Asymmetric Baxter-King filter

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Abstract

The paper proposes an extension of the symmetric Baxter-King band pass filter to an asymmetric Baxter-King filter. The optimal correction scheme of the ideal filter weights is the same as in the symmetric version, i.e, cut the ideal filter at the appropriate length and add a constant to all filter weights to ensure zero weight on zero frequency. Since the symmetric Baxter-King filter is unable to extract the desired signal at the very ends of the series, the extension to an asymmetric filter is useful whenever the real time estimation is needed. The paper uses Monte Carlo simulation to compare the proposed filter's properties in extracting business cycle frequencies to the ones of the original Baxter-King filter and Christiano-Fitzgerald filter. Simulation results show that the asymmetric Baxter-King filter is superior to the asymmetric default specification of Christiano-Fitzgerald filter in real time signal extraction exercises.

Keywords: real time estimation, Christiano-Fitzgerald filter, Monte Carlo simulation, band pass filter

JEL code: C13, C22

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1 Introduction

The paper proposes an extension of the symmetric Baxter-King band pass filter (Baxter and King, 1999) to an asymmetric Baxter-King filter. Such modification, to the best of my knowledge, has not been discussed in the literature. Symmetric filters are not applicable at the very ends of an input signal without the extension of the ends with forecasts. Thus, asymmetric band pass filters are necessary to extract the desired band of frequencies at the ends of an input signal, if forecasting is not used for extending the ends of the input signal.

The closest band pass filter to the Baxter-King filter is Christiano-Fitzgerald band pass filter (Christiano and Fitzgerald, 2003) which, in general, is asymmetric, and whose default specification is optimized for an input signal following a random walk (RW) process, but it allows the input signal to follow other data generating processes (DGP). However, Christiano and Fitzgerald (2003) argue that their default specification of the filter is a good approximation to many DGPs observed in macroeconomic time series and, thus, macroeconomists may opt for it. Although Christiano and Fitzgerald (2003) compares their filter to the symmetric Baxter-King filter, they do not elaborate on an asymmetric version of the Baxter-King filter.

This paper formally develops an asymmetric version of the Baxter-King filter and assesses its properties in extracting business cycle frequencies, in comparison to the symmetric Baxter-King filter, and symmetric and asymmetric default specification of Christiano-Fitzgerald filter, by using Monte Carlo simulation. The results show that, given the considered DGP and metric (estimated correlation of the true and extracted cycles at time t), the asymmetric Baxter-King filter is superior to the asymmetric Christiano-Fitzgerald filter at the very ends of a sample, thus indicating that the asymmetric Baxter-King filter should be preferred over the asymmetric Christiano-Fitzgerald filter in real time signal extraction exercises. Several other 'interesting' results are obtained, like, fixed-length symmetric filters outperforming their asymmetric counterparts, and shorter-length symmetric filters outperforming longer-length symmetric filters (which is contrary to the assertion by Christiano and Fitzgerald, 2003, and *a priori* belief of mine).

The paper is organized as follows. Section 2 develops the filter, Section 3 assesses the performance of the filter by means of Monte Carlo simulation, and Section 4 concludes.

2 The asymmetric Baxter-King filter

Consider the following orthogonal decomposition of the zero-mean covariance stationary stochastic process, x_t:

$$x_t = y_t + \tilde{x}_t. \tag{1}$$

The process, y_t, has power only in frequencies (measured in radians) belonging to the interval $\{[a_1, a_2] \cup [-a_2, -a_1]\} \subset (-\pi, \pi)$, where $0 < a_1 < a_2 < \pi$. The process, $\tilde{x}_t$, has power only in the complement of this interval in $(-\pi, \pi)$. By the spectral representation theorem,

$$y_t = b(L)x_t, \tag{2}$$

where the ideal band pass filter, $b(L)$, is

$$b(L) = \sum_{h=-\infty}^{\infty} b_h L^h, \quad L^h x_t = x_{t-h}, \tag{3}$$

with

$$\begin{aligned} b_h &= \frac{\sin(ha_2) - \sin(ha_1)}{\pi h}, \quad h = \pm 1, \pm 2, \ldots \\ b_0 &= \frac{a_2 - a_1}{\pi}, \quad a_1 = \frac{2\pi}{p_u}, \quad a_2 = \frac{2\pi}{p_l}, \end{aligned} \tag{4}$$

and $p_u, p_l \in (2, \infty)$ define the upper and lower bounds of the wave length of interest. With b_h's specified as in (4), the frequency response function of the ideal filter at frequency ω is

$$\begin{aligned} \beta(\omega) &= 1 \quad \text{for } \omega \in [a_1, a_2] \cup [-a_2, -a_1] \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{5}$$

Baxter and King (1999) have proposed to obtain a symmetric, fixed length approximation to the ideal filter, (3) and (4), by minimizing

$$\begin{aligned} &Q = \int_{-\pi}^{\pi} \delta(\omega)\delta(-\omega)d\omega \\ &\text{s.t.} \\ &\hat{\beta}(0) = \sum_{k=-K}^{K} \hat{b}_k = 0 \\ &\hat{b}_k = \hat{b}_{-k}, \end{aligned} \tag{6}$$

where $\delta(\omega) = \beta(\omega) - \hat{\beta}(\omega)$ is the discrepancy between the exact and the approximate filters at frequency ω, and the constraint $\hat{\beta}(0) = 0$ is to ensure zero weight on the trend frequency, in line with the assumption $a_1 > 0$. The solution to (6) is a truncation of the ideal filter symmetrically at length K, and addition of a constant $(-\sum_{k=-K}^{K} b_k)/(2K+1)$ to all filter weights to ensure $\hat{\beta}(0) = 0$. Baxter and King (1999) suggest the value of K to be about 3 years, i.e, K=12 for quarterly data, and K=36 for monthly data. The symmetry of the filter together with the condition $\hat{b}_k = \hat{b}_{-k}$ implies that the filter renders stationary time series that is integrated of oder 2 (I(2)) or less. Thus, the symmetric BK filter has trend-reduction property and, therefore, it can be applied to nonstationary, up to I(2) series.

Since the symmetric BK filter can not be used to extract the desired frequencies at the very end (for the first and the last K observations) of the input series, a natural extension of the Baxter and King (1999) filter is to allow the approximate filter to be asymmetric, to be able to use the filter in real time. In order to optimally approximate an ideal symmetric linear filter in a Baxter-King sense, the problem is to minimize

$$\begin{aligned} Q &= \int_{-\pi}^{\pi} \delta(\omega)\delta(-\omega)d\omega \\ &\text{s.t.} \\ \hat{\beta}(0) &= \sum_{h=-p}^{f} \hat{b}_h = 0. \end{aligned} \tag{7}$$

The condition $\hat{\beta}(0)$ ensures zero weight on zero frequency, thus this asymmetric filter also has a trend-reduction property, however, it alone, without symmetry, is not sufficient to render I(2) process stationary. Thus, the ability of the asymmetric BK filter of real time signal extraction comes at a cost of losing the power to eliminate two unit roots from the input series.

