Two-agent Nash implementation with partially-honest agents: Almost Full Characterizations

Lombardi, Michele

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Michele Lombardi†

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Abstract

In a two-agent society with partially-honest agents, we extend Dutta and Sen (2009)'s results of Nash implementation to the domain of weak orders. We identify the class of Nash implementable social choice correspondences with a "gap" between necessary and sufficient conditions, both when exactly one agent is partially-honest and when both agents are partially-honest. We also show that, on the domain of linear orders, the "gap" between our conditions gets closed and they become equivalent to those devised by Dutta and Sen. New implementing mechanisms are devised. In line with earlier works, the classic condition of monotonicity is no longer required, whereas a weak version of the standard punishment condition is required even when both agents are known to be partially-honest. We derive simpler sufficient conditions that are satisfied in a wide range of applications.

JEL classification: C72; D71.
Keywords: Two-agent Nash implementation, intrinsic preferences for honesty, permissive results.

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†Department of Quantitative Economics, Maastricht University, P.O. Box 616, NL-6200 MD Maastricht, Netherlands, phone: 0031-43-3883-761, fax: 0031-43-3882-000, e-mail: m.lombardi@maastrichtuniversity.nl.

1 Introduction

The (Nash) implementation problem consists in investigating the existence of a mechanism whose (Nash) equilibrium outcomes coincide with the optimal outcomes set by a given social choice correspondence (SCC), that is, SCC-optimal.[1] If such a mechanism exists, it is said to (Nash) implement the SCC. Seminal papers on two-agent implementation are those of Moore and Repullo (1990) and Dutta and Sen (1991) who independently refined Maskin's characterization result (Maskin, 1999)[2] by providing necessary and sufficient conditions for an SCC to be implementable. Notable parts of these conditions are the monotonicity condition and the punishment condition, as they limit our ability to implement SCCs more than others.

The two-agent implementation problem with partially-honest agents has recently been analyzed by Dutta and Sen (2009) on the assumption that agents' preferences are linear orders.[3] Their remarkable contribution is that the scope of implementation is enlarged as the stringent condition of monotonicity is not longer required, no even in the more problematic case of two agents. This paper extends their analysis to the domain of weak orders in view of its potential applications to bargaining and negotiating. It almost fully identifies the class of implementable SSCs, not only in the case that both agents are partially-honest but also in the more subtle case that exactly one agent is partially-honest.

Following Maskin (1999), the paper assumes that the message conveyed by each agent to the mechanism designer involves the announcement of a preference profile (i.e., agents' preferences over outcomes). A message is truthful if it involves the announcement of the true preference profile. Moreover, it retains Dutta and Sen (2009)'s idea that a *partially-honest agent* is an agent who strictly prefers to announce a truthful message rather than an untruthful one when the former (given a message announced by

[1] For excellent introductions to the theory of implementation see for instance Jackson (2001) and Maskin and Sjöström (2002).

[2] The first version appeared in 1977.

[3] The role of partial honesty has also been recently analyzed by Matsushima (2008) in a different set-up when there are more than two agents.

the other agent) produces an outcome which is at least as good as the one that would be achieved if the agent lied (keeping constant the other agent's message). To give an example, suppose that agent h is partially-honest and she believes that the other agent i will send the message m_i. Suppose that m_h is the truthful message of agent h while m'_h is the untruthful one. Suppose that the message profile (m_h, m_i) results in the outcome y, whereas the message profile (m'_h, m_i) leads to a different outcome x. Let the agent h be indifferent between x and y on the basis of her true preferences. Unlike an agent that is concerned solely with her outcomes, the agent h strictly prefers (m_h, m_i) to (m'_h, m_i). A different way of describing a partially-honest agent is that the agent at issue has preferences over message profiles in which she displays concerns for two dimensions in lexicographic order: (1) her outcome and (2) her truth-telling behavior.

The identification of implementable SCCs becomes more complicated and subtle when weak orders are considered. For instance, even the unanimity property which is satisfied by all SCCs used in social choice literature is violated on the domain of weak orders.[4] To explain this aspect, let us continue with our previous example by assuming that agents i and h rank outcomes x and y most highly and the SCC is Nash implementable with partially-honest agents. For simplicity's sake, assume that only agent h is partially-honest. Then, we have a mechanism and a corresponding message profile leading to x. Suppose that this message profile is (m'_h, m_i). When the preference domain consists only of linear orders, we deduce that $x = y$ (by the antisymmetry of the agents' preferences). Since the agent h strictly prefers (m_h, m_i) to (m'_h, m_i) and there cannot be any further profitable deviation, we can conclude that x is SCC-optimal. Notably, the antisymmetry of the agents' preferences is precisely what enables Dutta and Sen to employ some of the standard two-agent implementing conditions to fully identify the class of implementable SCCs. When indifference is permitted, however, outcomes x and y can be different outcomes. Since the agent h strictly prefers (m_h, m_i) to (m'_h, m_i) and the former message profile leads to y, we can no

[4] An SCC that satisfies *unanimity* if it selects alternatives that are ranked most highly by all agents.

longer conclude that x is SCC-optimal.

In the framework developed by Moore and Repullo (1990), we therefore devise new implementing conditions with a small "gap" between necessary and sufficient conditions. Our conditions are much weaker than the standard two-agent implementing conditions. In line with Dutta and Sen (2009), the condition of monotonicity is not longer required, whereas the condition guaranteeing the existence of the punishment outcome is required even in the case that both agents are known to be partially-honest. It may be worthwhile to emphasize that the "gap" between our conditions gets closed when the domain of preferences consists only of linear orders and, more importantly, they reduce to the necessary and sufficient conditions employed by Dutta and Sen (2009) on this domain.

The notable part of our necessity properties is the punishment condition, as all other parts incorporate weak versions of no-veto power and unanimity.[5] While the first part of this condition guarantees the existence of the punishment outcome the second part involves weak versions of the standard punishment condition. They state that if - under some requirements - the punishment outcome is an equilibrium outcome, it should be SCC-optimal. On the domain of linear orders, this second part is always satisfied (vacuously).

Our necessary conditions, however, are too weak to ensure the implementability of SCCs. Then, when both agents are partially-honest, we show that a slight strengthening of our conditions is sufficient for a full implementation. When exactly one agent is partially-honest and the domain of preferences is sufficiently rich, a full implementation is again ensured by a slight strengthening of the corresponding necessary conditions. Examples of preference domains which satisfy our richness condition would be the set of all profiles of weak orders, linear orders and single peaked preferences. It may be worth considering here that we also devise new implementing mechanisms in between those devised by Moore and Repullo (1990) and Busetto and Codognato (2009). Our mechanisms turn up to be an improve-

[5] An SCC that satisfies *no-veto power* must select an alternative if it is ranked most highly by all but at most one agent.

ment upon Dutta and Sen (2009)'s mechanisms in the sense that ours are endowed with a lower dimensional strategic space.

Given that all our conditions will be stated in choice-theoretical language, we also examine conditions and specify domain restrictions which allow us to give a quick answer to the question of implementability.

To begin with, we suppose the existence of a "bad outcome". On this supposition, we show that, on the one hand, an SCC satisfying the condition of restricted veto power is implementable when exactly one agent is partially-honest, whereas, on the other, when both agents are partially-honest, an SCC is implementable if it satisfies unanimity and a condition considerably weaker than restricted veto power. In this set-up, for instance, the class of non-monotonic and strong individually rational bargaining solutions such as the *Nash bargaining solution* defined on the class of utility possibility sets is implementable with partially-honest agents by setting the disagreement point as a bad outcome.[6]

Second, we consider the restriction on the domain of preferences introduced by Busetto and Codognato (2009). On this domain, we show that an SCC is implementable if it satisfies restricted veto power and non-empty lower intersection when exactly one agent is partially-honest, whereas unanimity and non-empty lower intersection suffice to ensure the implementability of SCCs when both agents are partially-honest. Consider a two-agent exchange economy with at least two divisible goods in which agents have continuous and strict monotonic preferences and in which indifference curves never touch the axes. Suppose that the SCC selects only interior allocations of the feasible set. In this set-up many interesting non-monotonic SCCs satisfy restricted veto power - and so unanimity - and non-empty lower intersection. A special instance of our results is that the $\omega-$*efficient-egalitarian correspondence* is implementable with partially-honest agents.

The paper is organized as follows. Section 2 describes the formal environment. Section 3 reports our analysis when exactly one agent is partially-honest, whereas Section 4 reports the analysis when both agents are partially-honest. Section 5 reports the implications of our results. Section 6 concludes.

[6] See Vartiainen (2007).

