\right).$$

Write $a \stackrel{c}{=} b$ when $a = b + c$. We denote by K a generic positive constant and by ρ a generic constant of the interval $(0,1)$. The results of Hamadeh and Zakoïan (2011) which will be needed for the proof of Theorem 2.1 are collected in the following lemma.

Lemma 5.1 *(Hamadeh & Zakoïan, 2011) Under Assumption* **A***,*

$$E|\epsilon_0|^{2s} < \infty, \quad E\sup_{\theta\in\Theta}\left\|\sigma_0^{2s}(\theta)\right\| < \infty, \quad E\sup_{\theta\in\Theta}\left\|\tilde{\sigma}_0^{2s}(\theta)\right\| < \infty \tag{6}$$

for some $s \in (0,1)$*;*

$$\sup_{\theta\in\Theta}\left\|\sigma_t^2(\theta) - \tilde{\sigma}_t^2(\theta)\right\| \le K\rho^t \sup_{\theta\in\Theta}\max\left\{\sigma_t^2(\theta), \tilde{\sigma}_t^2(\theta)\right\}, \tag{7}$$

where $K \ge 0$ *is measurable with respect to* $\{\epsilon_u, u < 0\}$ *and* $\rho \in (0,1)$*; for all* $\tau \ge 1$

$$E\left\|\sup_{\theta\in\Theta}\frac{1}{\sigma_t^\delta(\theta)}\frac{\partial\sigma_t^\delta(\theta)}{\partial\theta}\right\|^\tau < \infty, \quad E\left\|\sup_{\theta\in\Theta}\frac{1}{\sigma_0^\delta(\theta)}\frac{\partial^2\sigma_0^\delta(\theta)}{\partial\theta\partial\theta'}\right\|^\tau < \infty, \tag{8}$$

Table 2: For daily returns of stock market indices, p-values of portmanteau tests based on m squared residual autocovariances for the adequacy of the TARCH and GJR models of different orders.

	m											
	1	2	3	4	5	6	7	8	9	10	11	12
Portmanteau tests for adequacy of the TARCH$(0,5)$												
CAC	0.880	0.348	0.170	0.017	0.003	0.000	0.000	0.000	0.000	0.000	0.000	0.000
DAX	0.203	0.188	0.291	0.128	0.140	0.177	0.182	0.243	0.256	0.303	0.266	0.326
FTSE	0.468	0.125	0.015	0.009	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Nikkei	0.332	0.199	0.343	0.190	0.063	0.003	0.000	0.000	0.000	0.000	0.000	0.000
OMX	0.004	0.014	0.034	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SP500	0.319	0.094	0.005	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SPTSX	0.907	0.606	0.801	0.650	0.26	0.034	0.000	0.000	0.000	0.000	0.000	0.000
SSE	0.743	0.870	0.935	0.947	0.974	0.941	0.858	0.683	0.745	0.605	0.179	0.167
Portmanteau tests for adequacy of the GJR$(0,5)$												
CAC	0.159	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
DAX	0.047	0.011	0.012	0.005	0.004	0.007	0.01	0.017	0.023	0.035	0.035	0.052
FTSE	0.043	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Nikkei	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
OMX	0.446	0.118	0.134	0.001	0.001	0.001	0.003	0.000	0.000	0.000	0.000	0.000
SP500	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SPTSX	0.011	0.004	0.008	0.003	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SSE	0.647	0.359	0.363	0.284	0.254	0.336	0.38	0.386	0.484	0.458	0.161	0.181
Portmanteau tests for adequacy of the TARCH$(1,1)$												
CAC	0.846	0.007	0.015	0.033	0.059	0.059	0.086	0.112	0.160	0.131	0.178	0.235
DAX	0.402	0.595	0.410	0.563	0.474	0.602	0.704	0.793	0.846	0.899	0.933	0.952
FTSE	0.462	0.731	0.514	0.675	0.703	0.688	0.731	0.367	0.324	0.160	0.001	0.002
Nikkei	0.015	0.040	0.031	0.061	0.039	0.068	0.037	0.020	0.023	0.027	0.042	0.062
OMX	0.481	0.526	0.705	0.778	0.411	0.530	0.641	0.607	0.700	0.720	0.569	0.533
SP500	0.013	0.038	0.082	0.110	0.125	0.186	0.223	0.246	0.328	0.081	0.109	0.150
SPTSX	0.143	0.282	0.469	0.514	0.567	0.121	0.102	0.153	0.216	0.134	0.177	0.235
SSE	0.342	0.503	0.096	0.065	0.086	0.143	0.213	0.257	0.336	0.294	0.294	0.339
Portmanteau tests for adequacy of the GJR$(1,1)$												
CAC	0.151	0.185	0.333	0.424	0.555	0.610	0.716	0.790	0.849	0.839	0.891	0.920
DAX	0.121	0.293	0.482	0.651	0.749	0.839	0.907	0.944	0.970	0.980	0.990	0.991
FTSE	0.191	0.380	0.504	0.624	0.759	0.815	0.890	0.765	0.378	0.327	0.031	0.044
Nikkei	0.003	0.005	0.013	0.024	0.046	0.059	0.085	0.129	0.183	0.241	0.221	0.217
OMX	0.937	0.988	0.864	0.825	0.731	0.826	0.866	0.893	0.892	0.932	0.894	0.907
SP500	0.007	0.024	0.035	0.071	0.125	0.139	0.204	0.279	0.351	0.362	0.439	0.428
SPTSX	0.043	0.073	0.132	0.229	0.341	0.178	0.198	0.269	0.337	0.353	0.436	0.481
SSE	0.758	0.935	0.858	0.787	0.862	0.909	0.946	0.970	0.957	0.972	0.975	0.984

