# Finding Common Ground: Efficiency Indices

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by

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# FINDING COMMON GROUND: EFFICIENCY INDICES

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## Introduction

The last two decades have witnessed a revival in interest in the measurement of productive efficiency pioneered by Farrell (1957) and Debreu (1957). 1978 was a watershed year in this revival with the christening of DEA by Charnes, Cooper and Rhodes (1978) and the critique of Farrell technical efficiency in terms of axiomatic production and index number theory in Fare and Lovell (1978). These papers have inspired many others to apply these methods and to add to the debate on how best to define technical efficiency.

In this paper we try to pull together some of the variants that have arisen over these decades and show when they are equivalent. The specific cases we take up include: 1) the original Debreu-Farrell measure versus the Russell measure—the latter introduced by Färe and Lovell, and 2) the directional distance function and the additive measure. The former was introduced by Luenberger (1992) and the latter by Charnes, Cooper, Golany and Seiford (1985). We also provide a discussion of the associated cost interpretations.

## Basic Production Theory Details

In this section we introduce the basic production theory that we employ in this paper. We will be focusing on the input based efficiency measures here, but the analysis could readily be extended to the output oriented case as well.

To begin, technology may be represented by its input requirement sets

$$L(y) = \{x: \ x \ can \ produce \ y\}, \quad y \in \Re_+^M , \tag{1}$$

where $y \in \Re_+^M = \{y \in \Re^M : y_m \geq 0, m = 1,...,M\}$ denotes outputs and $x \in \Re_+^N$ denotes inputs. We assume that the input requirement sets satisfy the standard axioms, including: $L(0) = \Re_+^N$, and *L(y)* is a closed convex set with both inputs[2] and outputs[3] freely disposable (for details see Färe and Primont (1995)).

The subsets of *L(y)* relative toward which we measure efficiency are the isoquants

$$IsoqL(y) = \{x : x \in L(y), \lambda x \notin L(y), \lambda > 1\}, y \in \Re_+^M, \tag{2}$$

and the efficient subsets

$$EffL(y) = \{x : x \in L(y), x' \leq x, x' \neq x \Rightarrow x' \notin L(y)\}, y \in \Re_+^M. \tag{3}$$

Clearly, $EffL(y) \subseteq IsoqL(y)$ and as one can easily see with a Leontief technology, i.e., $L(y) = \{(x_1, x_2) : \min\{x_1, x_2\} \geq y\}$, the efficient subset may be a proper subset of the isoquant.

Next we introduce two function representations of *L(y),* namely the Shephard input distance function and the directional input distance function, and discuss some of their properties.

Shephard's (1953) input distance function is defined in terms of the input requirement sets *L(y)* as

$$D_i(y, x) = \sup\{\lambda : x/\lambda \in L(y)\}. \tag{4}$$

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[2] Inputs are freely disposable if $x' \geq x \in L(y) \Rightarrow x' \in L(y)$.

[3] Outputs are freely disposable if $y' \geq y \Rightarrow L(y') \subseteq L(y)$.

Among its important properties[4] we note the following

i) $D_i(y,x) \geq 1 \quad \textit{if and only if} \;\; x \in L(y),$ Representation

ii) $D_i(y,\lambda x) = \lambda D_i(y,x), \lambda > 0,$ Homogeneity

iii) $D_i(y,x) = 1 \quad \textit{if and only if} \;\; x \in IsoqL(y),$ Indication

Our first property shows that the distance function is a complete representation of the technology. Property ii) shows that the distance function is homogeneous of degree one in inputs, i.e., the variables which are scaled in (4). The indication condition shows that the distance function identifies the isoquants.