To solve (7), form the Lagrangian

$$\mathcal{L} = Q - \lambda\hat{\beta}(0) \tag{8}$$

with first order conditions (FOCs):

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\hat{b}_h} &= \frac{\partial Q}{\partial\hat{b}_h} - \lambda = 0 \\ \frac{\partial\mathcal{L}}{\partial\lambda} &= -\hat{\beta}(0) = 0. \end{aligned} \tag{9}$$

Since

$$\frac{\partial}{\partial \hat{b}_h}[\delta(\omega)\delta(-\omega)] = \frac{\partial \delta(\omega)}{\partial \hat{b}_h}\delta(-\omega) + \delta(\omega)\frac{\partial \delta(-\omega)}{\partial \hat{b}_h}, \tag{10}$$

and since the frequency response function of the approximating filter is $\hat{\beta}(\omega) = \sum_{h=-p}^{f} \hat{b}_h e^{-i\omega h}$, it follows that

$$\frac{\partial \delta(\omega)}{\partial \hat{b}_h} = -e^{-i\omega h}. \tag{11}$$

(7), (10) and (11) imply

$$\frac{\partial Q}{\partial \hat{b}_h} = -\int_{-\pi}^{\pi} [e^{-i\omega h}\delta(-\omega) + \delta(\omega)e^{i\omega h}]d\omega. \tag{12}$$

By the property $\int_{-\pi}^{\pi}[f(\omega) + f(-\omega)]d\omega = 2\int_{-\pi}^{\pi} f(\omega)d\omega$ (since $\int_{-\pi}^{\pi} f(\omega)d\omega = \int_{0}^{\pi} f(\omega)d\omega + \int_{-\pi}^{0} f(\omega)d\omega = \int_{0}^{\pi}[f(\omega) + f(-\omega)]d\omega$ is real, then $\int_{-\pi}^{\pi} f(\omega)d\omega = \int_{-\pi}^{\pi} f(-\omega)d\omega$, and the property follows), (12) becomes

$$\frac{\partial Q}{\partial \hat{b}_h} = -2\int_{-\pi}^{\pi} \delta(\omega)e^{i\omega h}d\omega. \tag{13}$$

By the property

$$\begin{aligned}\int_{-\pi}^{\pi} e^{i\omega n}e^{-i\omega m}d\omega = \int_{-\pi}^{\pi} e^{-i\omega(m-n)}d\omega &= 0 \quad \text{for } n \neq m \\ &= 2\pi \quad \text{for } n = m,\end{aligned} \tag{14}$$

obtain

$$\int_{-\pi}^{\pi} \delta(\omega)e^{i\omega h}d\omega = \int_{-\pi}^{\pi}\left[\sum_{k=-\infty}^{\infty} b_k e^{-i\omega k} - \sum_{j=-p}^{f} \hat{b}_j e^{-i\omega j}\right] e^{i\omega h}d\omega = 2\pi[b_h - \hat{b}_h]. \tag{15}$$

Given (15), the FOCs are

$$-4\pi[b_h - \hat{b}_h] - \lambda = 0. \tag{16}$$

If there is no constraint on $\hat{\beta}(0)$, the optimal approximate (in Baxter-King sense) filter is simply derived by truncation of the ideal filter's weights. If there is a constraint on $\hat{\beta}(0)$, then λ must be chosen so that the constraint is satisfied. For this purpose, rewrite (16) as

$$\hat{b}_h = b_h + \theta,$$

where $\theta = \lambda/(4\pi)$. In order to have $\hat{\beta}(0) = \sum_{h=-p}^{f} \hat{b}_h = 0$, the required adjustment is

$$\theta = \frac{-\sum_{h=-p}^{f} b_h}{p+f+1}, \tag{17}$$

which yields the same optimal weight adjustment scheme as in the symmetric Baxter-King filter case.

The next section describes results from Monte Carlo simulation to assess the performance of the proposed filter.

3 Comparing filters by means of Monte Carlo simulation

This section assesses the performance of the proposed filter to extract business cycle frequencies (corresponding to wave length between 1.5 and 8 years) in comparison to the original BK filter, as well as symmetric and asymmetric Christiano-Fitzgerald (CF) filter which is optimized for an input signal following a random walk (RW) process (Christiano and Fitzgerald, 2003). Thus, the asymmetric CF filter assumes that the first difference of the input signal is mean-zero covariance stationary process. The symmetric CF filter allows for the input signal to follow RW with drift.

Consider the following data generating process (DGP):

$$y_t = \mu_t + c_t, \tag{18}$$

where

$$\mu_t = \mu_{t-1} + \epsilon_t \tag{19}$$

$$c_t = \phi_1 c_{t-1} + \phi_2 c_{t-2} + \eta_t \tag{20}$$

$$\epsilon_t \sim nid(0, \sigma_\epsilon^2),\ \eta_t \sim nid(0, \sigma_\eta^2). \tag{21}$$

Equation (18) defines a series, y_t, as the sum of a permanent component (stochastic trend), μ_t, and a cyclical component, c_t. The trend, μ_t, in this case is specified as a random walk process. The dynamics of the cyclical component, c_t, is specified as a second order autoregressive (AR(2)) process so that the peak of the spectrum of c_t could be at zero frequency or at business cycle frequencies. Disturbances, ϵ_t and η_t, are assumed to be uncorrelated.

The spectrum of an AR(2) process is

$$f_c(\omega) = \frac{\sigma_\eta^2}{1 + \phi_1^2 + \phi_2^2 - 2\phi_1(1-\phi_2)\cos\omega - 2\phi_2\cos(2\omega)} \tag{22}$$

with a peak at frequency other than zero for

$$\phi_2 < 0 \,\text{and}\, \left|\frac{\phi_1(1-\phi_2)}{4\phi_2}\right| < 1 \tag{23}$$

with the corresponding frequency $\omega = \cos^{-1}[-\phi_1(1-\phi_2)/(4\phi_2)]$ (Box, Jenkins and Reinsel, 1994; Priestley, 1981).

Data are generated from (18) with $\phi_1 = 1.2$ and different values for ϕ_2 to control the location of the peak in the spectrum of the cyclical component. I also vary the ratio of standard deviations of the disturbances, $\sigma_\epsilon/\sigma_\eta$, to change the relative importance of components of y_t. Such DGP can create series with spectral characteristics typical to macroeconomic variables, such as gross domestic product and inflation (Watson, 1986; Guay and St-Amant, 2005). The idea of such simulation is taken from Guay and St-Amant (2005).

Particularly, 10000 samples of length 401 are created, with the first 200 observations of each sample dropped off as burn-in. The vector $[\phi_1, \phi_2]$ is set to five different values, as shown in Table 1.

ϕ_1	ϕ_2	Fundamental period of the cycle (yrs)
1.2	-0.25	$\approx \infty$
1.2	-0.35	$\gg 8$
1.2	-0.44	8.2
1.2	-0.5	3.5
1.2	-0.8	1.9

Table 1: Five different values of $[\phi_1, \phi_2]$ for the DGP.

