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Unilateral CVA for CDS in Contagion Model: with Volatilities and Correlation of Spread and Interest *

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Abstract

The price of financial derivative with unilateral counterparty credit risk can be expressed as the price of an otherwise risk-free derivative minus a credit value adjustment(CVA) component that can be seen as shorting a call option, which is exercised upon default of counterparty, on MtM of the derivative. Therefore, modeling volatility of MtM and default time of counterparty is key to quantification of counterparty risk. This paper models default times of counterparty and reference with a particular contagion model with stochastic intensities that is proposed by Bao et al. [1]. Stochastic interest rate is incorporated as well to account for positive correlation between spread and interest. Survival measure approach is adopted to calculate MtM of risk-free CDS and conditional survival probability of counterparty in defaultable environment. Semi-analytical solution for CVA is attained. Affine specification of intensities and interest rate concludes analytical expression for pre-default value of MtM. Numerical experiments at the last of this paper analyze the impact of contagion, volatility and correlation on CVA.

JEL classification: C15, C63, G12, G13

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1 Introduction

Counterparty credit risk arises from the fact that many financing transactions such as repos and financial derivatives are traded over the counter. For example, Lehman Brothers had a notional amount of $800 billion of OTC derivatives at the point of bankruptcy. After default of highly rated Lehmann Brothers as well as occurrence of financial failure in many other large financial institutions, such as Bear Stearns and AIG, counterparty risk has become a crucial issue in connection with valuation and risk management of credit derivatives. This paper deals with unilateral counterparty risk for a special class of derivatives–CDS–where only one counterparty of the transaction is assumed to defaultable on CDS and the other being default-free. As usual, the counterparty that calculates CVA is assumed to be default-free in this paper.

In contrast to unilateral counterparty risk, sometimes bilateral counterparty risk has to be considered, especially after the 2007 financial crisis. As Jon Gregory asserts in the introduction

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of his new book Gregory [6], "The 'too big to fail' mentality that seemingly existed in the market has been thoroughly discredited and the failure or financial instability of any institution large or small should be regarded as plausible", counterparty risk should be accounted for all counterparties for any financial derivatives traded over the counter, no matter how highly they are rated. However, we only deal with unilateral risk in this paper, while bilateral risk will be studied in the sequel paper.

Among vast literature dealing with counterparty risk in general, we only mention here the papers concerning valuation of CVA for CDS, especially two recent papers Brigo el al. [3] and Crépey et al. [5], as well as a paper that is most relevant to this paper, Jarrow et al. [9]. On the first stage of counterparty risk research, Huge et al. [7] propose a rating-based approach to price CDS counterparty risk, while Hull el al. [8] put the problem in the framework of static copula model. Thereafter, Jarrow et al. [9] propose a framework based on reduced-form model to incorporate contagion as model input, called contagion model or interacting intensity model, for the reason that contagion happens upon the default of one firm via increasing default intensities of other forms. Blanchet-Scalliet et al. [2] develop a Merton-type structural approach and derive closed form CVA for CDS.

Recently, Brigo et al. [3] propose a general framework to model unilateral counterparty risk, where CVA can be expressed as a call option, exercised upon the default of counterparty, on MtM of an otherwise default-free derivative with zero strike. Three key facts should be noted in the general formula for unilateral CVA. First, how the dependence of default times of counterparty and reference firm, if it's not default free, is modeled is one major factor while calculating CVA. Second, MtM of the risk free derivative, i.e. an equivalent derivative with the same cashflow except that no counterparty risk is accounted in the cashflow, is calculated with respect to the whole information, including default information of both firms. This is complex because reference's default time is present in cashflow and both firms' default information is included in the whole market information. Third, MtM of risk free derivative is always in complex form, even if explicit solution is available, not to mention calculating present value of MtM's call option that is payed upon default of counterparty. In Brigo's specific model, inter-dependence of default times are modeled via a static copula that couples unit exponential variables in Cox's construction of default times, while pre-intensities are assumed to be independent and interest rate is set to be constant. This specification allows for semi-analytical expression for conditional probability of reference's default time given market information. Moreover, a *default bucketing* technique is proposed in Brigo et al. [3] by assuming that the positive MtM is exercised on the next payment day after counterparty defaults.