2 The implementation problem

The set of outcomes is denoted by X and the set of agents is $N = \{1, 2\}$. The cardinality of X is $\#X \geq 2$. Let $\mathcal{R}(X)$ be the set of all weak orders on X.[7] Let $\mathcal{R}_i \subseteq \mathcal{R}(X)$ be the (non-empty) set of all admissible weak orders of agent $i \in N$.[8] Let $\mathcal{R}^2 \equiv \mathcal{R}_1 \times \mathcal{R}_2$ be the set all admissible weak order profiles (or states). An generic element of $\mathcal{R}^2$ is denoted by R, where its ith component is $R_i \in \mathcal{R}_i$, $i \in N$. The symmetric and asymmetric factors of any $R_i \in \mathcal{R}_i$ are, in turn, denoted P_i and I_i, respectively. Let $\mathcal{P}^2 \subseteq \mathcal{R}^2$ be the set of all admissible profiles of linear orders.[9] Let $L(R_i, x)$ denote agent i's lower contour set at $(R_i, x) \in \mathcal{R}_i \times X$, that is, $L(R_i, x) \equiv \{y \in X | (x, y) \in R_i\}$. Let $SL(R_i, x)$ denote agent i's strict lower contour set at $(R_i, x) \in \mathcal{R}_i \times X$, that is, $SL(R_i, x) \equiv \{y \in X | (x, y) \in P_i\}$. For any $R_i \in \mathcal{R}_i$, let $\max_{R_i} X$ be the set of maximal alternatives according to R_i, that is, $\max_{R_i} X := \{x \in X | (x, y) \in R_i \text{ for all } y \in X\}$.

A social choice correspondence (SCC) on $\mathcal{R}^2$ is a correspondence $F : \mathcal{R}^2 \rightrightarrows X$ with $\emptyset \neq F(R) \subseteq X$ for all $R \in \mathcal{R}^2$. An SCC F is *monotonic* if, for all $R, R' \in \mathcal{R}^2$ with $x \in F(R)$, $x \in F(R')$ whenever $L(R_i, x) \subseteq L(R'_i, x)$ for all $i \in N$.

A mechanism is a pair (M, g), where $M \equiv M_1 \times M_2$, with each M_i being a (non-empty) set, and $g : M \to X$. It consists of a message space M, where M_i is the message space for agent $i \in N$, and an outcome function g. Let $m_i \in M_i$ denote a generic message (or strategy) for agent i. A message profile is denoted $m \equiv (m_1, m_2) \in M$. Given a mechanism (M, g) and an $R \in \mathcal{R}^2$, the set of truthful messages for agent $i \in N$, denoted $T_i(R, F)$, is a correspondence $T_i : \{R\} \times S_i \rightrightarrows M_i$ such that $\emptyset \neq T_i(R, F) \subseteq M_i$, where S_i represents the other components of M_i.

[7] A weak order is a complete and transitive binary relation. A relation R on X is *complete* if, for all $x, x' \in X$, $(x, x') \in R$ or $(x', x) \in R$; *transitive* if, for all $x, x', x'' \in X$, if $(x, x') \in R$ and $(x', x'') \in R$, then $(x, x'') \in R$.

[8] The weak set inclusion is denoted by $\subseteq$.

[9] A linear order is a complete, transitive, and antisymmetric binary relation. A binary relation R on X is antisymmetric if, for all $x, x \in X$, $x = x'$ whenever $(x, x') \in R$ and $(x', x) \in R$.

For any $i \in N$ and $R \in \mathcal{R}^2$, let $\succcurlyeq_i^R$ be agent i's weak order over M under the state R. The asymmetric factor of $\succcurlyeq_i^R$ is denoted $\succ_i^R$, while the symmetric part is denoted $\sim_i^R$. Given a mechanism (M, g) and an $R \in \mathcal{R}^2$, (M, g, R) represents a (noncooperative) game. An agent $i \in N$ is a partially-honest agent if, for all games (M, g, R), for all $(m_i, m_j), (m_i', m_j) \in M$, with $i \neq j$:

(i) for any $m_i \in T_i(R, F)$ and $m_i' \notin T_i(R, F)$, if $(g(m_i, m_j), g(m_i', m_j)) \in R_i$, then $((m_i, m_j), (m_i', m_j)) \in \succ_i^R$;
(ii) otherwise, $(g(m_i, m_j), g(m_i', m_j)) \in R_i$ if and only if $((m_i, m_j), (m_i', m_j)) \in \succcurlyeq_i^R$.

Given an agent $i \in N$, if she is not partially-honest then, for each game (M, g, R), for all $m, m' \in M$:

$$\left(m, m'\right) \in \succcurlyeq_i^R \text{ if and only if } \left(g(m), g\left(m'\right)\right) \in R_i.$$

A mechanism (M, g) induces a class of (non-cooperative) games $\left\{\left(M, g, \succcurlyeq^R\right) | R \in \mathcal{R}^2\right\}$. Given a game $\left(M, g, \succcurlyeq^R\right)$, we say that $m = (m_i, m_j) \in M$ is a (pure strategy) Nash equilibrium at R if and only if, for all $i \in N$, $(m, (m_i', m_j)) \in \succcurlyeq_i^R$ for all $m_i' \in M_i$, with $i \neq j \in N$. Let $\mathbf{N}\left(M, g, \succcurlyeq^R\right)$ denote the set of Nash equilibria message profiles of $\left(M, g, \succcurlyeq^R\right)$, whereas $\mathbf{N}_g\left(M, g, \succcurlyeq^R\right)$ represents the corresponding set of Nash equilibrium outcomes.

The mechanism (M, g) is said to partially-honest implement F on $\mathcal{R}^2$ (in Nash equilibria) if and only if

$$\mathbf{N}_g\left(M, g, \succcurlyeq^R\right) = F(R) \text{ for all } R \in \mathcal{R}^2.$$

If such mechanism exists, then F is partially-honest (Nash) implementable.

Following Dutta and Sen (2009) we use the following informational assumptions throughout the paper.

Assumption A1 (for short, *A1*). There exists exactly one partially-honest agent in N. This is known to the mechanism designer. However, the identity of this agent is unknown to the mechanism designer.

Assumption A2 (for short, *A2*). Both agents are partially-honest. This is known to the mechanism designer.

The next condition basically requires that the class of admissible preferences is sufficiently rich. Examples of preferences domains satisfying the condition would be the set of all profiles of weak orders, linear orders, and single peaked preferences on X. Hence, our models are applicable to economic environments.

Condition D (for short, *D*). For any $i \in N$, $R \in \mathcal{R}^2$ and $x \in X$, $(R_i', R_j) \in \mathcal{R}^2$ whenever $R_i' \in \mathcal{R}_i(X)$ is such that $L(R_i', x) = L(R_i, x)$, with $\partial L(R_i', x) = \{x\}$.[10]

This condition has been used by Lombardi and Yoshihara (2010b) to provide almost full characterizations of partially-honest implementable SCCs in the case of "more than two agents" with strategy space reduction.[11]

3 Exactly one partially-honest agent

In this section we make the informational assumption that there is exactly one partially-honest agent; the mechanism designer is aware of this fact but ignores the identity of the partially-honest agent (*A1*). We begin by proving that if an SCC F is partially-honest implementable, then it must satisfy Condition $\mu 2^*$ below. Although such a condition is quite complex, it is in fact very weak.

Definition 1. An SCC F on $\mathcal{R}^2$ satisfies Condition $\mu 2^*$ if there exists $Y \subseteq X$, and, for each $i \in N$, $R \in \mathcal{R}^2$ and $x \in F(R)$, there is a set $C_i(R, x)$ such that $x \in C_i(R, x) \subseteq L(R_i, x) \cap Y$. Moreover, for each $R^* \in \mathcal{R}^2$, we have:

(i) (a) For each $(x, R, x', R') \in X \times \mathcal{R}^2 \times X \times \mathcal{R}^2$, with $x \in F(R)$ and $x' \in F(R')$, there is $e \equiv e(x, R, x', R') \in C_1(R, x) \cap C_2(R', x')$, with $e(x, R, x, R) = x$.

(b) If $R = R' = R^*$, $(x, R) \neq (x', R')$, $(e, x) \in I_1^*$ and $(e, x') \in I_2^*$, then $e \in F(R^*)$.

[10] $\partial L(R_i', x) = \{x\}$ means that, for all $x' \in L(R_i', x)$, with $x \neq x'$, $(x, x') \in P_i'$.

[11] See, for instance, Lombardi and Yoshihara (2010a) and the literature cited therein.

