Continuous time modeling of interest rates: An empirical study on the Turkish short rate

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Abstract

We proposed a continuous time ARMA known as CARMA(p,q) model for modeling the interest rate dynamics. CARMA(p,q) models have an advantage over their discrete time counterparts that they allow using Ito formulas and provide closed-form solutions for bond and bond option prices. We demonstrate the capabilities of CARMA(p,q) models by using Turkish short rate. The Turkish Republic Central Bank's benchmark bond prices are used to calculate short-term interest rates between the period of 15.07.2006 and 15.07.2008. ARMA(1,1) model and CARMA(1,0) model are chosen as best suitable models in modeling the Turkish short rate.

Introduction

The dynamics of interest rates play an important role in economics and finance, especially in macroeconomic policy making , derivative pricing, hedging and risk management for fixed income securities. There is a massive amount of literature which were devoted to modeling the dynamics of interest rates in developed markets. These include, among many others, Ait-Sahalia [1] [2], Stanton [12], Chapman and Pearson [8], Hong et.al [10], and Pritsker [11]

These studies show some important properties of spot interest rates in developed financial markets, especially in the U.S. markets. According to the studies, there exist a significant mean reverting for the U.S. interest rates, although existence of a non-linear drift is inconclusive. Chan et.al. [7] and Hong et.al. [10]

find that the interest rate volatility tends to be higher when the interest rate level is higher, which is often characterized by a constant eleasticity variance (CEV) specification. Moreover, it is also important to capture conditional heteroskedasticity of interest rates by stochastic volatility / GARCH models [3]. On the other hand, Gray [9] points out that regime switching and jump models help capturing volatilty clustering and particularly the excess kurtosis and heavy tails of spot interest rates.

While the interest rate dynamics has been well documented in the literature for the developed markets, there has been not enough study on interest rate dynamics in Turkey and other developing countries. The main purpose of this study is to characterize the behaviour of Turkish interest rates by continuous time econometric models.

1. Methodology

In this paper, we propose a model that the interest rate dynamics can be characterized by a continuous time auto-regressive moving average (CARMA) model. Discrete time ARMA models have been used in the literature of time series analysis (see e.g. Box and Jenkins). Also there are few examples of modeling the time series behaviour of interest rates with Box-Jenkins approach. But there is not any study, we are aware of, that employ a CARMA(p,q) in modeling the interest rate dynamics in Turkey. The advantage of continuous time ARMA model that we can use the Ito formula and derivative pricing tools. The CARMA(p,q) model allows for closed-form solution to bond and bond option prices.

The parameters of the CARMA(p,q) model can be found by using the autocovariance function as a corresponding tool between discrete and continuous model. But before we do this, we briefly outline the procedure for the general case CARMA(p,q). A CARMA(p,q) process with $0 \leq q < p$ is defined to be a stationary solution to the process :

$$a(D)Y_t = b(D)DW_t \, , \, t \geq 0 \qquad (1)$$

where D denotes differentiation with respect to t , and

$$a(z) = z^p + a_1 z^{p-1} + \cdots + a_p$$

$$b(z) = b_0 + b_1 + \cdots + b_q z^q \qquad (2)$$

Following Brockwell [5] and Benth et.al. [4], we interpret (1) as being equivalent to observation and state equations

$$Y_t = b^T X_t \tag{3}$$

$$dX_t = AX_t dt + e dW_t \tag{4}$$

An explicit solution of the state equation reads;

$$X_t = e^{At} X_0 + \int_0^t e^{A(t-u)} dW_u \tag{5}$$

$$\sum = E[X_0 X_0^T] = \int_0^{\infty} e^{Ay} e e^{Ty} dy \tag{6}$$

Stationary condition requires the real parts of eigenvalues of A must be negative.

$E[X_t] = 0 \ \ (t \geq 0)$

$E\left[X_{t+h}, X_i^T\right] = e^{Ah} \Sigma \ \ (h \geq 0)$

Thus we can express autocovariance function of CARMA(p,q) process as;

$$\gamma_y(h) = \Sigma_{\lambda i} \frac{b(\lambda) b(-\lambda)}{a^T(\lambda) a(-\lambda)} e^{\lambda(h)} \tag{7}$$

The autocovariance function of a discrete case ARMA(p,q) can be expressed as;

$$\gamma_y(h) = -\sigma^2 \sum_{j=1}^{p} \frac{\lambda_j^{|h|+1} \theta(\lambda) \theta(-\lambda)}{\phi(\lambda_j) \phi'(\lambda_j^{-1})} \tag{8}$$

Parameters $a_0, \dots, a_p, b_0, \dots, b_q$ of the CARMA(p,q) $(0 \leq q < p)$ can be found from the parameters $\Phi_0, \dots, \phi_p, \theta_0, \dots, \theta_q$ of the ARMA(p,q) by comparing the autocovariance functions of CARMA(p,q) (7) and of ARMA(p,q) (8).