Alternatively, Crépey et al. [5] propose a Markov chain copula model with joint defaults to account for wrong way risk. Although sounds unreasonable, simultaneous defaults can be interpreted in the way that at the default time of counterparty, there is positive probability of high spreads environment, in which case, the value of the CDS for a protection buyer is close (if not equal) to the loss given default of the firm. Markovian property of marginal default process in the framework of Markov chain copula model allows for explicit formula for MtM of risk-free CDS in an environment with default information, as well as analytical solution for CVA. However, one major drawback of Markov chain copula model is that it completely excludes contagion from the model, because joint default process in a contagion model can never be Markovian.

This paper models defaults of two firms by a specially designed contagion model with stochastic intensities, which is first proposed in Bao et al. [1]. As CVA can be seen as a call option on MtM of risk-free CDS, volatility of MtM is obviously a key model factor while calculating CVA. Modeling volatility of MtM is represented in two parts in this paper, the volatility from diffusion of Brownian Motions in pre-intensities, and the possibility of contagion from the other firm. In addition, stochastic interest rate is incorporated in our model, and positive correlation between spread and interest rate is modeled. Survival measure approach, which is designed in Bao et al. [1], is adopted to calculate MtM of risk-free CDS and conditional distribution of counterparty's default time in defaultable environment, and semi-analytical solution for CVA is attained. Early version of survival measure approach is referred to Collin-Dufresene et al. [4] and Schönbucher [12]. Affine specification of intensities and interest rate concludes analytical expression for pre-default value of MtM. Numerical results at the last of this paper analyze the impact of contagion, volatility and correlation on CVA.

The remaining sections are organized as follows. The general framework of pricing unilateral counterparty risk, i.e. calculating CVA, is reviewed in Section 2. A special contagion model is proposed in Section 3, and some key quantities are calculated in this section. Section 4 gives the major result in this paper, i.e. unilateral CVA for CDS. Section 5 proposes an affine specification of intensities and interest rate, and explicit formula for MtM's pre-default value is derived. Section 6 performs some numerical analysis and gives interpretation of model parameters. Section 7 concludes this paper.

2 Credit Valuation Adjustment

We review the general framework of pricing unilateral counterparty risk for CDS in this section, while the pricing formula is suitable for arbitrary OTC derivative. We adopt most of the notations from literature, especially in Brigo et al. [3]. Default times of reference firm and the counterparty are denoted as τ_1 and τ_2 respectively. Suppose we are in an economy $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q})$, where $\mathbb{G} = \{\mathcal{G}_t\}_{t\in\mathbb{R}^+}$ is the whole market information and $\mathbb{Q}$ is martingale measure. Suppose market information $\mathcal{G}_t$ is decomposed into two parts $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, with the RCLL and complete subfiltration $\mathcal{F}_t$ representing all default free information available in the market, and the RCLL and complete subfiltration $\mathcal{H}_t = \mathcal{H}_t^1 \vee \mathcal{H}_t^2$ representing default information, with $\mathcal{H}_t^i = \sigma\left(\left\{1_{\{\tau_i \leq s\}}\right\}_{s\leq t}\right), i = 1, 2$.