Turning to the directional input distance function introduced by Luenberger (1992)[5], we define it as

$$\vec{D}_i(y,x;g_x) = \sup\{\beta : (x - \beta g_x) \in L(y)\}, \tag{5}$$

where $g_x \in \Re^N_+$ is the directional vector in which inefficiency is measured. Here we choose $g_x = 1^N \in \Re^N_+$. This function $\vec{D}_i(y,x;1^N)$ has properties that parallel those of $D_i(y, x)$, and are listed below. For technical reasons the indication property is split into two parts. We note that we require inputs to be strictly positive in part a) of the indication property. The proofs of these properties are found in the appendix.

i) $\vec{D}_i(y,x;1^N) \geq 0 \quad \textit{if and only if} \;\; x \in L(y),$ Representation

ii) $\vec{D}_i(y,x+\alpha 1^N;1^N) = \vec{D}_i(y,x;1^N) + \alpha, \alpha > 0,$ Translation

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[4]For additional properties and proofs, see Färe and Primont (1995).

[5]In consumer theory he calls this the benefit function and in producer theory he uses the term shortage function.

iiia) *if* $\vec{D}_i(y,x;1^N)=0$ *and* $x_n>0, n=1,...N$, *then* $x\in IsoqL(y)$, Indication

iiib) $x\in IsoqL(y)$ *implies* $\vec{D}_i(y,x;1^N)=0$, Indication

Since we will be relating technical efficiency to costs, we also need to define the cost function, which for input prices $w\in\Re_+^N$ is

$$C(y,w)=\min\{wx : x\in L(y)\}. \tag{6}$$

The following dual relationships apply

$$\frac{C(y,x)}{wx}\leq 1/D_i(y,x) \tag{7}$$

and

$$\frac{C(y,x)-wx}{w1^N}\leq -\vec{D}_i(y,x;1^N). \tag{8}$$

Expression (7) which is the Mahler inequality, states that the ratio of minimum cost to observed cost is less than or equal to the reciprocal of the input distance function. Expression (8) states that the difference between minimum and observed cost, normalized by input prices, is no larger than the negative of the directional input distance function.

These two inequalities may be transformed to strict equalities by introducing allocative inefficiency as a residual.

## The Debreu-Farrell and Russell Equivalence

Our goal in this section is to find conditions on the technology $L(y), y\in\Re_+^M$, such that the Debreu-Farrell (Debreu (1957), Farrell (1957)) measure of technical efficiency coincides with the Russell (Färe and Lovell (1978)) measure. To establish these conditions we redefine the original Russell measure and introduce a multiplicative version. We do this by using the geometric mean as the objective function in its definition rather than an arithmetic mean. Thus our multiplicative Russell measure is defined as

$$R_M(y,x) = \min\left\{(\prod_{n=1}^{N}\lambda_n)^{1/N} : (\lambda_1 x_1,\ldots,\lambda_N x_N) \in L(y), 0 < \lambda_n \leq 1, n = 1,\ldots,N\right\} \tag{9}$$

The objective function here is $(\prod_{n=1}^{N}\lambda_n)^{1/N}$ in contrast to $\sum_{n=1}^{N}\lambda_n / N$ from the original specification in Färe and Lovell (1978). For technical reasons we assume here that inputs $x = (x_1, \ldots, x_n)$ are strictly positive, i.e., $x_n > 0$, $n = 1,\ldots,N$. More specifically in this section we assume that for $y \geq 0, y \neq 0, L(y)$ is a subset of the interior of $\Re_+^N$.[6]

Note that the Russell measure in (9) has the indication property

$$R_M(y,x) = 1 \text{ if and only if } x \in EffL(y) \tag{10}$$

Recall that the Debreu-Farrell measure of technical efficiency is the reciprocal of Shephard's input distance function, i.e.,

$$DF(y,x) = 1/D_i(y,x) \tag{11}$$

thus it is homogeneous of degree -1 in $x$ and it has the same indication property as $D_i(y, x)$.

[6] See Russell (1990) for a related assumption

Now assume that the technology is input homothetic[7], i.e.,

$$D_i(y,x) = D_i(1,x)/H(y) \tag{12}$$

and that the input aggregation function $D_i(1\ ,\ x)$ is a geometric mean, so that the distance function equals

$$D_i(y,x) = (\prod_{n=1}^{N} x_n)^{1/N} / H(y). \tag{13}$$

From (4) and the Representation property it is clear that the distance function takes the form above if and only if the input requirement sets are of the following form

$$L(y) = H(y)\cdot\left\{\hat{x} : (\prod_{n=1}^{N}\hat{x})^{1/N} \geq 1\right\},\ \hat{x} = \frac{x}{H(y)}\ . \tag{14}$$

The Russell characterization theorem can now be stated; the proof may be found in the appendix.

