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Abstract

In this paper we show that in order to aggregate individual efficiency scores into a group (e.g., industry) efficiency score, in such a way that the multiplicative structure of further decompositions is preserved with equal weights across components, the weighted *geometric* mean is required. We also show how the weights can be chosen using a variation of a theorem by Koopmans (1957). In the end, our paper provides a mathematically consistent and economic-theory justified way of aggregation of Farrell-type efficiency scores.

Key words: Farrell efficiency, Index Aggregation
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Introduction

In the classic paper: "The Measurement of Productive Efficiency," Farrell (1957) furnished a decomposition of his cost efficiency index or overall efficiency as he terms it. He shows that it can be multiplicatively partitioned into a technical component and an allocative component, which he calls the price efficiency. Farrell also discusses the aggregation of firm efficiency into industry efficiency measures, but he does not discuss under what conditions such aggregation can be performed while preserving the decomposition.

In this paper we show that in order to aggregate individual efficiency into a group (e.g., industry) efficiency, in such a way that the multiplicative structure of further decompositions is preserved with equal weights across components, the weighted *geometric* mean is required. We also show how the weights can be chosen using a variation of Koopmans's (1957) aggregation theorem. Thus our paper provides researcher with a mathematically consistent and economic-theory justified way of aggregating the Farrell-type efficiency indexes.

The Model

In this paper we derive our results for Farrell cost efficiency approach, but of course our methodology applies to other cases (such as revenue efficiency, etc.) as well.

Let r_k and s_k ($k=1, 2$) be firm k's two *component* measures of efficiency and let their product $q_k = r_k s_k$ be the Farrell cost measure of overall efficiency. In general, $k = 1, \ldots, K$, but for the sake of simplicity of notation, we will present the results for the case when $k = 1, 2$. Suppose we want to aggregate these measures into group (e.g., industry of K firms) measure while preserving the *multiplicative* structure of decomposition.

This results in the following functional equation,

$$V(q_1, q_2) = V(r_1, r_2)\ V(s_1, s_2), \qquad (1)$$

where V is some function aggregating individual indexes over k. Let us generalize this equation by introducing a set of parameters $z = (z_1, \ldots, z_J) \in \Re^J$, i.e.,

$$U(q_1, q_2; z) = U(r_1, r_2; z)\ U(s_1, s_2; z). \qquad (2)$$

The solution to this equation is (see Aczél, 1990, p.27 and Eichhorn 1978, p.94)

$$U(q_1, q_2; z) = q_1^{\alpha_1(z)} q_2^{\alpha_2(z)}$$

$$U(r_1, r_2; z) = r_1^{\alpha_1(z)} r_2^{\alpha_2(z)}, \qquad U(s_1, s_2; z) = s_1^{\alpha_1(z)} s_2^{\alpha_2(z)} \qquad (3)$$

where $\alpha_k(z)$, are arbitrary functions of z.

Thus, we have shown that aggregating the cost efficiency while preserving the decomposition and equal weights across components requires a weighted *geometric* mean procedure, i.e.,

$$\left(q_1^{\alpha_1(z)} q_2^{\alpha_2(z)}\right) = \left(r_1^{\alpha_1(z)} s_1^{\alpha_1(z)}\right) \times \left(r_2^{\alpha_2(z)} s_2^{\alpha_2(z)}\right)$$

$$= \left(r_1^{\alpha_1(z)} r_2^{\alpha_2(z)}\right) \times \left(s_1^{\alpha_1(z)} s_2^{\alpha_2(z)}\right) \quad (4)$$

where the group indexes are: $\left(q_1^{\alpha_1(z)} q_2^{\alpha_2(z)}\right)$, $\left(r_1^{\alpha_1(z)} r_2^{\alpha_2(z)}\right)$, $\left(s_1^{\alpha_1(z)} s_2^{\alpha_2(z)}\right)$, respectively.