We denote the maturity of CDS as T. Suppose recovery rates of the two firms are R_1 and R_2, and loss given default of them are $L_1 = 1 - R_1$ and $L_2 = 1 - R_2$ respectively. For the purpose of convenience, we call a CDS subject to counterparty risk "risky CDS" and an otherwise equivalent CDS but having no counterparty risk "risk-free CDS". Denote $\Pi^D(t,T)$ as the sum of all cashflow of risky CDS between t and T, all terms discounted by the stochastic discount factor $D(t,s) = exp\left\{-\int_t^s r_u du\right\}$, where r_t is stochastic short interest rate. $\Pi(t,T)$ is defined analogously for the risk-free CDS. If $\tau_2 > T$, the risky CDS investor will realize all cashflow promised in risk-free CDS. If $t < \tau_2 \leq T$, the investor could only realize the cashflow until default time τ_2, while the remaining cashflow should be marked to market and the MtM value would be settled. Note that the remaining cashflow is part of the risk-free CDS, while it is subject to the environment with default information of both firms. Thus we have $MtM(t,T) = E^{\mathbb{Q}}\left[\Pi(t,T)|\,\mathcal{G}_t\right]$, and

$$\begin{aligned}\Pi^D(t,T) = {} & 1_{\{\tau_2>T\}}\Pi\left(t,T\right) + \\ & 1_{\{t<\tau_2\leqslant T\}}\left[\Pi\left(t,\tau_2\right) + D\left(t,\tau_2\right)\left(R_2\left(MtM(\tau_2,T)\right)^+ - \left(MtM(\tau_2,T)\right)^-\right)\right] \quad (2.1)\end{aligned}$$

The above expression implies that if there is no early default of counterparty, all cashflow $\Pi(t,T)$ will be realized until maturity T. If the counterparty does default before maturity, all cashflow before τ_2 will be realized until τ_2 and the MtM value of remaining cashflow will be settled at τ_2. If $MtM(\tau_2,T) > 0$, only recovery value $R_2 MtM(\tau_2,T)$ will be payed to investor.

If $MtM(\tau_2, T) < 0$, the investor could not just walk away from the transaction, however, and should pay the entire amount $-MtM(\tau_2, T)$ to counterparty.

The following proposition summarizes the general pricing formula for unilateral counterparty risk, where $\Pi^D(t,T)$ and $\Pi(t,T)$ are supposed to be risky and risk-free discounted cashflow of arbitrary OTC derivatives, not just CDS.

Proposition 1. *At valuation time t, and on $\{\tau_2 > t\}$, the price of risky cashflow subject to counterparty risk is given by*

$$1_{\{\tau_2>t\}} E\left[\Pi^D(t,T)\middle|\mathcal{G}_t\right] = 1_{\{\tau_2>t\}} E\left[\Pi(t,T)\middle|\mathcal{G}_t\right] - 1_{\{\tau_2>t\}} CVA(t,T) \tag{2.2}$$

where $CVA(t,T)$ is expressed as

$$CVA(t,T) = 1_{\{\tau_2>t\}} CVA(t,T) = E\left[1_{\{t<\tau_2\leqslant T\}} L_2 \cdot D(t,\tau_2)\left(MtM(\tau_2,T)\right)^+ \middle|\mathcal{G}_t\right] \tag{2.3}$$

□.

Proof: Note that $\Pi(t,T) = \Pi(t,\tau_2) + D(t,\tau_2)\Pi(\tau_2,T)$, therefore