Theorem 1: Assume that $L(y)$ is interior to $\Re_{+}^{M}$ for $y \geq 0, y \neq 0$.

$$R_M(y,x) = DF(y,x)\ \textit{for all}\ x \in L(y)\ \textit{if and only if}\ D_i(y,x) = (\prod_{n=1}^{N} x_n)^{1/N} / H(y).$$

Thus for these two efficiency measures to be equivalent, technology must satisfy a fairly specific form of homotheticity - technology is of a restricted Cobb-Douglas form in which the inputs have equal weights. This makes intuitive sense,

[7] For details see Färe and Primont (1995).

since technology must be symmetric, but clearly not of the Leontief type. That is, technology must be such that the *IsoqL(y) =EffL(y).* Of course, it is exactly the Leontief type technology which motivated Färe and Lovell to introduce a measure that would use the efficient subset *EffL(y)* rather than the isoquant *IsoqL(y)* as the reference for establishing technical efficiency.

**The Directional Distance Function and the Additive Measure**

We now turn to some of the more recently derived versions of technical efficiency; specifically we derive conditions on the technology *L(y)*, $y \in \Re_+^M$ that are necessary and sufficient for the directional distance function to coincide with a "stylized' additive measure of technical efficiency.

The original additive measure introduced by Charnes, Cooper, Golany and Seiford (1985)(hereafter CCGS) simultaneously expanded outputs and contracted inputs. Here we focus on a version that contracts inputs only, but in the additive form of the original measure. Although the original measure was defined relative to a variable returns to scale technology, (see p. 97, CCGS), here we leave the returns to scale issue open and impose only those conditions itemized in Section 2. Finally, we normalize their measure by the number of inputs, *N*.

We are now ready to define the stylized additive model as

$$A(y,x) = \max\left\{ \sum_{n=1}^{N} s_n / N : (x_1 - s_1, \ldots, x_N - s_N) \in L(y) \right\}, \tag{15}$$

where $s_n \geq 0,\, n = 1, \ldots, N.$

This measure reduces each input $x_n$ so that the total reduction $\sum_{n=1}^{N} s_n / N$ is maximized. Intuitively, one can think of this problem as roughly equivalent to minimizing costs when all input prices are equal to one. We will discuss this link in the next section.

The additive measure and the modified Russell measure look quite similar, although the former uses an arithmetic mean as the objective and the modified Russell measure uses a geometric mean. The additive structure of *A(y, x)* suggests that the directional distance function - which also has an additive structure - may be related to it.[8] To make that link we begin by characterizing the technology for which these two measures would be equivalent. We begin by assuming that technology is translation input homothetic,[9] i.e., in terms of the directional distance function we may write

$$\vec{D}_i(y, x; 1^N) = \vec{D}_i(0, x; 1^N) - F(y). \tag{16}$$

Moreover, we assume that the aggregator function $\vec{D}_i(0, x; 1^N)$ is arithmetic mean so that the directional distance function may be written as

$$\vec{D}_i(y, x; 1^N) = \frac{1}{N}\sum_{n=1}^{N} x_n - F(y). \tag{17}$$

Note that from the properties of the directional distance function, it follows that it takes the form required above if and only if the underlying input requirement sets are of the form

[8] Larry Seiford noted the similarity at a North American Efficiency and Productivity Workshop.

[9] For details see Chambers and Färe (1998). Chambers and Färe assumed that F(y) depends on the directional vector $1^N$. Here we take it as fixed and omit it.

$$L(y) = \left\{ \tilde{x} : \frac{1}{N} \sum_{n=1}^{N} \tilde{x}_n \geq 0 \right\} + F(y), \tag{18}$$

where $\tilde{x} = (x_1 - F(y), \ldots, x_N - F(y))$.