2. Data Analysis

The data we are going to use in our analysis are the daily rates of the Turkish Republic Central Bank's benchmark bond between the period of 15.07.2006 and 15.07.2008 (729 observation). The interest rates have been calculated by the authors by using the bond prices. The following figure exhibits the time series behaviour of the bond rate during the sample period.

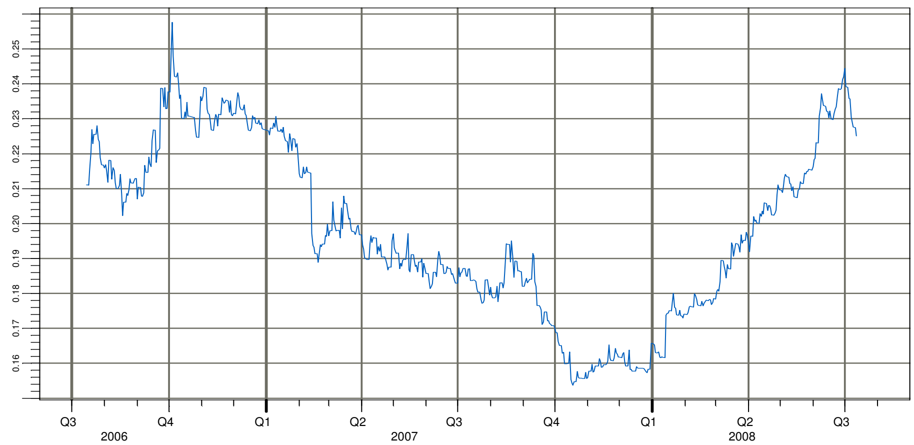

Figure 1: TCMB Benchmark Bond Rates

Source: Turkish Republic Central Bank's Web Page http://www.tcmb.gov.tr/

The series seem to have a non-stationary pattern therefore we need to employ unit-root tests to analyze the stationary structure of the series.

Table 1

Unit-Root Test

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		0.092674	0.7118
Test critical values:	1% level	-2.568151	
	5% level	-1.941260	
	10% level	-1.616406	

Source: Own Study

The unit-root tests of Augmented Dickey-Fuller indicates that the series have a unit-root by not rejecting the null hypothesis.Therefore, we need to transform them into stationary series by taking the first difference. The figure 2 below show the differenced series of the interest rates. The differenced series are stationary therefore we are going to use the first difference of the series in our analysis.

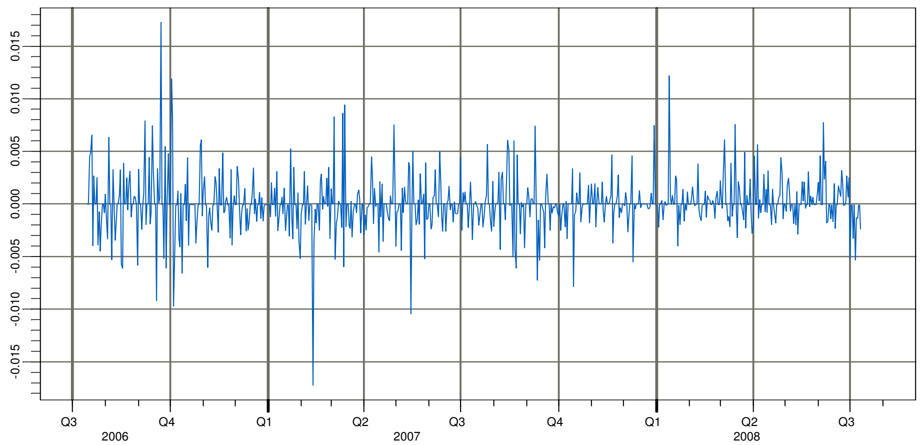

Figure 2: TCMB Benchmark Bond Rates (First Difference)
Source: Own Study

3. Empirical Results

In this part we firstly identify the discrete model for our sample data and then find the corresponding continuous time analog of the discrete time model. The Akaike information criteria (AIC) and Schwarz information criteria (BIC) suggest that we should use ARMA(1,1) model ,which has the lowest AIC and BIC statistics, for our data. The model estimation output can be seen in table 1 below.