(c) If $x \neq x'$, $C_1(R,x) \subseteq L(R_1^*, e)$, $C_2(R', x') \subseteq L(R_2^*, e)$, and: (1) if $R = R^* \neq R'$, $e = x$ and $(e, \hat{x}) \notin I_1^*$ for all $\hat{x} \in C_1(R,x) \setminus \{e\}$, then $e \in F(R^*)$; (2) if $R \neq R^* = R'$, $e = x'$ and $(e, \hat{x}) \notin I_2^*$ for all $\hat{x} \in C_2(R', x') \setminus \{e\}$, then $e \in F(R^*)$.

(ii) If $y \in C_i(R,x) \subseteq L(R_i^*, y)$ and $Y \subseteq L\left(R_j^*, y\right)$ for some $i, j \in N$, with $i \neq j$, and for all $\hat{x} \in C_i(R,x) \setminus \{y\} : (y, \hat{x}) \notin I_i^*$, for all $\hat{x} \in Y \setminus \{y\} : (y, \hat{x}) \notin I_j^*$, then $y \in F(R^*)$.

(iii) If $z \in Y \subseteq L(R_i^*, z)$ for all $i \in N$, and for each $i \in N$, for all $\hat{x} \in Y \setminus \{z\}$, $(\hat{x}, z) \notin I_i^*$, then $z \in F(R^*)$.

We can state our first result as follows.

Proposition 1. Let *A1* hold. A partially-honest implementable SCC F on $\mathcal{R}^2$ satisfies Condition $\mu 2^*$.

Proof. Let *A1* hold and let $h \in N$ be the partially-honest agent. Let (M, g) be a mechanism which partially-honest implements F on $\mathcal{R}^2$. Let $Y \equiv g(M)$. Take any $R \in \mathcal{R}^2$ and any $x \in F(R)$. Then, there is an equilibrium strategy $m^*(x,R) \equiv \left(m_i^*(x,R), m_j^*(x,R)\right) \in \mathbf{N}\left(M, g, \succcurlyeq^R\right)$ such that $g(m^*(x,R)) = x$. Moreover, $\{x\} \subseteq g\left(M_i, m_j^*(x,R)\right) \subseteq L(R_i, x) \cap Y$ for any $i \in N$. This is true even for h. In fact, if $m_h^*(x,R) \notin T_h(R,F)$, $m^*(x,R) \in \mathbf{N}\left(M, g, \succcurlyeq^R\right)$ implies that $(g(m^*(x,R)), g(m_h', m_\ell^*(x,R))) \in P_h$ for any $m_h' \in T_h(R,F)$, with $h \neq \ell \in N$. For $i, j \in N$, with $i \neq j$, let $C_i(R,x) \equiv g\left(M_i, m_j^*(x,R)\right)$ for all $i \in N$.

Take any $R^* \in \mathcal{R}^2$ and $(x, R, x', R') \in X \times \mathcal{R}^2 \times X \times \mathcal{R}^2$, with $x \in F(R)$ and $x' \in F(R')$. Let $e \equiv e(x, R, x', R') = g(m_1^*(x', R'), m_2^*(x, R))$ where $m_1^*(x', R')$ is the equilibrium message of agent 1 supporting x' as an equilibrium strategy when the state is R'. Then, $e \in C_1(R,x) \cap C_2(R', x')$, and so Condition $\mu^* 2$(i.a) is met. It is also clear that F meets Condition $\mu^* 2$(i.b) as every agent is truthful and e is optimal at state R^*. Moreover, it is also obvious that F satisfies Condition $\mu^* 2$(i.c) as, for instance, in the case $\mu^* 2$(i.c.1) the only deviator can be agent 1 whenever she is partially-honest, but her deviation to the truthful message results in the same outcome e as $(e, \hat{x}) \notin I_1^*$ for all $\hat{x} \in C_1(R,x) \setminus \{e\}$.

Take any $R^* \in \mathcal{R}^2$. Let $y \in C_i(R,x) \subseteq L(R_i^*, y)$ and $Y \subseteq L\left(R_j^*, y\right)$, for $i,j \in N$, with $i \neq j$, $(y,\hat{x}) \notin I_i^*$ for all $\hat{x} \in C_i(R,x) \setminus \{y\}$ and $(y,\hat{x}) \notin I_j^*$ for all $\hat{x} \in Y \setminus \{\hat{x}\}$. We show $y \in F(R^*) = N_g\left(M, g, \succcurlyeq^{R^*}\right)$. Assume, to the contrary, that $y \notin \mathbf{N}_g\left(M, g, \succcurlyeq^{R^*}\right)$. Let $g(m) = y$ with $m_j = m_j^*(x,R)$. It follows that $m \notin \mathbf{N}\left(M, g, \succcurlyeq^{R^*}\right)$. By our suppositions it follows that the only deviator is the partially-honest agent $h \in N$. Suppose $h = i$. Then, $m_h \notin T_h(R^*, F)$ and there is an $m_h' \in T_h(R^*, F)$ such that $g(m_h', m_j) = y$ and $((m_h', m_j), m) \in \succ_i^{R^*}$. Since there cannot be any further deviation, it follows that $y \in \mathbf{N}_g\left(M, g, \succcurlyeq^{R^*}\right) = F(R^*)$ which yields a contradiction. Similar argument applies if $h = j$. We conclude that F meets Condition μ^*2(ii).

Take any $R^* \in \mathcal{R}^2$ such that, for each $i \in N$, $z \in Y \subseteq L(R_i^*, z)$ and $(\hat{x}, z) \notin I_i^*$ for all $\hat{x} \in Y \setminus \{z\}$. As $z \in Y$ it follows that $g(m) = z$ for some $m \in M$. We show that $z \in F(R^*)$. Assume, to the contrary, that $z \notin F(R^*) = \mathbf{N}_g\left(M, g, \succcurlyeq^{R^*}\right)$, so that $m \notin \mathbf{N}\left(M, g, \succcurlyeq^{R^*}\right)$. As for each $i \in N$ we have $(\hat{x}, z) \notin I_i^*$ for all $\hat{x} \in Y \setminus \{z\}$ it follows that the only deviator is the partially-honest agent h. Without loss of generality, let $h = 1$. Then, $m_1 \notin T_1(R^*, F)$ and there is an $m_1' \in T_1(R^*, F)$ such that $g(m_1', m_2) = z$. Then, $((m_1', m_2), m) \in \succ_1^{R^*}$. Since there cannot be any further deviation, it follows that $z \in \mathbf{N}_g\left(M, g, \succcurlyeq^{R^*}\right) = F(R^*)$, a contradiction. We conclude that F meets Condition μ^*2(iii). □

We now turn to sufficient conditions under which an SCC F can be partially-honest implemented. The conditions are stated below.

Definition 2. An SCC F on $\mathcal{R}^2$ satisfies Condition $\mu 2^{**}$ if there exists $Y \subseteq X$, and, for each $i \in N$, $R \in \mathcal{R}^2$ and $x \in F(R)$, there is a set $C_i(R,x)$ such that $x \in C_i(R,x) \subseteq L(R_i, x) \cap Y$. Moreover, for each $R^* \in \mathcal{R}^2$, we have:

(i) (a) For each $(x, R, x', R') \in X \times \mathcal{R}^2 \times X \times \mathcal{R}^2$, with $x \in F(R)$ and $x' \in F(R')$, there is $e \equiv e(x, R, x', R') \in C_1(R,x) \cap C_2(R', x')$, with $e(x, R, x, R) = x$; and

(b) If $R = R' = R^*$, $(x,R) \neq (x', R')$, $(e,x) \in I_1^*$ and $(e, x') \in I_2^*$, then $e \in F(R^*)$.

(c) If $R^* \in \{R, R'\}$, $R \neq R'$, $x \neq x'$, $C_1(R,x) \subseteq L(R_1^*, e)$, $C_2(R', x') \subseteq L(R_2^*, e)$, then $e \in F(R^*)$.

(ii) If $y \in C_i(R,x) \subseteq L(R_i^*, y)$ and $Y \subseteq L\left(R_j^*, y\right)$, for $i, j \in N$, with $i \neq j$, and [for all $\hat{x} \in C_i(R,x) \setminus \{y\} : (y, \hat{x}) \notin I_i^*$, for all $\hat{x} \in Y \setminus \{y\} : (y, \hat{x}) \notin I_j^*$] or [$L(R_\ell, y) = L(R_\ell^*, y)$ for all $\ell \in N$], then $y \in F(R^*)$.
(iii) If $z \in Y \subseteq L(R_i^*, z)$ for all $i \in N$, and for each $i \in N$, $(\hat{x}, z) \notin I_i^*$ for all $\hat{x} \in Y \setminus \{z\}$, then $z \in F(R^*)$.