We are now ready to state our additive representation theorem (see appendix for proof),

Theorem 2:

$$\vec{D}_i(y, x; 1^N) = A(y, x) \ \ \textit{for all} \ \ x \in C(L(y)) = \left\{ \hat{x} : \hat{x} = x + \delta 1^N, x \in L(y), \delta \geq 0 \right\}$$

*if and only if* $\quad \vec{D}_i(y, x; 1^N) = \dfrac{1}{N} \sum_{n=1}^{N} x_n - F(y).$

Here we see that to obtain equivalence between the additive measure and the directional distance function, technology must be linear in inputs, i.e., the isoquants are straight lines with slope = -1 .

**Cost Interpretations**

The Debreu-Farrell measure has a dual interpretation, namely the cost deflated cost function. Here we show that the multiplicative Russell measure and the additive measure also have dual cost interpretations.[10]

[10] It is straightforward to show that the original (additive) Russell measure also has a cost interpretation, despite the claim by Kopp (1981, p. 450) that the Russell measure '...cannot be given a meaningful cost interpretation which is factor price invariant.' In this section, we provide such a cost interpretation.

Recall that we define the cost function

$$C(y,w) = \min\{wx : x \in L(y)\}, \tag{19}$$

where $w \in \Re_+^N$ are input prices. From the definition it follows that

$$C(y,w) \le wx, \forall x \in L(y). \tag{20}$$

Now since $DF(y,x)\ x \in L(y)$ it is also true that

$$C(y,w) \le w(DF(y,x)x) = wx(DF(y,x)) \tag{21}$$

and

$$C(y,w)/wx \le DF(y,x) \tag{22}$$

Expression (22) is the Mahler inequality expressed in terms of the cost efficiency measure *(C(y, w)/wx)* and the Debreu-Farrell measure of technical efficiency, *DF(y, x)*. This inequality may be closed by introducing a multiplicative measure of allocative efficiency, *AE(y, x, w)*, so that we have

$$C(y, w)/wx = DF(y, x)AE(y, x, w). \tag{23}$$

To introduce a cost interpretation of the multiplicative Russell measure we note that

$$(\lambda^*{}_1 x_{1,\ldots,} \lambda^*{}_N x_N) \in L(y), \tag{24}$$

where $\lambda^*_n$ $(n = 1, \ldots, N)$ are the optimizers in expression (9). From the assumption that the input requirement sets are subsets of the interior of $\Re_+^N$, it follows that $\lambda^*_n > 0$, $n = 1, \ldots, N$. By (20) and (24) we have

$$C(y,w) \leq (\lambda^*_1 w_1 x_1, \ldots, \lambda^*_N w_N x_N) \tag{25}$$

and by multiplication

$$C(y,w)/wx \leq \left(\prod_{n=1}^{N}\lambda_n^*\right)^{1/N}\left[\frac{\lambda^*_1 w_1 x_1}{\left(\prod_{n=1}^{N}\lambda_n^*\right)^{1/N} wx}+\cdots+\frac{\lambda^*_N w_N x_N}{\left(\prod_{n=1}^{N}\lambda_n^*\right)^{1/N} wx}\right] \tag{26}$$

or

$$C(y,w)/wx \leq R_M(y,x)\left[\frac{\lambda^*_1 w_1 x_1}{\left(\prod_{n=1}^{N}\lambda_n^*\right)^{1/N} wx}+\cdots+\frac{\lambda^*_N w_N x_N}{\left(\prod_{n=1}^{N}\lambda_n^*\right)^{1/N} wx}\right] \tag{27}$$

Expression (27) differs from the Mahler inequality (22) in that it contains a second term on the right hand side. This term may be called the Debreu-Farrell deviation, in that if $\lambda_1 = \ldots = \lambda_N$, the deviation equals one. That is, if the scaling factors $\lambda^*_n$ are equal for each $n$, then (27) coincides with (22). Again, the inequality (27) can be closed by introducing a multiplicative residual, which captures allocative inefficiency.