and there exists a neighborhood $\mathcal{V}(\theta_0)$ of θ_0 such that

$$E\left|\sup_{\theta\in V(\theta_0)}\frac{\sigma_0^\delta(\theta_0)}{\sigma_0^\delta(\theta)}\right|^\tau<\infty; \tag{9}$$

J is invertible and

$$\sqrt{n}(\hat{\theta}_n-\theta_0)\overset{o_P(1)}{=}J^{-1}\frac{1}{\sqrt{n}}\sum_{t=1}^n(\eta_t^2-1)\frac{1}{\sigma_t^2}\frac{\partial\sigma_t^2(\theta_0)}{\partial\theta} \tag{10}$$

Proof of Theorem 2.1

Introduce the vector $\boldsymbol{r}_m=(r_1,\dots,r_m)'$ where

$$r_h=n^{-1}\sum_{t=h+1}^n s_ts_{t-h},\qquad \text{with } s_t=\eta_t^2-1 \text{ and } 0<h<n.$$

Let $s_t(\theta)$ (respectively $\tilde{s}_t(\theta)$) be the random variable obtained by replacing η_t by $\eta_t(\theta)=\epsilon_t/\sigma_t(\theta)$ (respectively $\tilde{\eta}_t(\theta)=\epsilon_t/\tilde{\sigma}_t(\theta)$) in s_t. Let $r_h(\theta)$ (respectively $\tilde{r}_h(\theta)$) be obtained by replacing η_t by $\eta_t(\theta)$ (respectively $\tilde{\eta}_t(\theta)$) in r_h. The vectors $\boldsymbol{r}_m(\theta)=(r_1(\theta),\dots,r_m(\theta))'$ and $\tilde{\boldsymbol{r}}_m(\theta)=(\tilde{\boldsymbol{r}}_1(\theta),\dots,\tilde{\boldsymbol{r}}_m(\theta))'$ are such that $\boldsymbol{r}_m=\boldsymbol{r}_m(\theta_0)$ and $\hat{\boldsymbol{r}}_m=\tilde{\boldsymbol{r}}_m(\hat{\theta}_n)$.

We first study the asymptotic impact of the unknown initial values on the statistic $\hat{\boldsymbol{r}}_m$. We have $s_t(\theta)s_{t-h}(\theta)-\tilde{s}_t(\theta)\tilde{s}_{t-h}(\theta)=a_t+b_t$ with $a_t=\{s_t(\theta)-\tilde{s}_t(\theta)\}\,s_{t-h}(\theta)$ and $b_t=\tilde{s}_t(\theta)\{s_{t-h}(\theta)-\tilde{s}_{t-h}(\theta)\}$. Using (7) and $\inf_{\theta\in\Theta}\tilde{\sigma}_t^2\geq\inf_{\theta\in\Theta}\omega^{2/\delta}>0$, we have

$$|a_t|\leq K\rho^t\epsilon_t^2(\epsilon_{t-h}^2+1)\sup_{\theta\in\Theta}\max\left\{\sigma_t^2(\theta),\tilde{\sigma}_t^2(\theta)\right\}.$$