The value of $\sigma_\epsilon/\sigma_\eta$ is set to change from 0 to 9.9 with step size 0.15 (Watson (1986) estimated this ratio for the U.S. GNP to be 0.75).

I compare four filters in their capability to extract business cycle frequencies: i) symmetric, fixed-length BK filter with $K = 12$ (see (6)), ii) asymmetric BK filter described in Section 2, iii) symmetric, fixed-length CF filter with $K = 12$ for RW processes, and iv) default asymmetric specification of CF filter for RW processes.

The performance of filters is assessed by comparing the estimated correlation of the true cyclical component at time t with the estimated cyclical component at time t, $\hat{\rho}(c_t, \hat{c}_t)$, and by comparing the true AR(2) regression coefficients for the cycle with the fitted AR(2) regression coefficients. The distance measure $\hat{\rho}(c_t, \hat{c}_t)$ plays a key role from this point forward.

Figure 1 shows average estimated correlation between the true and estimated cyclical components, $\hat{\rho}(c_t, \hat{c}_t)$, for given values of $[\phi_1, \phi_2]$ and $\sigma_\epsilon/\sigma_\eta$. The correlation is estimated for the whole sample span except for the first and the last K=12 observations, since fixed-length symmetric filters do not produce the estimated cycle for those observations; these K=12 observations are deleted from

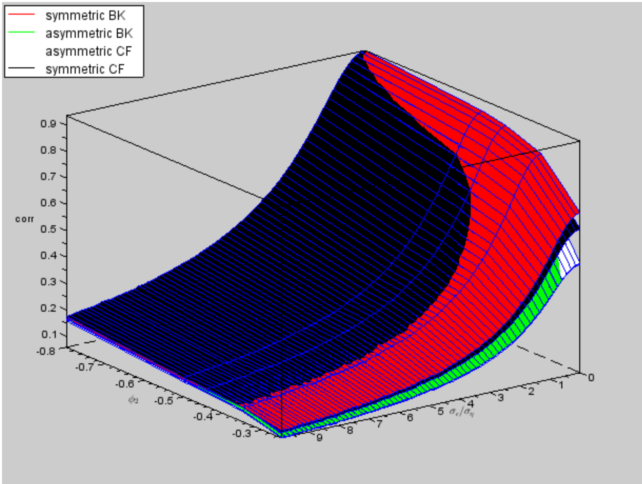

Figure 1: Average estimated correlation between the true and estimated cyclical components at time t for given $[\phi_1, \phi_2]$ and $\sigma_\epsilon/\sigma_\eta$ values. The correlation is estimated for the whole sample span except for the first and the last K=12 observations. The results show that, on average over the sample, symmetric filters are superior to asymmetric filters, and that asymmetric BK filter is superior to asymmetric CF filter.

the output of the asymmetric filters for a fair comparison between symmetric and asymmetric filters. Figure 1 shows a similar behavior between the filters - their performance decreases with $\sigma_\epsilon/\sigma_\eta$, which is an expected result. When $\sigma_\epsilon/\sigma_\eta = 0$, the input signal is the true cycle, so the output signal (estimated cycle) correlates highly with the input. As $\sigma_\epsilon/\sigma_\eta$ increases, the influence of the permanent component in the input increases, thus making harder for filters to extract the cycle, thus the estimated correlation between the true and estimated cycles, $\hat{\rho}(c_t, \hat{c}_t)$, decreases.

Figure 1 also shows that the performance of all filters decreases with an increasing ϕ_2. The value of $\phi_2 = -0.8$ together with $\phi_1 = 1.2$ corresponds to the length of the cycle 1.9 years, which is close to the usually defined minimum length of a business cycle, 1.5 years. The value of $\phi_2 = -0.44$ together with $\phi_1 = 1.2$ produces the cycle of length approximately 8.2 years, which is close to the usually defined maximum length of a business cycle, 8 years. With higher than $\phi_2 = -0.44$ values, the length of the true cycle rapidly increases. Although with $\phi_2 = -0.25$ the cycle still is considered stationary ($\phi_1 + \phi_2 < 1$, $\phi_2 - \phi_1 < 1$, and $|\phi_2| < 1$), it is a close approximation to a nonstationary process in a finite sample (Campbell and Perron, 1991). Thus, Figure 1 shows expected deterioration in performance of BK filters as ϕ_2 increases. The similar deterioration in performance of the CF filters with an increasing length of the cycle was less expected. Another unexpected result is the inferior performance of asymmetric filters to their shorter symmetric counterparts.

Figure 2 shows average correlation between the true and estimated cyclical components for given $[\phi_1, \phi_2]$ and $\sigma_\epsilon/\sigma_\eta$ values from fixed-length symmetric BK and CF filters. This figure shows that the distance between the two correlation surfaces is not high at any point. The observation from Figure 2 suggest that the performance of symmetric BK and CF filters is roughly the same regardless of the cycle length or the share of the permanent component.

Figure 3 shows the view from the top of Figure 2 to assess the regions of (although small, as seen in Figure 2) relative superiority of the fixed-length symmetric BK and CF filters. Figure 3 shows that fixed-length symmetric BK filter is superior to fixed-length symmetric CF filter for $0 \leq \sigma_\epsilon/\sigma_\eta < 0.5$ if cycle length is longer than 2 years. For most of the rest of the region, particularly - cycle length less than 8 years, given $\sigma_\epsilon/\sigma_\eta \geq 1$ - CF filter is slightly superior to the BK filter. For the remainder, i.e., $0.5 \leq \sigma_\epsilon/\sigma_\eta < 1$, CF filter shows superiority when cycle is relatively short (up to 3.5 years), and BK filter shows superiority when the cycle is longer than approximately 3.5 years.

Figure 4 shows comparison of $\hat{\rho}(c_t, \hat{c}_t)$ between fixed-length symmetric and asymmetric BK (Figure 4(a)) and CF (Figure 4(b)) filters. Figure 4 shows that symmetric filters are superior to asymmetric filters for all considered lengths of cycle and all proportions of permanent and cyclical components in the input series. When there is high influence of permanent component in the series, the performance of symmetric and asymmetric filters is close regardless of cycle length. As the influence of cyclical component rises, the performance of asymmetric filters (relative to symmetric ones) deteriorates with increasing length of cycle. Slightly more evident decrease of correlation between true and estimated cycles is for the asymmetric BK filter than for the CF filter. However, this is mainly due to the higher performance of symmetric BK filter compared to symmetric CF filter. The results in Figure 4 are in contrast to those drawn by Christiano and Fitzgerald (2003), who conclude that filters using all the data, which, therefore, are asymmetric and time-varying, improve the extraction of the desired frequencies, compared to fixed-length symmetric filters.

Figure 5 shows average correlation between the true and estimated cyclical components for given $[\phi_1, \phi_2]$ and $\sigma_\epsilon/\sigma_\eta$ values from asymmetric BK and CF filters. This figure shows that the distance between the two correlation surfaces is practically nil at all points. The observation from Figure 5 suggest that the performance, on average over the sample, of asymmetric BK and CF filters is almost the same regardless of the cycle length or the share of the permanent component.