Table 2

ARMA(1,1) Model Parameters

	Value	Standard Error	t-value
ar(1)	0.7045	0.1954	3.605
ma(1)	0.7664	0.1769	4.333

Source: Own Study

The discrete model in our case can be written as

$$r_t = 0.7045 r_{t-1} + 0.7664 e_{t-1} + e_t \tag{9}$$

After we have estimated a discrete time ARMA(1,1) model for our data, we move on to find the continuous time equivalent for this model. By using equations

(3) and (4), we find continuous time analog of the model as CARMA(1,0), and state equation reads as;

$$dX_t = -aX_t dt + bdWt \tag{10}$$

Since, the residuals from the discrete time equation are not normally distributed, we need to re-express equation (10) by replacing Wiener process with a Lévy process Benth et.al [4] and Brockwell [6]. The Lévy driven CARMA(1,0) model in this case reads as;

$$dX_t = -aX_t dt + bdLt \tag{11}$$

By using autocovariance functions of ARMA(1,1) and CARMA(1,0) models the parameters of the CARMA(1,0) model can be found as; $a = -ln\phi, b = -\sqrt{\frac{2\sigma^2}{1-\phi^2}}ln\phi$ [5]. The CARMA(1,0) model parameter in our case are $a = 0.35\ and\ b = 1.18$. Therefore, continuous time model for the Turkish short rates can be modeled as;

$$dX_t = -0.35X_t dt + 1.18dLt \tag{12}$$

By numerically solving equation (11) we simulate a series that follow Lévy driven CARMA(1,0). The figure 3 shows the simulated series from equation (12).

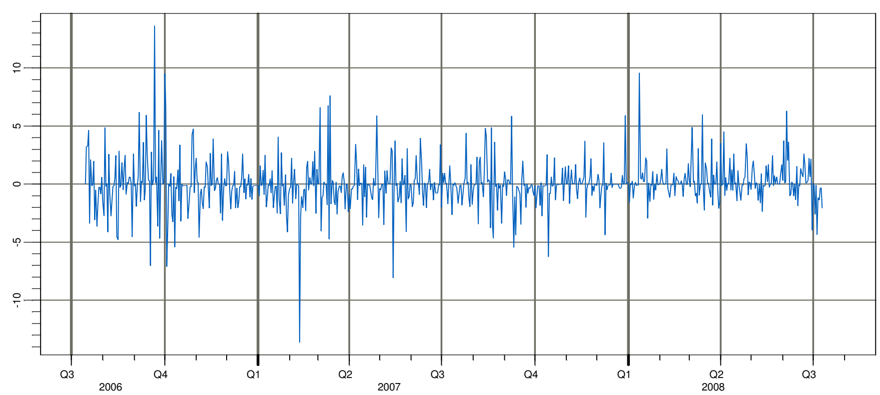

Figure 3: CARMA(1,0) Model Simulation
Source: Own Study

Moreover, we can have a clearer view about the model and its estimation power, by plotting the original series instead of differenced ones, and simulated series on the same plane. The figure 4 below indicates that CARMA(1,0) model is successful in modeling the interest rate dynamics as both real series and fitted series have same moving tendency but continuous time simulations are more volatile.

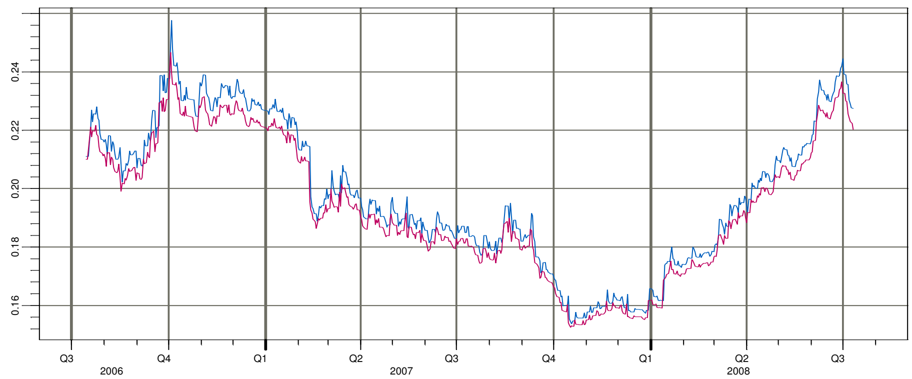

Figure 4: Real Series vs. CARMA(1,0) Fitted Series
Source: Own Study and Turkish Republic Central Bank's Web Page http://www.tcmb.gov.tr/

Conclusion

This paper is an analysis of the Turkish interest rates by using discrete and continuous time ARMA(p,q) models. We have used daily data of the Turkish benchmark bond rate as sample data between the period of 15.07.2006 and 15.07.2008. The CARMA(1,0) model is choosen as best canditate model for the sample data and it has been proven as a parsimonuous model for the Turkish short rate. CARMA(p,q) models have an advantage over discrete time counterparts that they allow us to use closed-form solutions for interest rate derivatives and bond pricing. Although we have used a Lévy driven CARMA(p,q) model for our analysis we do not consider the volatilty part. A future research on Contnuous time GARCH known as COGARCH(p,q) models could help us to develop our model and predict interest rate movements better.

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