The c_r and Hölder inequalities, together with (6), entail that for sufficiently small $s^*\in(0,1)$,

$$E\left|\frac{1}{\sqrt{n}}\sum_{t=1}^n\sup_{\theta\in\Theta}|a_t|\right|^{s^*}\leq Kn^{-s^*/2}\sum_{t=1}^n\rho^{ts^*}\to 0$$

as $n\to\infty$. It follows that $n^{-1/2}\sum_{t=1}^n\sup_{\theta\in\Theta}|a_t|=o_P(1)$. The same convergence holds for b_t and for the derivatives of a_t and b_t. We then obtain

$$\sqrt{n}\left\|\boldsymbol{r}_m-\tilde{\boldsymbol{r}}_m(\theta_0)\right\|=o_P(1),\qquad \sup_{\theta\in\Theta}\left\|\frac{\partial\boldsymbol{r}_m(\theta)}{\partial\theta'}-\frac{\partial\tilde{\boldsymbol{r}}_m(\theta)}{\partial\theta'}\right\|=o_P(1). \tag{11}$$

We now show that the asymptotic distribution of $\sqrt{n}\hat{\boldsymbol{r}}_m$ is a function of the joint asymptotic distribution of $\sqrt{n}\boldsymbol{r}_m$ and of the QMLE. Using (11), a Taylor expansion of $\boldsymbol{r}_m(\cdot)$ around $\hat{\theta}_n$ and θ_0 shows that

$$\begin{aligned}\sqrt{n}\hat{\boldsymbol{r}}_m &= \sqrt{n}\tilde{\boldsymbol{r}}_m(\theta_0)+\frac{\partial\tilde{\boldsymbol{r}}_m(\theta^*)}{\partial\theta'}\sqrt{n}(\hat{\theta}_n-\theta_0)\\ &\overset{o_P(1)}{=} \sqrt{n}\boldsymbol{r}_m+\frac{\partial\boldsymbol{r}_m(\theta^*)}{\partial\theta'}\sqrt{n}(\hat{\theta}_n-\theta_0)\end{aligned}$$

for some θ^* between $\hat{\theta}_n$ and θ_0. In view of (9), there exists a neighborhood $\mathcal{V}(\theta_0)$ of θ_0 such that

$$\sup_{\theta\in\mathcal{V}(\theta_0)} E\left|\frac{\partial^2 s_t(\theta)s_{t-h}(\theta)}{\partial\theta_i\partial\theta_j}\right| < \infty \quad \text{for all } i,j\in\{1,\dots,2q+p+1\}.$$

Using these inequalities, (8) and the assumption $E\eta_t^4 < \infty$, the ergodic theorem, the strong consistency of the QMLE, and a second Taylor expansion, we obtain

$$\frac{\partial \boldsymbol{r}_m(\theta^*)}{\partial\theta'} \overset{o_P(1)}{=} \frac{\partial \boldsymbol{r}_m(\theta_0)}{\partial\theta'} \to C_m := \begin{pmatrix} c_1' \\ \vdots \\ c_m' \end{pmatrix},$$

where

$$c_h = E\left\{s_{t-h}\frac{\partial s_t(\theta_0)}{\partial\theta}\right\} = -E\left\{s_{t-h}\frac{1}{\sigma_t^2(\theta_0)}\frac{\partial\sigma_t^2(\theta_0)}{\partial\theta}\right\}.$$

For the next to last equality, we use the fact that $E\left\{s_t\partial s_{t-h}(\theta_0)/\partial\theta\right\} = 0$. It follows that

$$\sqrt{n}\hat{\boldsymbol{r}}_m \overset{o_P(1)}{=} \sqrt{n}\boldsymbol{r}_m + C_m\sqrt{n}(\hat{\theta}_n-\theta_0). \tag{12}$$