Although this condition is rather long and complex, it is in fact weak. Condition $\mu 2$ (Moore and Repullo, 1990) is stronger than Condition $\mu 2^{**}$. In particular, the class of SCCs that satisfy our condition is larger than the one satisfying Condition $\mu 2$. One reason is that the class of SCCs satisfying Condition $\mu 2^{**}$ does contain non-monotonic SCCs.

It is clear that Condition $\mu 2^{**}$ implies Condition $\mu 2^{*}$. The exact difference between these two conditions is that Condition $\mu 2^{**}$(i.c) and Condition $\mu 2^{**}$(ii) are stronger than respectively Condition $\mu 2^{*}$(i.c) and Condition $\mu 2^{*}$(ii).

We show that Condition $\mu 2^{**}$ is sufficient to ensure the partially-honest implementability of SCCs if $\mathcal{R}^2$ satisfies the richness Condition D and the informational assumption *A1* holds. To obtain this result we devise a new mechanism that is in between the mechanism devised by Moore and Repullo (1990) and the one devised by Busetto and Codognato (2009). Each agent message space has the following specification: for $i \in N$,

$$M_i := \left\{ \left(R^i, x^i, y^i, k^i\right) \in \mathcal{R}^2 \times X \times Y \times \mathbb{Z}_+ : x^i \in F\left(R^i\right) \right\}, \qquad (1)$$

where $\mathbb{Z}_+$ is the set of nonnegative integers. Our message space specification is smaller than that specified by Dutta and Sen (2009) as it does not include the set consisting of the two elements "flag" and "no flag", that is, $\{F, NF\}$. Our next result also shows that the strategic choice in the set $\{F, NF\}$ can be omitted.

Proposition 2. Let D and *A1* hold. An SCC F on $\mathcal{R}^2$ is partially-honest implementable if it satisfies Condition $\mu 2^{**}$.

Proof. Let D and *A1* hold. Let $h \in N$ be the partially-honest agent. Suppose that F on $\mathcal{R}^2$ meets Condition $\mu 2^{**}$. Let (M, g) be a mechanism. For each $i \in N$, let the message space of agent i be that defined in (**??**).

Define the outcome function $g : M \to X$ as follows: For any $m \in M$,

Rule 1: If $\left(R^1, x^1\right) = \left(R^2, x^2\right)$ and $k^1 = k^2 = 0$, then $g(m) = x^1$.
Rule 2: If $k^1 > k^2 = 0$, then

$$g(m) = \begin{cases} y^1 & \text{if} \quad y^1 \in C_1\left(R^2, x^2\right) \\ e \equiv e\left(x^2, R^2, x^1, R^1\right) & \text{otherwise.} \end{cases}$$

Rule 3: If $k^2 > k^1 = 0$, then

$$g(m) = \begin{cases} y^2 & \text{if} \quad y^2 \in C_2\left(R^1, x^1\right) \\ e \equiv e\left(x^2, R^2, x^1, R^1\right) & \text{otherwise.} \end{cases}$$

Rule 4: If $\left(R^1, x^1\right) \neq \left(R^2, x^2\right)$ and $k^1 = k^2 = 0$, then

$$g(m) = \begin{cases} x^1 & \text{if} \quad x^1 = x^2 \\ e \equiv e\left(x^2, R^2, x^1, R^1\right) & \text{otherwise.} \end{cases}$$

Rule 5: If $k^1 \geq k^2 > 0$, then, $g(m) = y^1$.
Rule 6: Otherwise, $g(m) = y^2$.

We show that (M, g) partially-honest implements F. For, let $R \in \mathcal{R}^2$.

Since F satisfies Condition $\mu 2^{**}$, $F\left(\mathcal{R}^2\right) \subseteq Y$. Thus, for any $R \in \mathcal{R}^2$ and any $x \in F(R)$, $x \in Y$.

To show that $F(R) \subseteq \mathbf{N}_g\left(M, g, \succcurlyeq^R\right)$, let $x \in F(R)$ and suppose that, for all $\ell \in N$, $m_\ell = (R, x, x, 0) \in M_\ell$. *Rule 1* implies that $g(m) = x$. By the definition of g we have that any deviation of agent $i \in N$ will get him to an outcome in $C_i(R, x)$, so that $g(M_i, m_j) \subseteq C_i(R, x)$. Since $C_i(R, x) \subseteq L(R_i, x)$, such deviations are not profitable. As every agent is truthful as well, it follows that $x \in \mathbf{N}_g\left(M, g, \succcurlyeq^R\right)$.

Conversely, to show that $\mathbf{N}_g\left(M, g, \succcurlyeq^R\right) \subseteq F(R)$, let $m \in \mathbf{N}\left(M, g, \succcurlyeq^R\right)$. Consider the following cases.

Case 1: m corresponds to Rule 1.

Suppose that m falls into *Rule 1*. Then, $g(m) = x^1$. By the definition of g it follows that $m_h \in T_h(R, F)$. For, assume, to the contrary, that $m_h \notin T_h(R, F)$. Let $h = 1$. Then, by changing m_h with $m_h' = \left(R, x^h, x^1, k^h\right) \in T_h(R, F)$, with $x^h \in F(R)$ and $k^h > 0$, agent h induces *Rule 2* and obtains

$x^1 = g(m'_h, m_2) \in C_h(R^2, x^2)$. Therefore, $((m'_h, m_{-h}), m) \in \succ_h^R$ which contradicts that $m \in \mathbf{N}(M, g, \succcurlyeq^R)$. The same reasoning applies if $h = 2$ given that he can induce *Rule 3*. It follows that $x^1 \in F(R)$, as we sought.

Case 2: m corresponds to *Rule 2.*

Then, $g(m_1, M_2) = Y$ and $C_1(R^2, x^2) \subseteq g(M_1, m_2)$. Moreover, since $m \in \mathbf{N}(M, g, \succcurlyeq^R)$ it follows that $C_1(R^2, x^2) \subseteq L(R_1, g(m))$ and $Y \subseteq L(R_2, g(m))$. By the definition of g we have that $m_h \in T_h(R, F)$. As D holds, there exists $\hat{R} \in \mathcal{R}^2$ such that $L(R_\ell, g(m)) = L\left(\hat{R}_\ell, g(m)\right)$, with $\partial L\left(\hat{R}_\ell, g(m)\right) = \{g(m)\}$, for all $\ell \in N$. Condition $\mu 2^{**}$(ii) implies that $g(m) \in F\left(\hat{R}\right)$. Since F satisfies Condition $\mu 2^{**}$ there exists a profile $\left(C_\ell\left(\hat{R}, g(m)\right)\right)_{\ell \in N}$ such that $C_\ell\left(\hat{R}, g(m)\right) \subseteq L\left(\hat{R}_\ell, g(m)\right) \cap Y$ for any $\ell \in N$. As, by construction, $L(R_\ell, g(m)) = L\left(\hat{R}_\ell, g(m)\right)$ for any $\ell \in N$, Condition $\mu 2^{**}$(ii) implies that $g(m) \in F(R)$.

Case 3: m corresponds to *Rule 3.*

The proof can be obtained by simply readapting the proof of *Case 2*, so we omit it here.

Case 4: m corresponds to *Rule 4.*

Suppose that $g(m) = x^1 = x^2$. By the definition of g we have that $m_h \in T_h(R, F)$, and so $x^1 \in F(R)$.

Otherwise, let $x^1 \neq x^2$, and so $g(m) = e \in C_1(R^2, x^2) \cap C_2(R^1, x^1)$. By the definition of g we have that $C_i(R^j, x^j) \subseteq g(M_i, m_j)$ for $i, j \in N$, with $i \neq j$, and $m_h \in T_h(R, F)$. Since $m \in \mathbf{N}(M, g, \succcurlyeq^R)$ it follows that $C_1(R^2, x^2) \subseteq L(R_1, e)$ and $C_2(R^1, x^1) \subseteq L(R_2, e)$. We proceed according to whether $R^1 = R^2$ or $R^1 \neq R^2$.

Sub-case 1: $R^1 = R^2$

Then, $e \in C_1(R, x^2) \cap C_2(R, x^1)$. Since $x^1, x^2 \in F(R)$ it follows from Condition $\mu 2^{**}$ that $C_1(R, x^2) \subseteq L(R_1, x^2)$ and $C_2(R, x^1) \subseteq L(R_2, x^1)$. As we have also established that $C_1(R, x^2) \subseteq L(R_1, e)$ and $C_2(R, x^1) \subseteq L(R_2, e)$, it follows that $(e, x^2) \in I_1$ and $(e, x^1) \in I_2$. Condition $\mu 2^{**}$(i.b) implies that $e \in F(R)$.