Figure 6 shows the view from the top of Figure 5 to assess the regions of (small, as seen in Figure 5) relative superiority of the asymmetric BK and CF filters. Figure 6 shows that the asymmetric BK filter is superior to asymmetric CF filter. The reason of the slight superiority of the BK filter will be evident below, when comparing the performances at the ends of a sample. A slightly surprising finding from Figure 6 is the inability of asymmetric CF filter to perform better than the asymmetric BK filter in the region of high influence of the permanent component (corresponds to lower part of the graph).

Now, let us compare the performance of the asymmetric filters for the K=12 observations of the sample, where the fixed-length symmetric filters can not be applied. Figures 7 to 10 show the estimated correlation of the true and estimated cycles at each of the K=12 observations, calculated across the 10000 replications, and averaged over both symmetric ends. Figures 7 to 10 show the filters give close result at points closer to the center of the sample. Indeed, at the center of the sample, where the asymmetric filters become symmetric, the correlation graph looks similar to Figure 7(a), thus not shown here. As the estimation point approaches the end of the sample, filters become more asymmetric, and the difference in their performance becomes more obvious. Thus, the asymmetric filters perform roughly equally well at points that are at least about 3 years (for quarterly data) away from the end of the sample. Otherwise, the asymmetric BK filter becomes increasingly superior to the asymmetric CF filter for any cycle length and for any share of permanent component in the input signal considered in the simulation. Thus, based on Figures 7 to 10, it is recommended to use the asymmetric BK filter rather than the asymmetric CF filter for the business-cycle frequency extraction in real time, i.e, at the very end of the sample, given that the considered DGP and metric are appropriate in practice.

Figure 4 shows that, on average in the sample, the fixed-length symmetric filters show higher performance than their asymmetric counterparts. This result is somewhat unexpected, since Figure 4 says the symmetry or the shorter length of the fixed-length filters are more important than the higher length of an asymmetric filter, for better performance. Since the asymmetric filters become symmetric in the middle of a sample, it is tempting to compare the performance of filters in the middlepoint of the sample. Thus, Figure 11 shows correlation surfaces estimated at the middlepoint of the sample, where all filters are symmetric, but fixed-length filters are shorter than the asymmetric filters, with the latter spanning the whole sample length. The results show that the shorter filters outperform the longer ones, which is contrary to what most (know to me) literature takes as granted. More simulations results indicate (not shown here) that, for BK filter, $K = 13$ gives the best results, outperforming the one with $K = 12$, regardless of wave length of the cycle, or the influence of permanent component in the input signal. For CF filter, $K = 12$ appears to be optimal, unless $\sigma_\epsilon/\sigma_\eta > 1$, for which $K = 13$ seems to yield higher $\hat{\rho}(c_t, \hat{c}_t)$.

Figures 12 and 13 show the estimated regression coefficients on the first and second lag, respectively, from fitting an AR(2) model on the cyclical component extracted by the four filters. Figures 12 and 13 show both estimated regression coefficients converge to a constant as the influence of the permanent component on the input series increases, regardless of the true length of the cycle. While it is expected that $\hat{\phi}_1$ would be approximately constant, since ϕ_1 is always set to 1.2, it is not that plausible for $\hat{\phi}_2$ to converge to a constant, regardless of the value of ϕ_2. The results also show that, regardless of the considered true length of the cycle and regardless of the influence of the permanent component in the input series, $\hat{\phi}_1 \in (1.4, 1.7)$ and $\hat{\phi}_2 \in (-0.96, -0.84)$, i.e, ϕ_1 is always overestimated and ϕ_2 is always underestimated.

Table 2 shows the length of the cycle extracted by the filters, when the influence of the permanent component in the input series is sufficiently high, i.e., about $\sigma_\epsilon/\sigma_\eta > 5$. In such case, the length of the extracted cycle is about constant, regardless of the true length of the cycle.

	ϕ_1	ϕ_2	Fundamental period of the cycle (yrs)
true	1.2	-0.25	$\approx \infty$
	1.2	-0.35	$\gg 8$
	1.2	-0.44	8.2
	1.2	-0.5	3.5
	1.2	-0.8	1.9
symmetric BK	1.699	-0.886	3.56
asymmetric BK	1.689	-0.884	3.48
asymmetric CF	1.696	-0.879	3.60
symmetric CF	1.623	-0.848	3.23

Table 2: The true AR(2) parameters and cycle length, and the estimated AR(2) parameters and cycle length by the four filters, when the influence of the permanent component in the input series is sufficiently high, i.e., about $\sigma_\epsilon/\sigma_\eta > 5$. In such case, the estimated AR(2) parameters and the length of the extracted cycle are about constant, regardless of the true length, or existence, of the cycle.

4 Conclusions

This paper proposes an extension of the symmetric BK band pass filter to an asymmetric case. Such modification, to the best of my knowledge, has not been discussed in the literature. It turns out that the optimal correction scheme of the ideal filter weights is the same as in the symmetric version, i.e, cut the ideal filter at the appropriate length and add a constant to all filter weights to ensure zero weight on zero frequency. Since the symmetric BK filter is unable to extract the band of frequencies at the very end of the series without extending the ends with forecasts, an asymmetric filter is useful whenever the real time estimation is needed.

This paper formally develops an asymmetric version of the Baxter-King filter and assesses its properties in extracting business cycle frequencies, in comparison to the symmetric Baxter-King filter, and symmetric and asymmetric default specification of Christiano-Fitzgerald filter, by using Monte Carlo simulation. The results are as follows.

The performance in terms of correlation between the true and estimated cycles at time t of all considered filters decreases with $\sigma_\epsilon/\sigma_\eta$, which is an expected result, since an increasing presence of a trend makes it harder for filters to extract the cycle.

The performance of the filters decreases with increasing length of the cycle.

An unexpected result is the inferior performance of asymmetric filters to their shorter fixed-length symmetric counterparts for all lengths of the cycle and all considered/meaningful $\sigma_\epsilon/\sigma_\eta$. When there is high influence of permanent component in the series, the performance of symmetric and asymmetric filters is close regardless of cycle length. As the influence of cyclical component rises, the performance of asymmetric filters (relative to symmetric ones) deteriorates with increasing length of cycle. This result is in contrast to those drawn by Christiano and Fitzgerald (2003), who conclude that filters using all the data, which, therefore, are asymmetric and time-varying, improve the extraction of the desired frequencies, compared to fixed-length symmetric filters. More simulations results indicate that, for BK filter, $K = 13$ gives the best results, outperforming the one with $K = 12$, regardless of wave length of the cycle, or the influence of permanent component in the input signal. For CF filter, $K = 12$ appears to be optimal, unless $\sigma_\epsilon/\sigma_\eta > 1$, for which $K = 13$ seems to yield higher $\hat{\rho}(c_t, \hat{c}_t)$.