$$\begin{aligned}
1_{\{\tau_2>t\}} E\left[\Pi^D(t,T)\middle|\mathcal{G}_t\right] &= E\left[1_{\{\tau_2>t\}}\Pi(t,T) - 1_{\{t<\tau_2\leqslant T\}} D(t,\tau_2)\Pi(\tau_2,T)\middle|\mathcal{G}_t\right] \\
&\quad + E\left[1_{\{t<\tau_2\leqslant T\}} D(t,\tau_2)\left[R_2\cdot\left(MtM(\tau_2,T)\right)^+ - \left(MtM(\tau_2,T)\right)^-\right]\middle|\mathcal{G}_t\right] \\
&= E\left[1_{\{\tau_2>t\}}\Pi(t,T) - 1_{\{t<\tau_2\leqslant T\}} D(t,\tau_2) E\left(\Pi(\tau_2,T)\middle|\mathcal{G}_{\tau_2}\right)\middle|\mathcal{G}_t\right] \\
&\quad + E\left[1_{\{t<\tau_2\leqslant T\}} D(t,\tau_2)\left[R_2\cdot\left(MtM(\tau_2,T)\right)^+ - \left(MtM(\tau_2,T)\right)^-\right]\middle|\mathcal{G}_t\right] \\
&= E\left[1_{\{\tau_2>t\}}\Pi(t,T) - 1_{\{t<\tau_2\leqslant T\}} D(t,\tau_2) MtM(\tau_2,T)\middle|\mathcal{G}_t\right] \\
&\quad + E\left[1_{\{t<\tau_2\leqslant T\}} D(t,\tau_2)\left[R_2\cdot\left(MtM(\tau_2,T)\right)^+ - \left(MtM(\tau_2,T)\right)^-\right]\middle|\mathcal{G}_t\right] \\
&= 1_{\{\tau_2>t\}} E\left[\Pi(t,T)\middle|\mathcal{G}_t\right] - E\left[1_{\{t<\tau_2\leqslant T\}} L_2\cdot D(t,\tau_2)\left(MtM(\tau_2,T)\right)^+\middle|\mathcal{G}_t\right]
\end{aligned}$$

where the second equality holds by using "tower property" of conditional expectation, and the fact that $1_{\{t<\tau_2\leqslant T\}} D(t,\tau_2)$ is $\mathcal{G}_{\tau_2}$ measurable.

□

Equations (2.2) and (2.3) show that arbitrage free price of an OTC derivative subject to unilateral counterparty risk can be expressed as arbitrage price of an equivalent risk-free derivative minus the credit valuating adjustment component which can be seen as a call option on MtM of risk-free derivative with strike price 0 and delivery time τ_2. Specifically, this paper considers valuation of CVA at time 0 for CDS. Thus the objective of this paper is calculating the following quatility

$$CVA(0,T) = E\left[1_{\{\tau_2\leqslant T\}} L_2 \cdot D(0,\tau_2)\left(MtM^{CDS}_{Seller}(\tau_2,T)\right)^+\right] \tag{2.4}$$

where $MtM^{CDS}_{Seller}(t,T)$ is mark-to-market value of CDS in the view of CDS seller with respect to whole market information $\mathcal{G}_t$, i.e.

$$\begin{aligned}
MtM^{CDS}_{Seller}(t,T) &= E^{\mathbb{Q}}\left[\Pi^{CDS}_{Seller}(t,T)\middle|\mathcal{G}_t\right] \\
&= E^{\mathbb{Q}}\left[S\cdot\int_t^T D(t,s)\, 1_{\{\tau_1>s\}} ds - L_1\cdot\int_t^T D(t,s)\, dH^1_s\middle|\mathcal{G}_t\right]
\end{aligned} \tag{2.5}$$

with $\Pi^{CDS}_{Seller}(t,T)$ discounted cashflow for risk-free CDS seller. Equations (2.4) and (2.5) show that the key issue in CVA valuation is modeling default dependence between τ_1 and τ_2, which is the main subject of next section.

3 Modeling Default Dependence

As indicated at the last of previous section and concluded from equations (2.4-2.5), the major task for valuating CVA is to model default dependence between τ_1 and τ_2, and furthermore, deriving MtM for risk-free CDS in the environment with default information of both firms based on the dependence structure. This paper models the default dependence by a special contagion model proposed in Bao et al. [1] with stochastic intensities. τ_1 and τ_2 are constructed as

$$\tau_i = \inf\left\{ t > 0 \,\middle|\, \int_0^t \lambda_s^i ds \geq E_i \right\}, \quad i = 1,2 \tag{3.6}$$