Sub-case 2: $R^1 \neq R^2$

Recall that $C_1\left(R^2, x^2\right) \subseteq L\left(R_1, e\right)$, $C_2\left(R^1, x^1\right) \subseteq L\left(R_2, e\right)$ and $x^1 \neq x^2$. As $m_h \in T_h\left(R, F\right)$ it follows that either $R^1 = R$ or $R^2 = R$. Condition $\mu 2^{**}$(i.c) implies that $e \in F\left(R\right)$.

Case 5: m corresponds to *Rule 5.*

Then, $g\left(m_1, M_2\right) = Y$ and $g\left(M_1, m_2\right) = Y$. Since $m \in \mathbf{N}\left(M, g, \succcurlyeq^R\right)$ it follows that $Y \subseteq L\left(R_1, g\left(m\right)\right)$ and $Y \subseteq L\left(R_2, g\left(m\right)\right)$. By the definition of g we have that $m_h \in T_h\left(R, F\right)$. As D holds, there exists $\hat{R} \in \mathcal{R}^2$ such that $L\left(R_\ell, g\left(m\right)\right) = L\left(\hat{R}_\ell, g\left(m\right)\right)$, with $\partial L\left(\hat{R}_\ell, g\left(m\right)\right) = \{g\left(m\right)\}$, for all $\ell \in N$. Condition $\mu 2^{**}$(iii) implies that $g\left(m\right) \in F\left(\hat{R}\right)$. Since F satisfies Condition $\mu 2^{**}$ there exists a profile $\left(C_\ell\left(\hat{R}, g\left(m\right)\right)\right)_{\ell \in N}$ such that $C_\ell\left(\hat{R}, g\left(m\right)\right) \subseteq L\left(\hat{R}_\ell, g\left(m\right)\right) \cap Y$ for any $\ell \in N$. As, by construction, $L\left(R_\ell, g\left(m\right)\right) = L\left(\hat{R}_\ell, g\left(m\right)\right)$ for any $\ell \in N$, Condition $\mu 2^{**}$(ii) implies that $g\left(m\right) \in F\left(R\right)$.

Case 6: m corresponds to *Rule 6.*

The proof can be obtained by simply readapting the proof of *Case 5*, so we omit it here. $\square$

4 Both agents partially-honest

In this section we assume that both agents are partially-honest and that this fact is known to the mechanism designer (*A2*). We begin by stating our necessary condition for Nash implementation below.

Definition 3. An SCC F on $\mathcal{R}^2$ satisfies Condition $\mu 2^{\circ}$ if there exists $Y \subseteq X$, and, for each $i \in N$, $R \in \mathcal{R}^2$ and $x \in F\left(R\right)$, there is a set $C_i\left(R, x\right)$ such that $x \in C_i\left(R, x\right) \subseteq L\left(R_i, x\right) \cap Y$. Moreover, for each $R^* \in \mathcal{R}^2$, we have:
(i) (a) For each $\left(x, R, x', R'\right) \in X \times \mathcal{R}^2 \times X \times \mathcal{R}^2$, with $x \in F\left(R\right)$ and $x' \in F\left(R'\right)$, there is $e \equiv e\left(x, R, x', R'\right) \in C_1\left(R, x\right) \cap C_2\left(R', x'\right)$, with $e\left(x, R, x, R\right) = x$;
(b) If $R = R' = R^*$, $\left(x, R\right) \neq \left(x', R'\right)$, $\left(e, x\right) \in I_1^*$ and $\left(e, x'\right) \in I_2^*$, then $e \in F\left(R^*\right)$.

(ii) If $R = R^*$, $y = x$, $y \in C_i(R, x) \subseteq L(R_i^*, y)$ and $Y \subseteq L\left(R_j^*, y\right)$ for some $i, j \in N$, with $i \neq j$, then $y \in F(R^*)$;
(iii) If $z \in Y \subseteq L(R_i^*, z)$ for all $i \in N$, $(\hat{x}, z) \notin I_i^*$ for all $\hat{x} \in Y \setminus \{z\}$, then $z \in F(R^*)$.

Our next result shows that Condition $\mu 2^\circ$ is met by any partially-honest implementable SCC when *A2* holds.

Proposition 3. Let *A2* hold. A partially-honest implementable SCC F on $\mathcal{R}^2$ satisfies Condition $\mu 2^\circ$.

Proof. Since it is clear that F meets Condition $\mu 2^\circ$(ii) if it is partially-honest implementable and the proof of the remaining parts of Condition $\mu 2^\circ$ can easily be obtained by readapting the arguments provided in Proposition 1, we omit the formal proof here. □

We turn to sufficient conditions for partially-honest implementation. The sufficient condition can be stated as follows:

Definition 4. An SCC F on $\mathcal{R}^2$ satisfies Condition $\mu 2^{\circ\circ}$ if there exists $Y \subseteq X$, and, for each $i \in N$, $R \in \mathcal{R}^2$ and $x \in F(R)$, there is a set $C_i(R, x)$ such that $x \in C_i(R, x) \subseteq L(R_i, x) \cap Y$. Moreover, for each $R^* \in \mathcal{R}^2$, we have:
(i) (a) For each $(x, R, x', R') \in X \times \mathcal{R}^2 \times X \times \mathcal{R}^2$, with $x \in F(R)$ and $x' \in F(R')$, there is $e \equiv e(x, R, x', R') \in C_1(R, x) \cap C_2(R', x')$, with $e(x, R, x, R) = x$; and
(b) If $R = R' = R^*$, $(x, R) \neq (x', R')$, $(e, x) \in I_1^*$ and $(e, x') \in I_2^*$, then $e \in F(R^*)$.
(ii) If $R = R^*$, $y \in C_i(R, x) \subseteq L(R_i^*, y)$ and $Y \subseteq L\left(R_j^*, y\right)$ for some $i, j \in N$, with $i \neq j$, then $y \in F(R^*)$;
(iii) If $z \in Y \subseteq L(R_i^*, z)$ for all $i \in N$, then $z \in F(R^*)$.

Condition $\mu 2^{\circ\circ}$ is much weaker than the classical Condition $\mu 2$ (Moore and Repullo, 1990). It is worth mentioning that the class of SCC satisfying Condition $\mu 2^{\circ\circ}$ is larger than that satisfying Condition $\mu 2$ as the former includes non-monotonic SCCs. The differences between Condition $\mu 2^{\circ\circ}$ and Condition $\mu 2^\circ$ are that $\mu 2^{\circ\circ}$(ii) and $\mu 2^{\circ\circ}$(iii) are stronger than respectively

Condition $\mu2^\circ$(ii) and Condition $\mu2^\circ$(iii). Since Condition $\mu2^{\circ\circ}$ is slightly stronger than Condition $\mu2^\circ$, our next result is almost a complete characterization of Nash implementable SCCs when both agents are partially-honest. It is worth mentioning that the devised mechanism differs from that employed in Proposition 2 in the definition of the outcome function g.

Proposition 4. Let *A2* hold. An SCC F on $\mathcal{R}^2$ is partially-honest implementable if it satisfies Condition $\mu2^{\circ\circ}$.

Proof. Let *A2* hold. The mechanism is slightly different form that used in Proposition 2 in the sense that if a message profile falls into *Rule 2* the outcome function is:

$$g(m) = \begin{cases} x^1 & \text{if} \quad x^1 = x^2 \\ y^1 & \text{if} \quad y^1 \in C_1\left(R^2, x^2\right), \, x^1 \neq x^2 \\ e \equiv e\left(x^2, R^2, x^1, R^1\right) & \text{otherwise.} \end{cases}$$

The outcome function is defined similarly if a message profile falls into *Rule 3.* We show that (M, g) partially-honest implements F whenever F satisfies Condition $\mu2^{\circ\circ}$. For, let $R \in \mathcal{R}^2$.

Since F satisfies Condition $\mu2^{\circ\circ}$, $F\left(\mathcal{R}^2\right) \subseteq Y$. Thus, for any $R \in \mathcal{R}^2$ and any $x \in F(R)$, $x \in Y$.

The proof that $F(R) \subseteq \mathbf{N}_g\left(M, g, \succcurlyeq^R\right)$ is similar to that of Proposition 2, so we omit it here. Conversely, to show that $\mathbf{N}_g\left(M, g, \succcurlyeq^R\right) \subseteq F(R)$, let $m \in \mathbf{N}\left(M, g, \succcurlyeq^R\right)$. Consider the following cases.

Case 1: m corresponds to Rule 1.

By a similar argument provided in Proposition 2 it follows $x^1 \in F(R)$.

Case 2: m corresponds to Rule 2.