Although the performance of symmetric BK and CF filters is roughly the same regardless of the cycle length or the share of the permanent component, the symmetric BK filter is superior to the symmetric CF filter for $\sigma_\epsilon/\sigma_\eta$ between 0 and 0.5, given the length of the cycle is more than 2 years. For cycle length less than 8 years, given that $\sigma_\epsilon/\sigma_\eta$ is less than 1, the symmetric CF filter is slightly superior to the symmetric BK filter (as expected). For the remainder of the parameter region, i.e, $\sigma_\epsilon/\sigma_\eta$ between 0.5 and 1, the symmetric CF filter shows superiority when cycle is relatively short (up to about 3.5 years), otherwise symmetric BK filter is superior.

The asymmetric filters perform similarly at points closer to the center of the sample. Indeed, at the center of the sample, where the asymmetric filters become symmetric, the correlation surface over the parameter space looks similar to the one at the observation K=12, counting from the end of the sample. As the estimation point approaches the end of the sample, filters become more asymmetric, and the difference in their performance becomes more obvious. Thus, the asymmetric filters perform roughly equally well at points that are at least about 3 years away from the end of the sample. Otherwise, the asymmetric BK filter becomes increasingly superior to the asymmetric CF filter for any cycle lengths and for any share of permanent component in the input signal considered in the simulation. Thus, based on these results, it is recommended to use the asymmetric BK filter rather than the asymmetric CF filter for the business-cycle frequency extraction in real time, i.e, at the very end of the sample, given that the DGP and the metric, $\hat{\rho}(c_t, \hat{c}_t)$, considered in the exercise are relevant in practice.

Finally, both estimated AR(2) regression coefficients converge to a constant as the influence of the permanent component on the input series increases ($\sigma_\epsilon/\sigma_\eta > 5$), regardless of the true length of the cycle. While it is expected that the estimate of ϕ_1 would be approximately constant, since the true ϕ_1 is always set to 1.2, it is not that plausible for the estimate of ϕ_2 to converge to a constant, regardless of the true value of ϕ_2. The according length of the extracted cycle converges to about 3.2 (fixed-length CF) to 3.6 (asymmetric CF) years, regardless of the length (or presence) of the true cyclical component in the input signal. This result shows the potential limits of all the considered filters.

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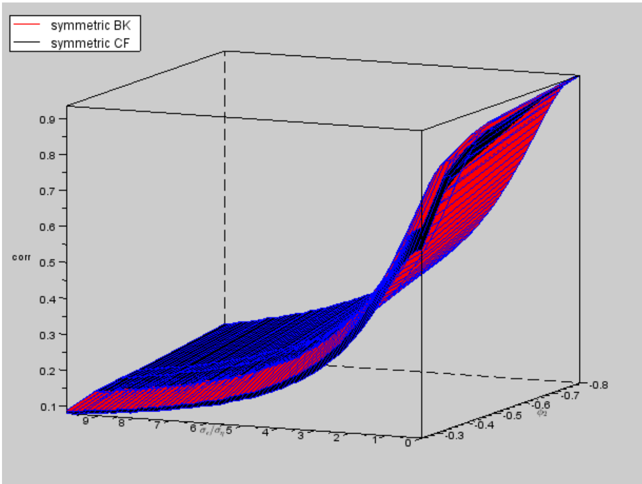


Figure 2: Average correlation between the true and estimated cyclical components at time t for given $[\phi_1, \phi_2]$ and $\sigma_\epsilon/\sigma_\eta$ values from fixed-length symmetric BK and CF filters. The correlation is estimated for the whole sample interval except for the first and the last K=12 observations. The performance of the filters is similar, regardless of wave length of the cycle or the influence of the permanent component.

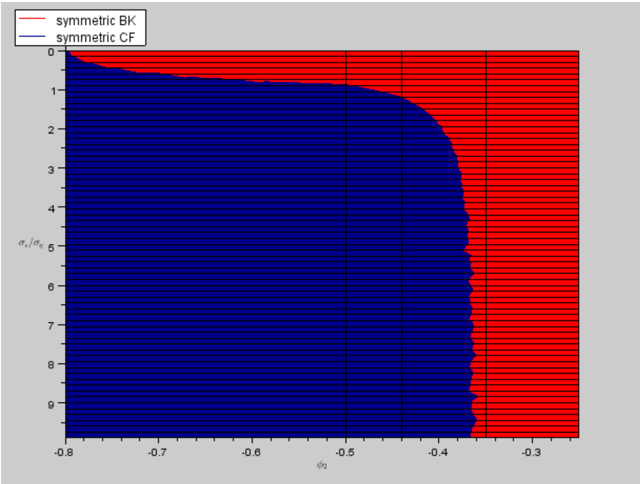
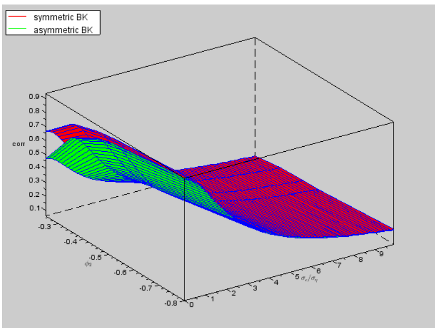
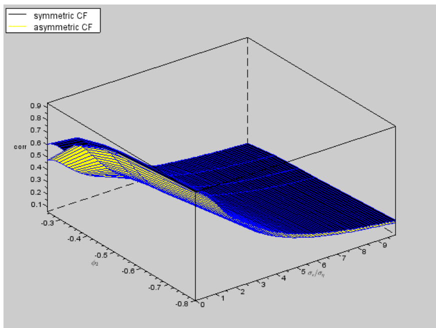


Figure 3: The view from the top of Figure 2, showing relative superiority of the fixed-length symmetric BK and CF filters. The horizontal axis represent cycle length, while the vertical axis represent the importance of permanent component in the series. Fixed-length symmetric BK filter is superior to fixed-length symmetric CF filter for $0 \leq \sigma_\epsilon/\sigma_\eta < 0.5$ if cycle length is longer than 2 years. For most of the rest of the region, particularly - cycle length less than 8 years, given $\sigma_\epsilon/\sigma_\eta \geq 1$ - CF filter is slightly superior to the BK filter. For the remainder, i.e., $0.5 \leq \sigma_\epsilon/\sigma_\eta < 1$, CF filter shows superiority when cycle is relatively short (up to 3.5 years), and BK filter shows superiority when the cycle is longer than approximately 3.5 years.



(a) BK



(b) CF

Figure 4: A comparison of performance (in terms of average $\hat{\rho}(c_t, \hat{c}_t)$ over the sample) of symmetric versus asymmetric filters. The left figure compares BK filters, and the right figure compares CF filters. The results show that symmetric filters are superior to their asymmetric counterparts.