where λ_t^i's are specified as

$$\begin{cases} \lambda_t^1 = \alpha_t^1 + \beta_t^1 \cdot 1_{\{\tau_2 \leqslant t\}} \\ \lambda_t^2 = \alpha_t^2 + \beta_t^2 \cdot 1_{\{\tau_1 \leqslant t\}} \end{cases}, \quad with \quad \begin{cases} \beta_t^1 = \eta_2 \cdot \alpha_t^2 \\ \beta_t^2 = \eta_1 \cdot \alpha_t^1 \end{cases} \tag{3.7}$$

with α_t^i's and β_t^i's being $\mathbb{F}$-adapted non-negative processes. E_i's are mutually independent unit mean exponential variables that are independent from $\mathbb{F}$. Therefore, τ_i's can be seen as constructed in a HBPR framework (see Bao et al. [1] for detailed discussion) with λ_t^i's being $\mathbb{G}^{-i} = \mathbb{F} \vee \mathbb{H}^{j,j\neq i}$-adapted stochastic hazard processes, and thus $M_t^i = H_t^i - \int_0^{t\wedge\tau_i} \lambda_s^i ds$ being basic $\mathbb{G}$ -martingale for default time τ_i.

Contagion model (3.6-3.7) effectively incorporates volatility into MtM of risk-free CDS in two ways, diffusion of pre-default intensities α_t^i's and contagion effect from defaulted firm to survival firm. For example, before default of counterparty, the intensity of reference firm is α_t^1, which is an $\mathbb{F}$-adapted diffusion, and the volatility of MtM is mainly attributed to volatilities of α_t^1 and α_t^2 in this case. Once the counterparty defaults, the default risk spreads from counterparty to reference firm immediately through a sudden jump of reference's intensity in the amount proportional to counterparty's pre-default intensity α_t^2. Unlike traditional design of jump that is proportional to a firm's own pre-default intensity, such as Leung et al. [11], we assume jump of one firm's intensity is proportional to the other firm's pre-default intensity. This implies that contagion from one firm to another is represented not only by a sudden jump in its intensity, but also by transferring defaulted firm's pre-default intensity to the survival firm. The major advantage of this design is that explicit formulas for marginal survival probability of τ_2 and joint survival probability of τ_1 and τ_2 conditional on default-free information are available.

Solving a contagion model such as (3.6-3.7) faces an obstacle of looping default problem. Three alternative approaches are proposed in literature, i.e. total hazard approach in Yu [14] and Yu [15], Markov chain approach in Leung et al. [11] and Walker [13], survival measure approach in Leung et al. [10]. This paper adopts survival measure approach for its convenience in dealing with contagion model with stochastic intensities, especially the case in presence of stochastic interest rate. The following lemma exhibits the definition and properties of survival measures used hereafter, whose proof is referred to Bao et al. [1].

Lemma 1. *For contagion model with stochastic intensities (3.6-3.7), define the following two survival measures*

$$\left.\frac{d\mathbb{Q}_i}{d\mathbb{Q}}\right|_{\mathcal{G}_t} = 1_{\{\tau_i > t\}} \cdot \exp\left\{ \int_0^t \lambda_s^i ds \right\}, \quad \forall t \leq T, \quad i = 1,2 \tag{3.8}$$

then stochastic hazard processes λ_t^1 and λ_t^2 can be significantly simplified under the two survival measures to become

$$\begin{cases} \lambda_t^2 = \alpha_t^2 \sim \mathbb{Q}_1 - a.s. \\ \lambda_t^1 = \alpha_t^1 \sim \mathbb{Q}_2 - a.s. \end{cases} \quad and \quad \begin{cases} \lambda_t^2 = 0 \sim \mathbb{Q}_2 - a.s. \\ \lambda_t^1 = 0 \sim \mathbb{Q}_1 - a.s. \end{cases} \tag{3.9}$$

Moreover, if α_t^k's and β_t^k's are assumed to be $\mathbb{F}$-adapted non-negative Itô diffusion processes, and $\mathbb{F}$ is assumed to be expanded by Brownian motion W_t, then distributions of α_t^k's and β_t^k's under $(\mathbb{Q}_i, \mathbb{F})$, $i = 1, 2$, are the same as under $(\mathbb{Q}, \mathbb{F})$. $\square$.