Turning to the additive measure, we note that

$$(x_1 - s_1^*, \ldots, x_N - s_N^*) \in L(y) \tag{28}$$

where $s_n^*, n = 1,\ldots,N$ are the optimizers in problem (15). Thus from cost minimization we have

$$C(y, w) \le wx - ws^*, \tag{29}$$

where $s^* = (s_1^*, \ldots, s_N^*)$.From (29) we can derive two dual interpretations: a ratio and a difference version.

The ratio interpretation is

$$C(y, w) / wx \le 1 - \frac{ws^*}{wx}, \tag{30}$$

which bears some similarity to the Farrell cost efficiency model in (22). Now if $w = (1, \ldots, 1)$, then it follows that the additive model is related to costs as

$$\frac{C(y, 1^N)}{\sum_{n=1}^{N} x_n} \le 1 - \frac{\sum_{n=1}^{N} s_n^* / N}{\sum_{n=1}^{N} x_n / N} = 1 - \frac{A(y, x)}{\sum_{n=1}^{N} x_n / N} \tag{31}$$

In this case we see that Debreu-Farrell cost efficiency (the left-hand side) is not larger than one minus a normalized additive measure.

The second cost interpretation is

$$C(y,w) - wx \leq -ws^*, \tag{32}$$

and when $w = (1, \ldots, 1)$ we obtain

$$\frac{C(y,1^N) - \sum_{n=1}^{N} x_n}{N} \leq -A(y,x) \tag{33}$$

If we compare this result to (8), we see again, the close relationship between the additive measure and the directional distance function.

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**Appendix**

Proof of (2.5):

i) See Chambers, Chung and Färe (1998, p. 354) for a similar proof.

ii)

$$\begin{aligned}\vec{D}_i(y, x+\alpha 1^N; 1^N) &= \sup\{\beta : (x-\beta 1^N + \alpha 1^N) \in L(y)\}\\ &= \sup\{\beta : (x-(\beta+\alpha)1^N) \in L(y)\}\\ &= +\alpha + \sup\{\hat{\beta} : (x-\beta 1^N \in L(y)\}(\hat{\beta} = \beta - \alpha)\\ &= \vec{D}_i(y, x; 1^N) + \alpha \,.\end{aligned}$$

iiia) We give a contrapositive proof. Let $x \in L(y)$ with $x_n > 0, n = 1,\ldots,N$ and $x \notin IsoqL(y)$. Then *Di(y, x)* > 1, and by strong disposability, there is an open neighborhood $N_\varepsilon(x)$ of $x$ ($\varepsilon = \min\{x_1 - D_i(y,x)x_1,\ldots,x_N - D_i(y,x)x_N\}$) such that $N_\varepsilon(x) \in L(y)$. Thus $\vec{D}_i(y,x;1^N) > 0$ proving iiia).

iiib) Again we give a contrapositive proof. Let $\vec{D}_i(y,x;1^N) > 0$ then $x - \vec{D}_i(y,x;1^N)1^N \in L(y)$ and since the directional vector is $1^N = (1,\ldots,1)$, each $x_n, n = 1,\ldots,N$ can be reduced while still in *L(y)*. Thus $D_i(y, x) > 1$ and by the Indication property for $D_i(y, x)$, $x \notin IsoqL(y)$. This completes the proof.

Remark on the proof of iiia): The following figure shows that when the directional vector has all coordinates positive, for example $1^N$, then $x_n > 0, n = 1,\ldots,N$ is required. In the Figure 1, input vector *a* has $x_1 = 0$, and $\vec{D}_i(y,x;1^N) = 0$ , but *a* is not on the isoquant.