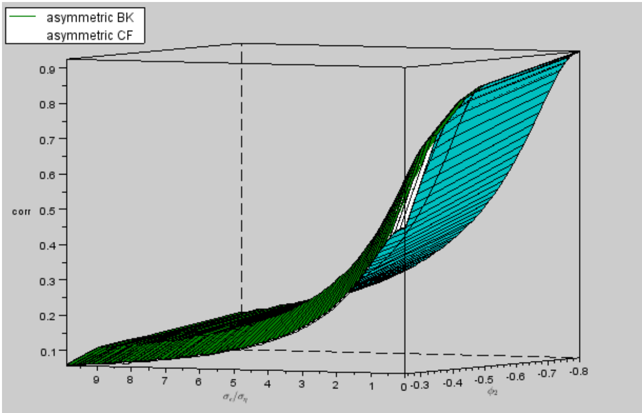


Figure 5: Average correlation between the true and estimated cyclical components at time t for given $[\phi_1, \phi_2]$ and $\sigma_\epsilon/\sigma_\eta$ values from asymmetric filters. The correlation is estimated for the whole sample interval except for the first and the last K=12 observations. Results show very similar performance of the filters - the correlation surfaces are almost identical.

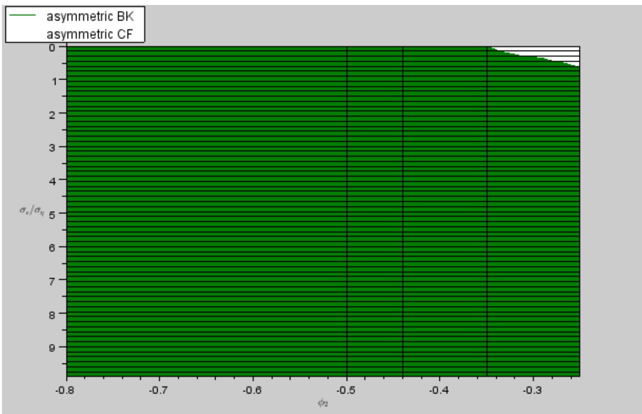
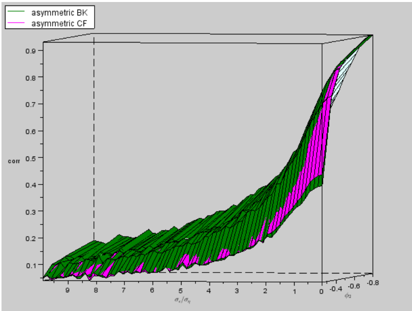
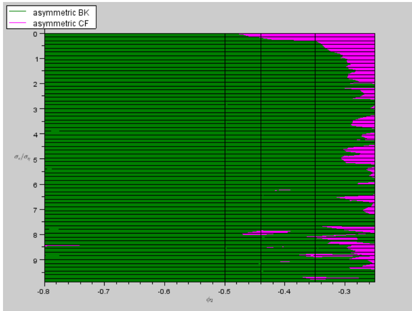


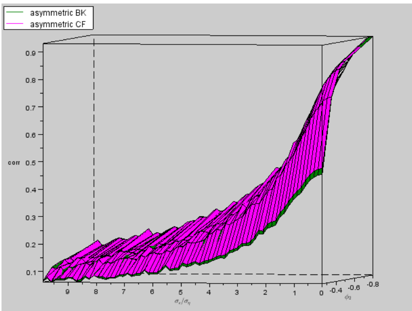
Figure 6: The view from the top of Figure 5, showing relative superiority of the asymmetric BK and CF filters. The horizontal axis represent cycle length, while the vertical axis represent the importance of permanent component in the series. Even if the average performance of the asymmetric filters over the sample is very close, this figure shows that asymmetric BK filter is persistently superior to the asymmetric CF filter, regardless of wave length of the cycle, or the influence of permanent component in the input signal. This result is due to BK filter's superiority at the ends of a sample, see below.



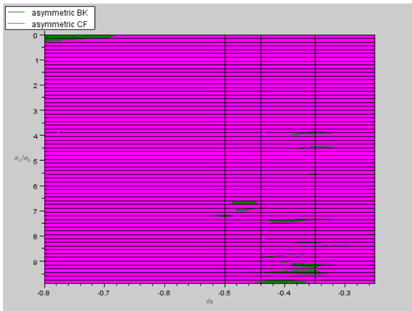
(a) Correlation at obs. 12



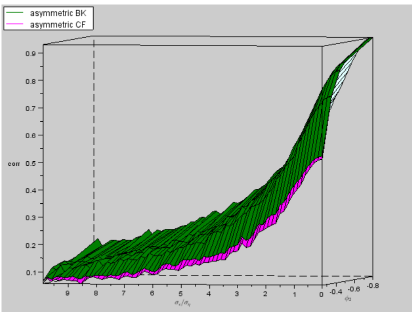
(b) view at 7(a) from the top



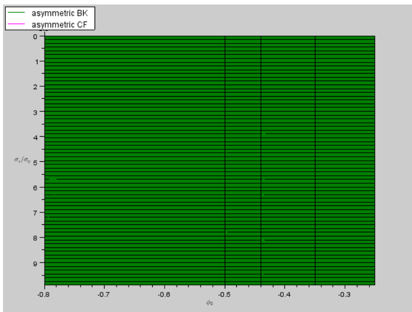
(c) Correlation at obs. 11



(d) view at 7(c) from the top

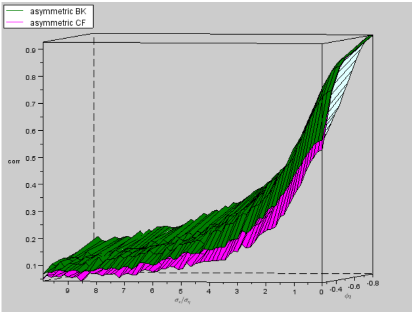


(e) Correlation at obs. 10

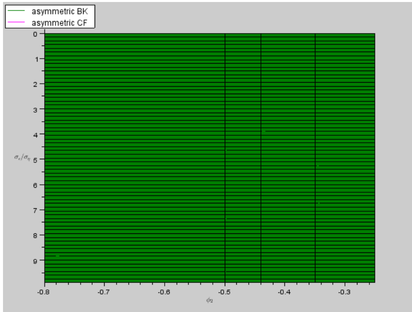


(f) view at 7(e) from the top

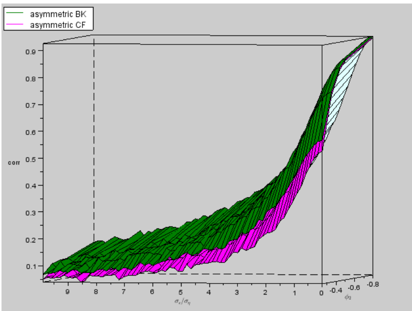
Figure 7: Estimated correlation of the true and extracted cycles at time t, $\hat{\rho}(c_t, \hat{c}_t)$, by asymmetric BK and CF filters at observations number 12 to 10, counting from the end of the series. The results show a close performance of the two filters, although the BK filter performs slightly better than the CF filter.