Equation (3.9) shows the major advantage of survival measure approach. Intensities of the two default times are significantly simplified because default indicators are eliminated. However, using this measure change needs a survival indicator in the cashflow under original martingale measure $\mathbb{Q}$ due to Bayesian formula. The following lemma displays all the quantities that will be used in the sequel sections. Specifically, we calculate present value of a survival claim and a recovery value, which are basic building blocks for the two legs of a CDS, on the event of $\{\tau_1 > t, \tau_2 \leq t\}$.

Lemma 2. *Under the contagion model with stochastic intensities (3.6-3.7), given a stochastic interest rate r_t, we have*

(1). On the event of $\{\tau_1 > t, \tau_2 \leqslant t\}$, present value of general survival claim $1_{\{\tau_1 > T\}} \cdot Z_T$, with $Z_T \in \mathcal{F}_T$, calculated with respect to market information $\mathcal{G}_t$ can be expressed as

$$\begin{aligned} & 1_{\{\tau_1 > t, \tau_2 \leqslant t\}} \cdot E^{\mathbb{Q}} \left[D(t,T) \, 1_{\{\tau_1 > T\}} \cdot Z_T \middle| \mathcal{G}_t \right] \\ = \ & 1_{\{\tau_1 > t, \tau_2 \leqslant t\}} \cdot E^{\mathbb{Q}} \left[\exp\left\{ - \int_t^T \left(\alpha_s^1 + \beta_s^1 + r_s \right) ds \right\} \cdot Z_T \middle| \mathcal{F}_t \right] \end{aligned} \tag{3.10}$$

Particularly, present value of unit survival claim $1_{\{\tau_1 > T\}}$ is

$$\begin{aligned} & 1_{\{\tau_1 > t, \tau_2 \leqslant t\}} \cdot E^{\mathbb{Q}} \left[D(t,T) \, 1_{\{\tau_1 > T\}} \middle| \mathcal{G}_t \right] \\ = \ & 1_{\{\tau_1 > t, \tau_2 \leqslant t\}} \cdot E^{\mathbb{Q}} \left[\exp\left\{ - \int_t^T \left(\alpha_s^1 + \beta_s^1 + r_s \right) ds \right\} \middle| \mathcal{F}_t \right] \\ \equiv \ & 1_{\{\tau_1 > t, \tau_2 \leqslant t\}} \cdot \tilde{P}(t,T) \end{aligned} \tag{3.11}$$

(2). On the event of $\{\tau_1 > t, \tau_2 \leqslant t\}$, present value of unit recovery value $1_{\{t < \tau_1 \leq T\}}$, paid at default time τ_1, with respect to market information $\mathcal{G}_t$ can be represented as

$$\begin{aligned} & 1_{\{\tau_1 > t, \tau_2 \leqslant t\}} \cdot E^{\mathbb{Q}} \left[\int_t^T D(t,s) \, dH_s^1 \middle| \mathcal{G}_t \right] \\ = \ & 1_{\{\tau_1 > t, \tau_2 \leqslant t\}} \cdot \int_t^T E^{\mathbb{Q}} \left[\exp\left\{ - \int_t^s \left(\alpha_u^1 + \beta_u^1 + r_u \right) du \right\} \left(\alpha_s^1 + \beta_s^1 \right) \middle| \mathcal{F}_t \right] ds \\ \equiv \ & 1_{\{\tau_1 > t, \tau_2 \leqslant t\}} \cdot \int_t^T \bar{P}(t,s) ds \end{aligned} \tag{3.12}$$