Next, for practical purposes, we want to determine the weights $\alpha_1(z)$ and $\alpha_2(z)$. For this, define the group input requirement set as

$$\overline{L}(y^1, y^2) = L^1(y^1) + L^2(y^2) \quad (5)$$

where y^1 and y^2 are output vectors for each firm and where $L^1(y^1)$ and $L^2(y^2)$ are the firms input requirement sets, i.e.,

$$L^k(y^k) = \{x^k \in \Re_+^N : x^k \ \mathit{can\ produce}\ y^k\}, \qquad y^k \in \Re_+^M,\ k = 1, 2. \quad (6)$$

Given a vector of input prices $w \in \Re_+^N$, equal across firms, the group and the firm's cost functions are given by

$$\overline{C}(y^1, y^2, w) = \min_x \{wx : x \in \overline{L}(y^1, y^2)\}, \quad (7)$$

and

$$C^k(y^k, w) = \min_{x^k} \{wx^k : x^k \in L^k(y^k)\}, \quad (8)$$

respectively. The following statement is a variation of Koopmans' (1957) aggregation theorem

$$\overline{C}(y^1, y^2, w) = C^1(y^1, w) + C^2(y^2, w), \quad (9)$$

(proof of this statement can be found in Färe, Grosskopf and Zelenyuk (2004) and for the revenue analogue in Färe and Zelenyuk (2003)). From the last expression it follows that the group cost efficiency index is the share weighted average of the efficiencies of all the firms within the group, i.e.,

$$Q \equiv \frac{\overline{C}(y^1, y^2, w)}{\sum_{n=1}^{N} w_n (x_{1n} + x_{2n})} = \frac{C^1(y^1, w)}{\sum_{n=1}^{N} w_n x_{1n}} S^1 + \frac{C^2(y^2, w)}{\sum_{n=1}^{N} w_n x_{2n}} S^2, \quad (10)$$

where the weights are the firm's cost shares, i.e.,

$$S^k = \frac{\sum_{n=1}^{N} w_n x_{kn}}{\sum_{n=1}^{N} w_n (x_{1n} + x_{2n})}, \qquad k = 1, 2. \tag{11}$$

It might be worth noting that an advantage of these weights is that they are not ad hoc (although might be exactly what one would expect them to be) but derived from the aggregation structure (5) that we imposed on the group technology, the economic optimization (here cost minimization) behavior and equal across the firms input prices—all needed to prove (9).

As a result, the group cost efficiency (or group overall efficiency) index would be

$$Q = q_1 S^1 + q_2 S^2, \tag{12}$$

Next, taking the first order Taylor series approximation of (3) around $q_1 = q_2 = 1$ (which is a natural point for the Farrell-type efficiency index to be approximated around), we obtain

$$U(q_1, q_2; z) \cong 1^{\alpha_1(z)} 1^{\alpha_2(z)} + \alpha_1(z)\left(1^{\alpha_1(z)-1} 1^{\alpha_2(z)}\right) + \alpha_2(z)\left(1^{\alpha_1(z)} 1^{\alpha_2(z)-1}\right)$$

i.e.,

$$U(q_1, q_2; z) \cong \alpha_1(z) q_1 + \alpha_2(z) q_2 \tag{13}$$

By equating (12) and (13) we get

$$\alpha_1(z) = S^1, \;\; \alpha_2(z) = S^2.$$

which gives us particular weights that can be used for the geometric aggregation obtained in (3).

Concluding Remarks

We have presented a practical way to aggregate the overall Farrell efficiencies of individual firms into the group (e.g., industry) efficiency index so that the decomposition that exists on the disaggregated level is also preserved on the aggregate level. Such aggregation is based on the weighted geometric mean. To determine economically meaningful weights we turned to a cost function analog of the Koopmans (1957) theorem on aggregation of profit functions and obtain weights for our aggregation to be the observed cost shares of individual firms in the group. Such approach should prove to be useful for researchers challenged with a question of efficiency of industries as well as various groups (e.g., regulated vs. non-regulated, foreign vs. domestic, etc) within such industries.

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