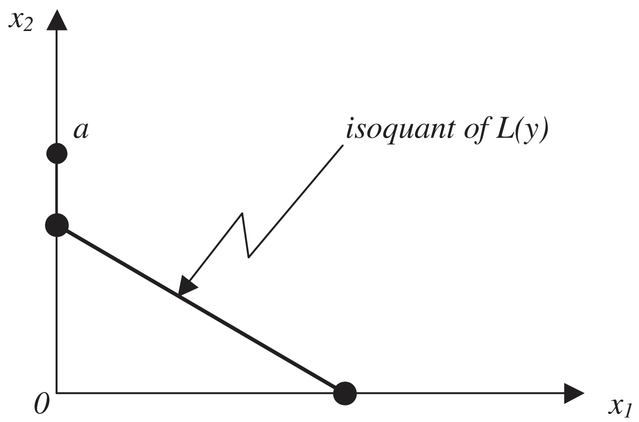


Figure 1. Remark on the proof of iiia).

This problem may be avoided by choosing the directional vector to have ones only for positive *x's*.

Proof of Theorem 1:

Assume first that the technology is as in (13), then

$$R_M(y,x) = \min\left\{\left(\prod_{n=1}^{N} \lambda_n\right)^{1/N} : (\lambda_1 x_1,\ldots,\lambda_N x_N) \in L(y),\, 0 < \lambda_n \le 1,\, n = 1,\ldots,N\right\}$$

$$= \min\left\{\left(\prod_{n=1}^{N} \lambda_n\right)^{1/N} : D_i(\lambda_1 x_1,\ldots,\lambda_N x_N) \ge 1,\, 0 < \lambda_n \le 1,\, n = 1,\ldots,N\right\}$$

$$= \min\left\{\left(\prod_{n=1}^{N} \lambda_n\right)^{1/N} : \left(\prod_{n=1}^{N} \lambda_n x_n\right)^{1/N} / H(y) \ge 1,\, 0 < \lambda_n \le 1,\, n = 1,\ldots,N\right\}$$

$$= \min\left\{\left(\prod_{n=1}^{N} \lambda_n\right)^{1/N} : \left(\prod_{n=1}^{N} \lambda_n\right)^{1/N} \ge H(y) / \left(\prod_{n=1}^{N} x_n\right)^{1/N} 1,\, 0 < \lambda_n \le 1,\, n = 1,\ldots,N\right\}$$

$$= H(y) / \left(\prod_{n=1}^{N} x_n\right)^{1/N} = 1/D_i(y,x).$$

Since *DF(y, x) =*1 */Di(y, x)* we have shown that ( 3) implies $R_M(y, x) = DF(x, y)$.

To prove the converse we first show that

$$R_M(y, \delta_1 x_{1,\ldots,} \delta_N x_N) = R_M(y,x) / \left(\prod_{n=1}^{N} \delta_n\right)^{1/N},\, 0 < \delta_n \le 1,\, n = 1,\ldots,N. \qquad (34)$$

To see this,

$$R_M(y, \delta_1 x_1, \ldots, \delta_N x_N) = \min\left\{ \left(\prod_{n=1}^{N} \lambda_n\right)^{1/N} : (\lambda_1 \delta_1 x_1, \ldots, \lambda_N \delta_N x_N) \in L(y), \right.$$
$$\left. 0 < \lambda_n \le 1,\ 0 < \delta_n \le 1,\ n = 1, \ldots, N \right\}$$

$$= \left(\prod_{n=1}^{N} \delta_n\right)^{-1/N} \min\left\{ \left(\prod_{n=1}^{N} \lambda_n \delta_n\right)^{1/N} : (\lambda_1 \delta_1 x_1, \ldots, \lambda_N \delta_N x_N) \in L(y), \right.$$
$$\left. 0 < \lambda_n \le 1,\ 0 < \delta_n \le 1,\ n = 1, \ldots, N \right\}$$

$$= \left(\prod_{n=1}^{N} \delta_n\right)^{-1/N} \min\left\{ \left(\prod_{n=1}^{N} \hat{\lambda}_n\right)^{1/N} : (\hat{\lambda}_1 \delta_1 x_1, \ldots, \hat{\lambda}_N \delta_N x_N) \in L(y), \right.$$
$$\left. 0 < \hat{\lambda}_n \le 1,\ 0 < \delta_n \le 1,\ n = 1, \ldots, N \right\}$$

$$= R_M(y, x)\left(\prod_{n=1}^{N} \delta_n\right)^{-1/N}$$

where $\hat{\lambda}_n = \lambda_n \delta_n, n = 1, \ldots, N$. Thus (34) holds.