Then, $g(m_1, M_2) = Y$ and $C_1\left(R^2, x^2\right) \subseteq g(M_1, m_2)$. Moreover, since $m \in \mathbf{N}\left(M, g, \succcurlyeq^R\right)$ it follows that $C_1\left(R^2, x^2\right) \subseteq L(R_1, g(m))$ and $Y \subseteq L(R_2, g(m))$. By the definition of g we have that $m_i \in T_i(R, F)$ for all $i \in N$. If $x^1 = x^2$, then $g(m) = x^1 \in F(R)$. Otherwise, let $x^1 \neq x^2$. Nothing has to be proved if $g(m) \in \left\{x^1, x^2\right\}$. Therefore, let $g(m) \notin \left\{x^1, x^2\right\}$. Suppose that $g(m) = e \in C_1\left(R, x^2\right) \cap C_2\left(R, x^1\right)$. Since $\left(e, x^2\right) \in I_1$ and $\left(e, x^1\right) \in I_2$, Condition $\mu2^\circ$(i.b) implies that $g(m) \in F(R)$. Finally, let

$g(m) \in C_1(R, x^2)$ and $g(m) \notin \{x^2, x^1, e\}$. Condition $\mu 2^{\circ}$(ii) implies that $g(m) \in F(R)$.

Case 3: m corresponds to *Rule 3*.

The proof can be obtained by simply readapting the proof of *Case 2*, so we omit it here.

Case 4: m corresponds to *Rule 4*.

Then, $C_i(R^j, x^j) \subseteq g(M_i, m_j)$ for $i, j \in N$, with $i \neq j$. Moreover, we have that $m_\ell \in T_\ell(R, F)$ for all $\ell \in N$. Since $m \in \mathbf{N}(M, g, \succsim^R)$ it follows that $C_1(R^2, x^2) \subseteq L(R_1, g(m))$ and $C_2(R^1, x^1) \subseteq L(R_2, g(m))$. Since $m_\ell \in T_\ell(R, F)$ for all $\ell \in N$, it must be that $x^1 \neq x^2$, otherwise a contradiction. Then, $g(m) = e \in C_1(R^2, x^2) \cap C_2(R^1, x^1)$. Since $x^1, x^2 \in F(R)$ it follows from Condition $\mu 2^{\circ\circ}$ that $(e, x^2) \in I_1$ and $(e, x^1) \in I_2$. Condition $\mu 2^{\circ\circ}$(i.b) implies that $g(m) \in F(R)$.

Case 5: m corresponds to *Rule 5*.

Then, $g(m_1, M_2) = Y$ and $g(M_1, m_2) = Y$. Since $m \in \mathbf{N}(M, g, \succsim^R)$ it follows that $Y \subseteq L(R_1, g(m))$ and $Y \subseteq L(R_2, g(m))$. Moreover, all agents are reporting truthfully, i.e., $m_\ell \in T_\ell(R, F)$ for all $\ell \in N$. Condition $\mu 2^{\circ\circ}$(iii) implies that $g(m) \in F(R)$.

Case 6: m corresponds to *Rule 6*.

The proof can be obtained by simply readapting the proof of *Case 5*, so we omit it here. □

5 Implications

In this section we briefly discuss the implications of our results and the relationship between our conditions and those provided by Dutta and Sen (2009).

Under the informational assumption *A1* (resp., *A2*) and the assumption that the domain of F is the set of all admissible profiles of linear orders $\mathcal{P}^2$, Dutta and Sen (2009) prove that Condition β^1 (resp., Condition β^2) is not only necessary but also sufficient for Nash implementability of SCCs. These conditions are the following.

Definition 5 (Dutta and Sen, 2009, p. 12). An SCC F on $\mathcal{P}^2$ satisfies Condition β^1 if there exists $Y \subseteq X$, and, for each $i \in N$, $R \in \mathcal{P}^2$ and $x \in F(R)$, there is a set $C_i(R, x)$ such that $x \in C_i(R, x) \subseteq L(R_i, x) \cap Y$. Moreover, for each $R^* \in \mathcal{R}^2$, we have:
(i) For each $(x, R, x', R') \in X \times \mathcal{P}^2 \times X \times \mathcal{P}^2$, with $x \in F(R)$ and $x' \in F(R')$, there is $e \equiv e(x, R, x', R') \in C_1(R, x) \cap C_2(R', x')$, with $e(x, R, x, R) = x$.
(ii) If $y \in C_i(R, x) \subseteq L(R_i^*, y)$ and $Y \subseteq L\left(R_j^*, y\right)$ for some $i, j \in N$, with $i \neq j$, then $y \in F(R^*)$.
(iii) If $z \in Y \subseteq L(R_i^*, z)$ for all $i \in N$, then $z \in F(R^*)$.

Definition 6 (Dutta and Sen, 2009, p. 9). An SCC F on $\mathcal{P}^2$ satisfies Condition β^2 if there exists $Y \subseteq X$, and, for each $i \in N$, $R \in \mathcal{P}^2$ and $x \in F(R)$, there is a set $C_i(R, x)$ such that $x \in C_i(R, x) \subseteq L(R_i, x) \cap Y$. Moreover, for each $R^* \in \mathcal{R}^2$, we have:
(i) For each $(x, R, x', R') \in X \times \mathcal{P}^2 \times X \times \mathcal{P}^2$, with $x \in F(R)$ and $x' \in F(R')$, there is $e \equiv e(x, R, x', R') \in C_1(R, x) \cap C_2(R', x')$, with $e(x, R, x, R) = x$.
(ii) If $z \in Y \subseteq L(R_i^*, z)$ for all $i \in N$, then $z \in F(R^*)$.

In the next two propositions we show that when the domain of preferences consists only of linear orders the "gap" between our necessary conditions and sufficient conditions gets closed, and more importantly, these conditions reduce to Dutta and Sen's conditions.

Proposition 5. Let F be an SCC defined on $\mathcal{P}^2$. Then, the following statements are equivalent:
(a) Condition $\mu 2^{**}$;
(b) Condition $\mu 2^{*}$;
(c) Condition β^1.

Proof. Let F be an SCC defined on $\mathcal{P}^2$. It is clear that Condition $\mu 2^{**}$ implies Condition $\mu 2^{*}$ which, in turn, implies Condition β^1**.** Therefore, we show that Condition β^1 implies Condition $\mu 2^{**}$. It is also clear that β^1(i) and β^1(iii) imply $\mu 2^{**}$(i.a) and $\mu 2^{**}$(iii), respectively. Conditions $\mu 2^{**}$(i.b) and $\mu 2^{**}$(i.c) are always vacuously satisfied when F is defined on $\mathcal{P}^2$, and so β^1 implies them trivially. Finally, we show that β^1 implies condition $\mu 2^{**}$(ii). Let $R, R^* \in \mathcal{P}^2$, $x \in F(R)$ and $y \in C_i(R, x) \subseteq L(R_i^*, y)$ and $Y \subseteq L\left(R_j^*, y\right)$, for $i, j \in N$, with $i \neq j$, and [for all $\hat{x} \in C_i(R, x) \setminus \{y\} : (y, \hat{x}) \notin I_i^*$, for all $\hat{x} \in Y \setminus \{y\} : (y, \hat{x}) \notin I_j^*$] or [$L(R_\ell, y) = L(R_\ell^*, y)$ for all $\ell \in N$]. We show that $y \in F(R^*)$. Since $R^* \in \mathcal{P}^2$, it is always true that, for all $\hat{x} \in C_i(R, x) \setminus \{y\}$, $(y, \hat{x}) \notin I_i^*$, and $(y, \hat{x}) \notin I_j^*$ for all $\hat{x} \in Y \setminus \{y\}$. Condition β^1(ii) implies that $y \in F(R^*)$. Therefore, suppose also that it holds that $L(R_\ell, y) = L(R_\ell^*, y)$ for all $\ell \in N$. Then, $x \in C_i(R, x) \subseteq L(R_i, y) = L(R_i^*, y)$. It follows from the antisymmetry of R_i that $x = y$. Again, Condition β^1(ii) implies that $y \in F(R^*)$. $\square$

Proposition 6. Let F be an SCC defined on $\mathcal{P}^2$. Let F be an SCC defined on $\mathcal{P}^2$. Then, the following statements are equivalent:
(a) Condition $\mu 2^{\circ\circ}$;
(b) Condition $\mu 2^{\circ}$;
(c) Condition β^2.