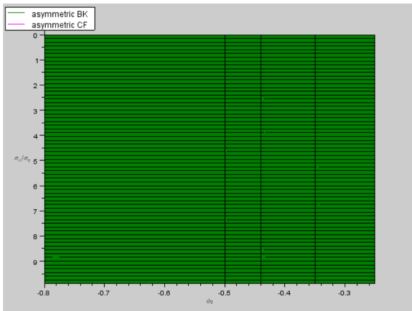
(a) Correlation at obs. 9



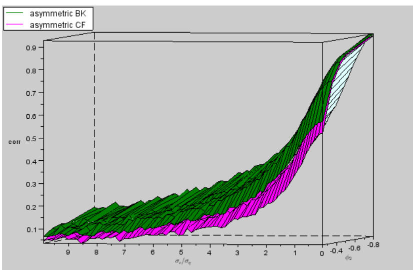
(b) view at 8(a) from the top



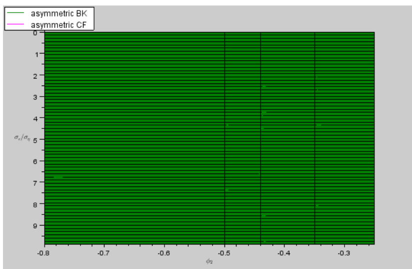
(c) Correlation at obs. 8



(d) view at 8(c) from the top

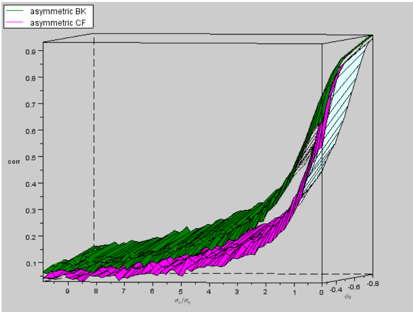


(e) Correlation at obs. 7

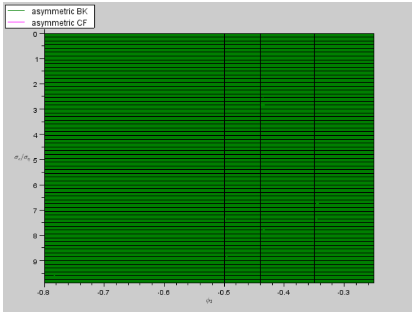


(f) view at 8(e) from the top

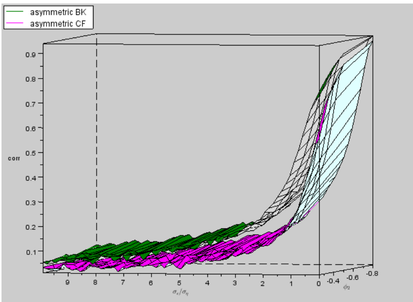
Figure 8: Estimated correlation of the true and extracted cycles at time t, $\hat{\rho}(c_t, \hat{c}_t)$, by asymmetric BK and CF filters at observations number 9 to 7, counting from the end of the series. Going further away from the center of the sample, the difference of the performance of the filters starts showing up, with the BK filter being persistently better.



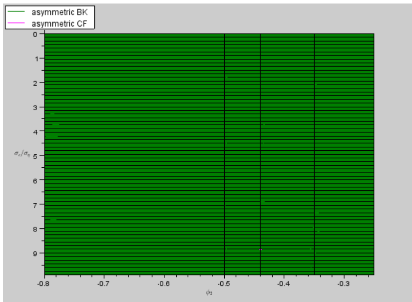
(a) Correlation at obs. 6



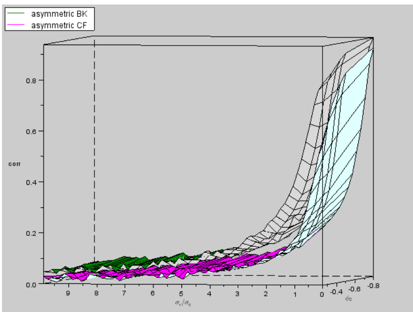
(b) view at 9(a) from the top



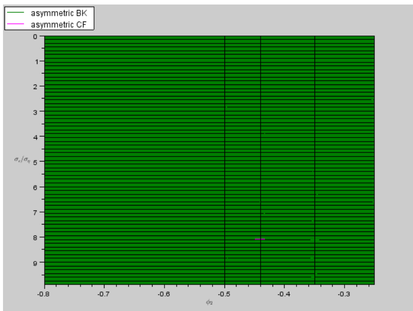
(c) Correlation at obs. 5



(d) view at 9(c) from the top

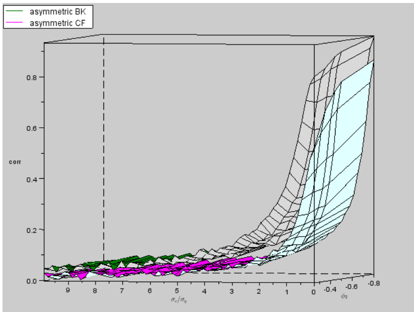


(e) Correlation at obs. 4

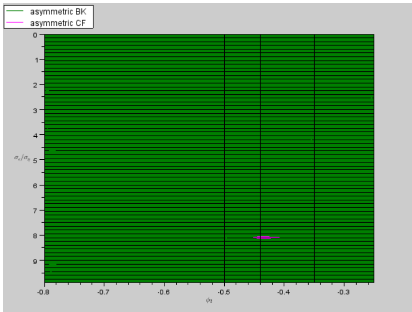


(f) view at 9(e) from the top

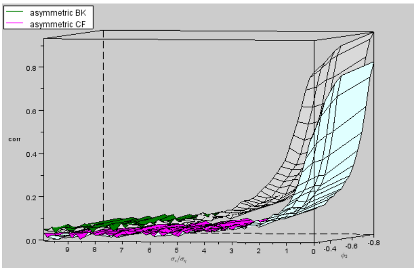
Figure 9: Estimated correlation of the true and extracted cycles at time t, $\hat{\rho}(c_t, \hat{c}_t)$, by asymmetric BK and CF filters at observations number 6 to 4, counting from the end of the series. Clearly, the BK filter is superior to the CF filter at observations close to the ends of the sample.