Next, assume that the Debreu-Farrell and the multiplicative Russell measures are equal, then

$$R_M(y, \delta_1 x_1, \ldots, \delta_N x_N) = R_M(y, x) / \left(\prod_{n=1}^{N} \delta_n\right)^{1/N} = DF(y, \delta_1 x_1, \ldots, \delta_N x_N)$$

thus

$$R_M(y, x) = DF(y, \delta_1 x_1, \ldots, \delta_N x_N)\left(\prod_{n=1}^{N} \delta_n\right)^{1/N}$$

and

$$DF(y, x) = DF(y, \delta_1 x_1, \ldots, \delta_N x_N)\left(\prod_{n=1}^{N} \delta_n\right)^{1/N}$$

Now we take $\delta_n = 1/x_n, n = 1,\ldots,N$ then

$$DF(y,x) = DF(y,1,\ldots,1)\left(\prod_{n=1}^{N}\delta_n\right)^{1/N}$$

Moreover, since the Debreu-Farrell measure is independent of units of measurement (Russell (1987), p. 215),[11] $x_n$ can be scaled so that $x_n > 0, n = 1,\ldots,N$. Thus by taking $H(y) = DF(y,1,\ldots,1)$, and using (11) we have proved our claim.

Proof of Theorem 2:

First consider

$$A(y, x_1 - \delta_1, \ldots, x_N - \delta_N) =$$

$$= \max\left\{\frac{1}{N}\sum_{n=1}^{N} s_n : (x_1 - \delta_1 - s_1, \ldots, x_N - \delta_N - s_N) \in L(y)\right\},$$

$$= \max\left\{\frac{1}{N}\sum_{n=1}^{N}(s_n - \delta_n + \delta_n) : (x_1 - (\delta_1 + s_1), \ldots, x_N - (\delta_N + s_N)) \in L(y)\right\},$$

$$= -\frac{1}{N}\sum_{n=1}^{N}\delta_n + A(y,x),$$

where $s_n \geq 0, \delta_n \geq 0, n = 1,\ldots,N$.

[11] This was pointed out to us by R.R. Russell.

This is equivalent to

$$A(y,x)=\frac{1}{N}\sum_{n=1}^{N}\delta_n + A(y,x_1-\delta_1,\ldots,x_N-\delta_N)$$

Take $\delta_n = x_n$ and define $-F(y) = A(y,0)$, then since equality between the directional distance function and the additive measure holds,

$$\vec{D}_i(y,x;1^N) = A(y,x) = \frac{1}{N}\sum_{n=1}^{N} x_n - F(y).$$

Next, let $x \in \overline{C}(L(y))$, then for some $x \in IsoqL(y)$, and $\delta \geq 0$,

$$\vec{D}_i(y,x;1^N) = \vec{D}_i(y,\hat{x}+\delta 1^N;1^N) = \vec{D}_i(y,\hat{x};1^N)+\delta.$$

Since $\hat{x} \in IsoqL(y)$, $\vec{D}_i(y,x;1^N) = \delta$.

Next,

$$A(y,x) = \max\left\{\frac{1}{N}\sum_{n=1}^{N} s_n : \sum_{n=1}^{N}(x_n - s_n)/N - F(y) \geq 0\right\}$$

$$= \max\left\{\frac{1}{N}\sum_{n=1}^{N} s_n : \sum_{n=1}^{N}(\hat{x}_n + \delta - s_n)/N - F(y) \geq 0\right\}$$

$$= \max\left\{\frac{1}{N}\sum_{n=1}^{N} s_n : \delta + \sum_{n=1}^{N}\hat{x}_n / N - F(y) \geq \frac{1}{N} s_N\right\}$$

$$= \delta,$$

since $\hat{x} \in IsoqL(y)$, thus $\vec{D}_i(y,x;1^N) = A(y,x)$.