Proof. Let F be an SCC defined on $\mathcal{P}^2$. As it is clear that Condition $\mu 2^{\circ\circ}$ implies Condition $\mu 2^{\circ}$ which, in turn, implies Condition β^2, we have only to show that Condition β^2 implies Condition $\mu 2^{\circ\circ}$. It is also clear that β^2(i) and β^2(ii) imply $\mu 2^{\circ\circ}$(i.a) and $\mu 2^{\circ\circ}$(iii), respectively. Furthermore, Condition $\mu 2^{\circ\circ}$(i.b) is always vacuously satisfied when F is defined on $\mathcal{P}^2$, and so β^2 implies it trivially. Finally, we show that Condition $\mu 2^{\circ\circ}$(ii) is implied by β^2. Let $R, R^* \in \mathcal{P}^2$, with $R = R^*$, $x \in F(R)$, $y \in C_i(R, x) \subseteq L(R_i^*, y)$ and $Y \subseteq L\left(R_j^*, y\right)$ for some $i, j \in N$, with $i \neq j$. Since $y \in C_i(R, x) \subseteq L(R_i, x)$ and $R = R^*$ it follow from antisymmetry of R_i that $x = y$. It trivially follows that $y \in F(R^*)$. $\square$

Condition $\mu 2^{**}$ (resp., Condition $\mu 2^{\circ\circ}$) imposes non-trivial restrictions on F. For example, as Condition $\mu 2^{**}$ (resp., Condition $\mu 2^{\circ\circ}$) implies Condition β^2 which, in turn, is violated by the *Pareto SCC* (Dutta and Sen, 2009, p. 14), it follows that this SCC is not Nash implementable in the presence of partially-honest agents. Despite it, our results are very permissive. In the following we justify this assertion by providing sufficient conditions which allow us to give a quick answer to question of implementability.

One avenue is to introduce a bad outcome $b \in X$ and make the following assumption.

Assumption A3 (for short, *A3*; Moore and Repullo, 1990, p. 1093). There exists a bad outcome $b \in X$ such that for any $R \in \mathcal{R}^2$ and $i \in N$, $(x, b) \in P_i$ for all $x \in F\left(\mathcal{R}^2\right) \equiv \left\{y \in X | y \in F\left(R'\right) \text{ for some } R' \in \mathcal{R}^2\right\}$.

There are economic environments in which it is easy to find a bad outcome. Consider an exchange economy in which agents have strict monotonic preferences and the SCC assigns only positive consumption bundles. Under free disposability, one can define the null consumption bundle as the bad outcome.

If there is a bad outcome we can set $e\left(x, R, x', R'\right) = b$ for each $\left(x, R, x', R'\right) \in X \times \mathcal{R}^2 \times X \times \mathcal{R}^2$ to satisfy Condition $\mu 2^{**}$(i) and Condition $\mu 2^{\circ\circ}$(i) vacuously. Then, Condition $\mu 2^{**}$(ii) and Condition $\mu 2^{**}$(iii) (resp., Condition $\mu 2^{\circ\circ}$(ii) and Condition $\mu 2^{\circ\circ}$(iii)) are sufficient for Nash implementability in the presence of partially-honest agents.

Even though these conditions can easily be checked by using the algorithm provided by Sjöström (1991), the following conditions, when combined with *A3*, are enough to ensure Condition $\mu 2^{**}$ (resp., Condition $\mu 2^{\circ\circ}$) and are easier to check.

An SCC F on $\mathcal{R}^2$ satisfies *restricted veto power* if, for all $i \in N$, $R \in \mathcal{R}^2$, $x \in X$ and $x' \in F\left(\mathcal{R}^2\right) \equiv \left\{y \in X | y \in F\left(R\right) \text{ for some } R \in \mathcal{R}^2\right\}$, $x \in F\left(R\right)$ whenever $X \subseteq L\left(R_j, x\right)$ for all $j \in N \backslash \{i\}$ and $\left(x, x'\right) \in R_i$. An SCC F on $\mathcal{R}^2$ satisfies *weak restricted veto power* if, for all $i \in N$, $R \in \mathcal{R}^2$, $x \in X$ and $x' \in F\left(R\right)$, $x \in F\left(R\right)$ whenever $X \subseteq L\left(R_j, x\right)$ for all $j \in N \backslash \{i\}$ and $\left(x, x'\right) \in R_i$. An SCC F on $\mathcal{R}^2$ satisfies *unanimity* if, for all $R \in \mathcal{R}^2$, $x \in F\left(R\right)$ whenever $x \in \max_{R_\ell} X$ for all $\ell \in N$.

Unanimity is standard. Restricted veto power is used by Moore and Repullo (1990, p. 1093) for analyzing the two-agent case under *A3* while weak restricted veto power is new and is considerably weaker than restricted veto power.

We can now state the following results.

Corollary 1. Let *A1* and *A3* hold. An SCC F on $\mathcal{R}^2$ is partially-honest implementable if it satisfies restricted veto power.

Proof. Let *A1* and *A3* hold. Suppose that F on $\mathcal{R}^2$ satisfies restricted veto power. It suffices to show that *A3* and restricted veto power imply Condition $\mu 2^{**}$. Let $Y = X$; and for any $R \in \mathcal{R}^2$ and $x \in F(R)$, let $C_i(R,x) = L(R_i, x)$ for each $i \in N$. Since *A3* holds, for each $(x, R, x', R') \in X \times \mathcal{R}^2 \times X \times \mathcal{R}^2$, with $x \in F(R)$ and $x' \in F(R')$, let $e(x, R, x', R') = b$ if $(x, R) \neq (x', R')$, otherwise $e(x, R, x', R') = x$. Then, Condition $\mu 2^{**}$(i) is satisfied. To show Condition $\mu 2^{**}$(ii), let $R, R^* \in \mathcal{R}^2$, $x \in F(R)$, $y \in C_i(R,x) \subseteq L(R_i^*, y)$ and $Y \subseteq L\left(R_j^*, y\right)$ for some $i, j \in N$, with $i \neq j$, and $[(y, \hat{x}) \notin I_i^*$ for all $\hat{x} \in C_i(R,x) \setminus \{y\}$ and $(y, \hat{x}) \notin I_j^*$ for all $\hat{x} \in Y \setminus \{y\}]$ or $[L(R_\ell, y) = L(R_\ell^*, y)$ for all $\ell \in N]$. Then , it is easy to see that agent i cannot veto y as the latter is maximal for j in Y under R^* and $(y, x) \in R_i^*$. We conclude that restricted veto power implies Condition $\mu 2^{**}$(ii). Finally, to show Condition $\mu 2^{**}$(iii) let $R^* \in \mathcal{R}^2$, $z \in Y \subseteq L(R_i^*, y)$ for all $i \in N$, and $(y, \hat{x}) \notin I_i^*$ for all $\hat{x} \in Y \setminus \{z\}$. Then, neither agent 1 or agent 2 can veto z as it is not strictly worse for both of them under the profile R^* than any other outcome in $F\left(\mathcal{R}^2\right)$. □

Corollary 2. Let *A2* and *A3* hold. An SCC F on $\mathcal{R}^2$ is partially-honest implementable if it satisfies weak restricted veto power and unanimity.

Proof. Let *A2* and *A3* hold. Suppose that F on $\mathcal{R}^2$ satisfies weak restricted veto power and unanimity. It suffices to show that *A3*, weak restricted veto power and unanimity imply Condition $\mu 2^{\circ\circ}$. Let $Y = X$; and for any $R \in \mathcal{R}^2$ and $x \in F(R)$, let $C_i(R,x) = L(R_i, x)$ for each $i \in N$. Since *A3* holds, for each $(x, R, x', R') \in X \times \mathcal{R}^2 \times X \times \mathcal{R}^2$, with $x \in F(R)$ and $x' \in F(R')$, let $e(x, R, x', R') = b$ if $(x, R) \neq (x', R')$, otherwise $e(x, R, x', R') = x$. Then, Condition $\mu 2^{\circ\circ}$(i) is satisfied. To show Condition $\mu 2^{\circ\circ}$(ii), let $R, R^* \in \mathcal{R}^2$, $R = R^*$, $x \in F(R)$, $y \in C_i(R,x) \subseteq L(R_i, y)$ and $Y \subseteq L(R_j, y)$ for some $i, j \in N$, with $i \neq j$. Nothing has to be proved if $y = x$. Then, let $y \neq x$. It follows readily from weak restricted veto power that $y \in F(R)$. Finally, as unanimity implies Condition $\mu 2^{\circ\circ}$(iii), we conclude that Condition $\mu 2^{\circ\circ}$ is satisfied. □

For instance, suppose that two agents bargain over the division of one unit of a perfectly divisible good and if they do not reach an agreement they both receive nothing. In this framework, non-monotonic strong individually rational bargaining solutions[12] defined on the class of utility possibility sets such as the *Nash bargaining solution* are special cases of Corollary 1 and Corollary 2 by setting the disagreement point $d = (0, 0)$ as a bad outcome.[13]

Another interesting weak domain restriction is the following.