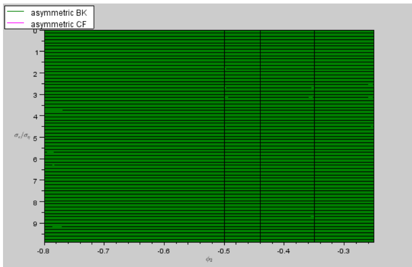
(a) Correlation at obs. 3



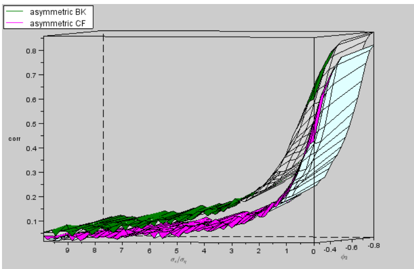
(b) view at 10(a) from the top



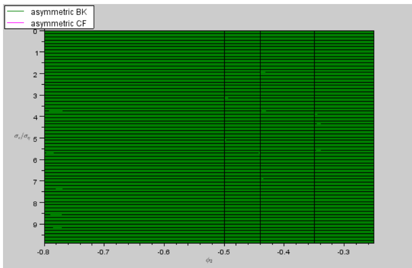
(c) Correlation at obs. 2



(d) view at 10(c) from the top

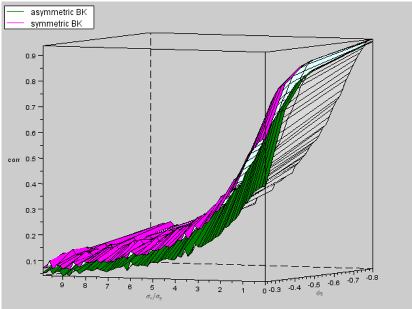


(e) Correlation at obs. 1

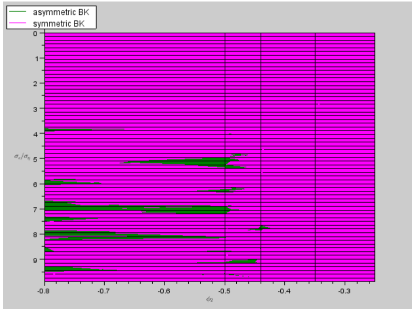


(f) view at 10(e) from the top

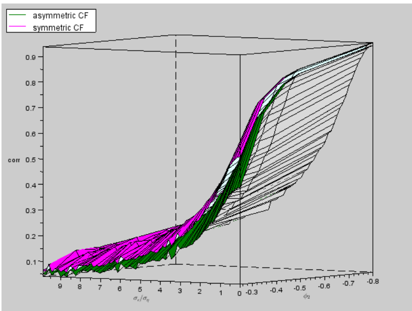
Figure 10: Estimated correlation of the true and extracted cycles at time t, $\hat{\rho}(c_t, \hat{c}_t)$, by asymmetric BK and CF filters at observations number 3 to 1, counting from the end of the series. If the considered DGP is relevant in practice, the asymmetric BK filter should be given preference over the asymmetric default specification of CF filter in real time signal extraction exercises.



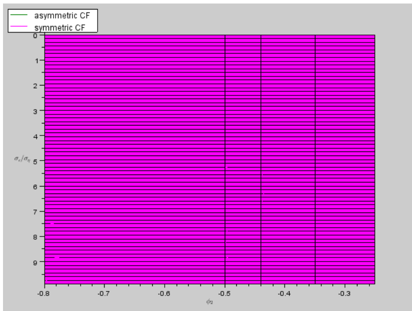
(a) fixed-length symmetric and asymmetric BK filters



(b) view at 11(a) from the top



(c) fixed-length symmetric and asymmetric CF filters



(d) view at 11(c) from the top

Figure 11: Correlation surfaces estimated at the middlepoint of the sample, where all filters are symmetric, but fixed-length filters are shorter than the asymmetric filters, with the latter spanning the whole sample length. The results show that the shorter filters outperform the longer ones. More simulations results indicate (not shown here) that, for BK filter, $K = 13$ gives the best results, outperforming the one with $K = 12$, regardless of wave length of the cycle, or the influence of permanent component in the input signal. For CF filter, $K = 12$ appears to be optimal, unless $\sigma_\epsilon/\sigma_\eta > 1$, for which $K = 13$ seems to yield higher $\hat{\rho}(c_t, \hat{c}_t)$.

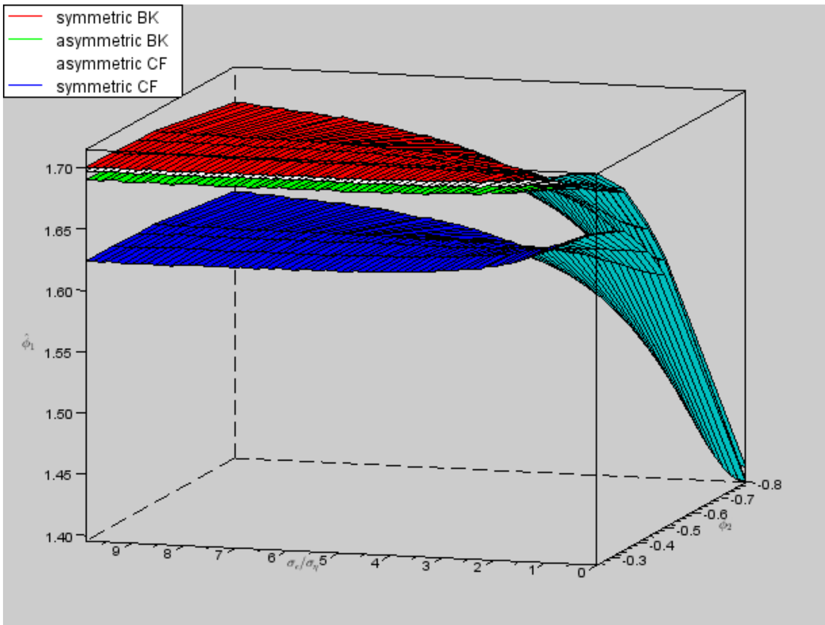


Figure 12: Estimated ϕ_1 by the four filters for various ϕ_2 and $\sigma_\epsilon/\sigma_\eta$ values. ϕ_1 is always overestimated.

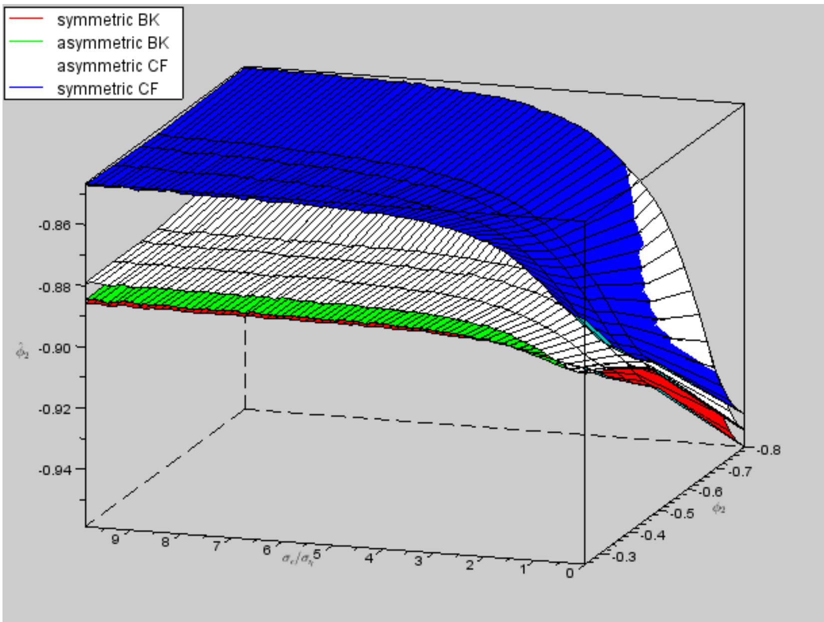


Figure 13: Estimated ϕ_2 by the four filters for various ϕ_2 and $\sigma_\epsilon/\sigma_\eta$ values. The estimated ϕ_2 converges to a constant as the influence of the permanent component on the input series increases, regardless of the true length of the cycle. While it is expected that $\hat{\phi}_1$ would be approximately constant, since ϕ_1 is always set to 1.2, it is not that plausible for $\hat{\phi}_2$ to converge to a constant, regardless of the value of ϕ_2. ϕ_2 is always underestimated.