We now derive the asymptotic distribution of $\sqrt{n}(\boldsymbol{r}_m, \hat{\theta}_n-\theta_0)$. Note that $\boldsymbol{r}_m \overset{o_P(1)}{=} n^{-1}\sum_{t=1}^n s_t\boldsymbol{s}_{t-1:t-m}$ where $\boldsymbol{s}_{t-1:t-m} = (s_{t-1},\dots,s_{t-m})'$. In view of (10), the central limit theorem applied to the martingale difference

$$\left\{\left(s_t\frac{1}{\sigma_t^2}\frac{\partial\sigma_t^2(\theta_0)}{\partial\theta'}, s_t\boldsymbol{s}_{t-1:t-m}'\right)'; \sigma\left(\eta_u,\, u\le t\right)\right\}$$

shows that

$$\begin{aligned}\sqrt{n}\begin{pmatrix}\hat{\theta}_n-\theta_0\\ \boldsymbol{r}_m\end{pmatrix} &\overset{o_P(1)}{=} \frac{1}{\sqrt{n}}\sum_{t=1}^n s_t\begin{pmatrix} J^{-1}\frac{1}{\sigma_t^2}\frac{\partial\sigma_t^2(\theta_0)}{\partial\theta} \\ \boldsymbol{s}_{t-1:t-m}\end{pmatrix}\\ &\xrightarrow{\mathcal{L}} \mathcal{N}\left\{0, \begin{pmatrix}(\kappa_\eta-1)J^{-1} & \Sigma_{\hat{\theta}_n\boldsymbol{r}_m}\\ \Sigma_{\hat{\theta}_n\boldsymbol{r}_m}' & (\kappa_\eta-1)^2 I_m\end{pmatrix}\right\},\end{aligned} \tag{13}$$

where

$$\Sigma_{\hat{\theta}_n\boldsymbol{r}_m} = (\kappa_\eta-1)J^{-1}E\frac{1}{\sigma_t^2}\frac{\partial\sigma_t^2(\theta_0)}{\partial\theta}\boldsymbol{s}_{t-1:t-m}' = -(\kappa_\eta-1)J^{-1}C_m'.$$

Using together (12) and (13), we obtain

$$\sqrt{n}\hat{\boldsymbol{r}}_m \xrightarrow{\mathcal{L}} \mathcal{N}(0,D), \quad D = (\kappa_\eta-1)^2 I_m - (\kappa_\eta-1)C_mJ^{-1}C_m'.$$

We now show that D is invertible. Because the law of η_t^2 is non degenerated, we have $\kappa_\eta > 1$. We thus have to show the invertibility of

$$(\kappa_\eta-1)I_m - C_mJ^{-1}C_m' = E\mathbf{V}\mathbf{V}', \quad \mathbf{V} = \boldsymbol{s}_{-1:-m} + C_mJ^{-1}\frac{2}{\delta}\frac{1}{\sigma_0^\delta}\frac{\partial\sigma_0^\delta(\theta_0)}{\partial\theta}.$$

If this matrix is singular then there exists $\lambda = (\lambda_1, \ldots, \lambda_m)'$ such that $\lambda \neq 0$ and

$$\lambda' \mathbf{V} = \lambda' \boldsymbol{s}_{-1:-m} + \mu' \frac{1}{\sigma_0^\delta} \frac{\partial \sigma_0^\delta(\theta_0)}{\partial \theta} = 0 \quad \text{a.s.,} \tag{14}$$

with $\mu = (2/\delta)\lambda' C_m J^{-1}$. Note that $\mu = (\mu_1, \ldots, \mu_{2q+p+1})' \neq 0$. Otherwise $\lambda' \boldsymbol{s}_{-1:-m} = 0$ a.s., which implies that there exists $j \in \{1, \ldots, m\}$ such that s_{-j} is measurable with respect to $\sigma\{s_t,\ t \neq -j\}$. This is impossible because the s_t's are independent and non degenerated. Note that $\epsilon_t^+ = \sigma_t \eta_t^+$ and $\epsilon_t^- = \sigma_t \eta_t^-$. Denoting by R_t *any* random variable measurable with respect to $\sigma\{\eta_u,\ u \leq t\}$, and noting that (3)-(4) holds true when $\tilde{\sigma}_t$ is replaced by σ_t, we have

$$\mu' \frac{\partial \sigma_0^\delta(\theta_0)}{\partial \theta} = \mu_2 \sigma_{-1}^\delta (\eta_{-1}^+)^\delta + \mu_{q+2} \sigma_{-1}^\delta (-\eta_{-1}^-)^\delta + R_{-2},$$

and

$$\begin{aligned} \sigma_0^\delta \lambda' \boldsymbol{s}_{-1:-m} &= \left(\alpha_{01+}\sigma_{-1}^\delta (\eta_{-1}^+)^\delta + \alpha_{01-}\sigma_{-1}^\delta (-\eta_{-1}^-)^\delta + R_{-2}\right)\left(\lambda_1 \eta_{-1}^2 + R_{-2}\right) \\ &= \lambda_1 \sigma_{-1}^\delta \left\{\alpha_{01+}(\eta_{-1}^+)^{\delta+2} + \alpha_{01-}(-\eta_{-1}^-)^{\delta+2}\right\} + R_{-2}\eta_{-1}^2 + R_{-2}. \end{aligned}$$