Assumption Q (for short, AQ; Busetto and Codognato, 2009). $\mathcal{R}^2$ is such that, for each $R^* \in \mathcal{R}^2$, we have:
(i) $\max_{R_i^*} SL(R_i, x) \cap \max_{R_j^*} SL(R_i, x) = \emptyset$, for all $i, j \in N$, $i \neq j$, $R \in \mathcal{R}^2$, and $x \in X$;
(ii) $\max_{R_1^*} SL(R_1, x) \cap \max_{R_2^*} SL(R_2', x') = \emptyset$, for each $(x, R, x', R') \in X \times \mathcal{R}^2 \times X \times \mathcal{R}^2$, with $(x, R) \neq (x', R')$.

This domain restriction is very mild and much weaker than *Assumption E* imposed by Moore and Repullo (1990, p. 1095) and *Assumptions 5.1-5.2* imposed by Dutta and Sen (1991, p. 125) whenever X is a subset of a finite-dimensional Euclidean space.[14] For example, this restriction is satisfied in environments with continuous and locally non-satiated preferences or in environments in which the set of outcomes is a space of lotteries over a finite set of outcomes and agents preferences over lotteries are represented by von Neumann-Morgenstern utility functions. Given AQ we can define a condition that, when combined with others, is enough to ensure Condition $\mu 2^{**}$ (resp., Condition $\mu 2^{\circ\circ}$).

Definition 7. An SCC F on $\mathcal{R}^2$ satisfies the *non-empty lower intersection* if for any $(x, R, x', R') \in X \times \mathcal{R}^2 \times X \times \mathcal{R}^2$, with $x \in F(R)$ and $x' \in F(R')$, we have that $SL(R_1, x) \cap SL(R_2', x') \neq \emptyset$.

This property appears in Moore and Repullo (1990) and Dutta and Sen (1991) and holds in many environments. For example, it holds in an exchange economy for which indifference curves never touch the axes and for which the SCC recommends only interior allocations.

[12] A bargaining solution is strong individually rational if it provides agents with agreements which give them utilities higher than those they derive from the disagreement point d.

[13] For Nash bargaining solution defined on the class of utility possibility sets see Vartiainen (2007).

[14] The formal arguments are provided in Busetto and Codognato (2009).

We can now state our final results.

Corollary 3. Let *A1* and *AQ* hold. An SCC F on $\mathcal{R}^2$ is partially-honest implementable if it satisfies restricted veto power and non-empty lower intersection.

Proof. Let *A1* and *AQ* hold. Suppose that F on $\mathcal{R}^2$ satisfies restricted veto power and non-empty lower intersection. We show that F is partially-honest implementable. It suffices to show that Condition $\mu 2^{**}$ is implied by the non-empty lower intersection property and restricted veto power when combined with our domain restriction. For each $i \in N$, $(x, R) \in X \times \mathcal{R}^2$, and $x \in F(R)$, let $C_i(R, x) = SL(R_i, x) \cup \{x\}$ and $Y = \bigcup_{i \in N} \bigcup_{R \in \mathcal{R}^2} \bigcup_{x \in F(R)} C_i(R, x)$. It is easy to verify that $C_i(R, x) \subseteq L(R_i, x) \cap Y$. For each $(x, R, x', R') \in X \times \mathcal{R}^2 \times X \times \mathcal{R}^2$, with $x \in F(R)$ and $x' \in F(R')$, let $e(x, R, x', R') \in SL(R_i, x) \cap SL\left(R'_j, x'\right)$ if $(x, R) \neq (x', R')$, otherwise $e(x, R, x', R') = x$. It is easy to see that $\mu 2^{**}$(i) is satisfied as F meets the non-empty lower intersection property. Then, it remains to show that $\mu 2^{**}$(ii)-$\mu 2^{**}$(iii) are met as well. It is clear that restricted veto power implies $\mu 2^{**}$(iii). Finally, to show that F satisfies $\mu 2^{**}$(ii) let $y \in C_i(R, x) \subseteq L(R^*_i, y)$ and $Y \subseteq L\left(R^*_j, y\right)$, for $i, j \in N$, with $i \neq j$, and [for all $\hat{x} \in C_i(R, x) \setminus \{y\} : (y, \hat{x}) \notin I^*_i$, for all $\hat{x} \in Y \setminus \{y\} : (y, \hat{x}) \notin I^*_j$] or [$L(R_\ell, y) = L(R^*_\ell, y)$ for all $\ell \in N$]. Suppose that $y \in C_i(R, x) \setminus \{x\}$. Then, $y \in \max_{R^*_i} SL(R_i, x) \cap \max_{R^*_j} SL(R_i, x)$ which contradicts assumption *AQ(i)*. Then, let $y = x$. Restricted veto power implies $x \in F(R^*)$, as sought. □

Corollary 4. Let *A2* and *AQ* hold. An SCC F on $\mathcal{R}^2$ is partially-honest implementable if it satisfies unanimity and non-empty lower intersection.

Proof. Let *A2* and *AQ* hold. Suppose that F on $\mathcal{R}^2$ satisfies non-empty lower intersection and unanimity. We show that F is partially-honest implementable. It suffices to show that Condition $\mu 2^{\circ\circ}$ is implied by the non-empty lower intersection property and unanimity when combined with our domain restriction. Define $C_i(R, x)$, Y and $e(x, R, x', R')$ in the same way done in Corollary 3. It is easy to see that $\mu 2^{\circ\circ}$(i) is satisfied as F meets the non-empty lower intersection property. Then, we have only to show that $\mu 2^{\circ\circ}$(ii)-$\mu 2^{\circ\circ}$(iii) are met as well. To show that F satisfies $\mu 2^{\circ\circ}$(ii) let $R = R^{*}$, $y \in C_i(R, x) \subseteq L(R_i^{*}, y)$ and $Y \subseteq L\left(R_j^{*}, y\right)$, for $i, j \in N$, with $i \neq j$. If $y = x$, then nothing else has to be proved. Then, $y \in \max_{R_i^{*}} SL(R_i, x) \cap \max_{R_j^{*}} SL(R_i, x)$ which contradicts assumption *AQ(i)*. Finally, as unanimity implies $\mu 2^{\circ\circ}$(iii), we conclude that Condition $\mu 2^{\circ\circ}$ is satisfied. $\square$

For instance, consider a two-agent exchange economy with $\ell \geq 2$ divisible goods in which agents have continuous and strict monotonic preferences and in which indifference curves never touch the axes (for instance, Cobb-Douglas preferences). Suppose that the SCC F selects only interior allocations of the feasible set. In this setting, restricted veto power - and so unanimity - and non-empty lower intersection are met by F. An example of non-monotonic F would be the $\omega-$*efficient-egalitarian correspondences* which under our assumptions on preferences always exists (Pazner and Schmeidler, 1978).[15] Corollary 3 and Corollary 4 imply that the $\omega-$*efficient-egalitarian correspondences* is partially-honest implementable.

6 Conclusion

This paper examined the problem of fully implementing SCCs in a two-agent society when agents are partially-honest. We formalized the problem by requiring that the domain of preferences consists of all weak orders. We presented almost necessary and sufficient conditions for implementation, both when exactly one agent is partially-honest and when both agents are partially-honest. We then proceeded to relate our conditions to the conditions earlier devised by Dutta and Sen (2009). While our conditions and Dutta and Sen's conditions do not imply each other on the domain of weak orders we found that our conditions are equivalent to their conditions on the domain of linear orders. Moreover, our conditions are much weaker than the classic conditions devised by Moore and Repullo (1990) and Dutta and Sen (1991). In line with earlier results of Dutta and Sen (2009) and Matsushima (2008), the results presented in this paper confirm that the consideration of

[15] $\omega \in \mathbb{R}_{+}^{\ell}$ is the vector of resources available for distribution, a.k.a. the social endowment.

partially-honest agents in implementation theory drastically improves the scope and quality of implementation, though limits still remain. In particular, the classic condition of monotonicity is no longer required while what still limits our ability to implement is the punishment condition. Finally, we identified sufficient conditions and weak domain restrictions which allowed us to give a quick answer to the question of implementability in a wide range of applications.

We have not been able to provide necessary and sufficient conditions, given that indifference *wides* the "gap" between preferences over outcomes and preferences over message profiles for partially-honest agents. We conjecture that a full characterization is impossible on the domain of weak orders if conditions on SCCs are to be formulated in standard terms. However, we expect that a full characterization of partially-honest implementable SCCs can be obtained by devising a new suitable definition of implementation. This is left for further research.

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