Thus (14) entails $\lambda_1 \sigma_{-1}^\delta \left\{\alpha_{01+}(\eta_{-1}^+)^{\delta+2} + \alpha_{01-}(-\eta_{-1}^-)^{\delta+2}\right\} + R_{-2}(\eta_{-1}^+)^\delta + R_{-2}(-\eta_{-1}^-)^\delta + R_{-2}\eta_{-1}^2 + R_{-2} = 0$ a.s., which is equivalent to the two equations

$$\lambda_1 \sigma_{-1}^\delta \alpha_{01+} (\eta_{-1}^+)^{\delta+2} + R_{-2}(\eta_{-1}^+)^\delta + R_{-2}(\eta_{-1}^+)^2 + R_{-2} = 0 \quad \text{a.s.} \tag{15}$$

and

$$\lambda_1 \sigma_{-1}^\delta \alpha_{01-} (-\eta_{-1}^-)^{\delta+2} + R_{-2}(-\eta_{-1}^-)^\delta + R_{-2}(-\eta_{-1}^-)^2 + R_{-2} = 0 \quad \text{a.s.} \tag{16}$$

Note that an equation of the form $a|x|^{\delta+2} + b|x|^\delta + cx^2 + d = 0$ cannot have more than 3 positive roots or more than 3 negative roots, except if $a = b = c = d = 0$. By Assumption **B**, Equations (15) and (16) thus imply $\lambda_1(\alpha_{01+} + \alpha_{01-}) = 0$. Let $\lambda'_{2:m} = (\lambda_2, \ldots, \lambda_m)'$. If $\lambda_1 = 0$ then (14) implies

$$\begin{aligned} &\left(\alpha_{01+}\sigma_{-1}^\delta (\eta_{-1}^+)^\delta + \alpha_{01-}\sigma_{-1}^\delta (-\eta_{-1}^-)^\delta\right) \lambda'_{2:m} \boldsymbol{s}_{-2:-m} \\ &= \mu_2 \sigma_{-1}^\delta (\eta_{-1}^+)^\delta + \mu_{q+2}\sigma_{-1}^\delta (-\eta_{-1}^-)^\delta + R_{-2} \quad \text{a.s.,} \end{aligned}$$

which is equivalent to

$$\alpha_{01+}\sigma_{-1}^\delta (\eta_{-1}^+)^\delta \lambda'_{2:m} \boldsymbol{s}_{-2:-m} = \mu_2 \sigma_{-1}^\delta (\eta_{-1}^+)^\delta + R_{-2} \quad \text{a.s.,}$$

and a similar equation involving $(-\eta_{-1}^-)^\delta$. Subtracting the conditional expectation with respect to $\sigma\{\eta_t,\ t \leq -2\}$ in both sides of the previous equation, we obtain

$$(\alpha_{01+}\lambda'_{2:m}\boldsymbol{s}_{-2:-m} - \mu_2)\left\{(\eta_{-1}^+)^\delta - E(\eta_{-1}^+)^\delta\right\} = 0 \quad \text{a.s.}$$

which entails $\alpha_{01+} = \mu_2 = 0$. Symmetrically, $\alpha_{01-} = 0$. For APARCH$(p, 1)$ models, it is impossible to have $\alpha_{01+} = \alpha_{01-} = 0$, because of the assumption $\mathcal{A}_{\theta_0+}(1) + \mathcal{A}_{\theta_0-}(1) \neq 0$. The invertibility of D is thus shown in this case. In the general case, we show by induction that (14) entails $\alpha_{01+} + \alpha_{01-} = \cdots = \alpha_{0p+} + \alpha_{0p-} = 0$.

It is easy to show that $\hat{D} \to D$ in probability (and even almost surely) as $n \to \infty$ The conclusion follows $\square$

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Author's addresses:

Michel Carbon, 2450, rue de bilbao, app.201, Québec (Québec) G2C 0E4, Canada

Christian Francq, Université Lille 3, UFR MSES, BP 60149, 595653 Villeneuve d'Ascq, France