(3). Survival probability of τ_2 conditional on default-free information $\mathcal{F}_T$ can be expressed as

$$\mathbb{Q}\left(\tau_2 > s \middle| \mathcal{F}_T \right) = \frac{1}{1 - \eta_1} \left(\exp\left\{ - \int_0^s \left[\alpha_u^2 + \beta_u^2 \right] du \right\} - \eta_1 \exp\left\{ - \int_0^s \left[\alpha_u^1 + \alpha_u^2 \right] du \right\} \right) \tag{3.13}$$

for $\eta_1 \neq 1$ and $\forall s \in (0, T]$. When $\eta_1 = 1$, $\mathbb{Q}\left(\tau_2 > s \middle| \mathcal{F}_T \right)$ is defined as limit of expression (3.13) as $\eta_1 \to 1$, i.e.

$$\mathbb{Q}\left(\tau_2 > s \middle| \mathcal{F}_T \right) = \int_0^s \alpha_u^1 du \cdot \exp\left\{ - \int_0^s \left[\alpha_u^2 + \beta_u^2 \right] du \right\} + \exp\left\{ - \int_0^s \left[\alpha_u^1 + \alpha_u^2 \right] du \right\} \tag{3.14}$$

(4). Joint probability $\mathbb{Q}\left\{ \tau_1 > T, S < \tau_2 \leq T \middle| \mathcal{F}_T \right\}$ of the two firms conditional on default-free information $\mathcal{F}_T$ can be expressed as

$$
\begin{aligned}
&\mathbb{Q}\left\{\tau_1>T, S<\tau_2\leqslant T\right|\mathcal{F}_T\}\\
=&\begin{cases}
\dfrac{1}{1-\eta_2}\exp\left\{-\displaystyle\int_0^T\left[\alpha_s^1+\alpha_s^2\right]ds\right\}\left[\exp\left\{\displaystyle\int_S^T\left(1-\eta_2\right)\alpha_u^2du\right\}-1\right], \eta_2\neq 1,\eta_2>0\\
\exp\left\{-\displaystyle\int_0^T\left[\alpha_s^1+\alpha_s^2\right]ds\right\}\left[\displaystyle\int_S^T\alpha_u^2du+1\right], \qquad \eta_2=1
\end{cases}\\
\approx&\ (T-S)\exp\left\{-\int_0^T\left[\alpha_s^1+\alpha_s^2\right]ds\right\}\alpha_T^2, \forall\ \eta_2>0.
\end{aligned}
\tag{3.15}
$$

when $T-S$ is small enough. □.

Proof: (1). To derive pricing formula for general survival claim $1_{\{\tau_1>T\}}\cdot Z_T$ in contagion model (3.6-3.7), we change measure from $\mathbb{Q}$ to $\mathbb{Q}_1$. The Bayesian formula for absolutely continuous measure change is referred to Appendix A in Bao et al. [1]. For notational convenience, we misuse $\mathcal{G}_t$ in case $\bar{\mathcal{G}}_t^1$ or $\bar{\mathcal{G}}_t^2$ should be used without changing the results. $\bar{\mathbb{G}}^i$'s are the natural filtration expanded by $\mathbb{G}$ and the null sets under survival measures $\mathbb{Q}_i$'s. Therefore,

$$
\begin{aligned}
&1_{\{\tau_1>t,\tau_2\leqslant t\}}\cdot E^{\mathbb{Q}}\left[D(t,T)1_{\{\tau_1>T\}}\cdot Z_T\right|\mathcal{G}_t]\\
=&\ 1_{\{\tau_1>t,\tau_2\leqslant t\}}\cdot E^{\mathbb{Q}_1}\left[D(t,T)\exp\left\{-\int_t^T\left(\alpha_s^1+\beta_s^1\cdot 1_{\{\tau_2\leqslant s\}}\right)ds\right\}\cdot Z_T\middle|\mathcal{G}_t\right]\\
=&\ 1_{\{\tau_1>t,\tau_2\leqslant t\}}\cdot E^{\mathbb{Q}_1}\left[\exp\left\{-\int_t^T\left(\alpha_s^1+\beta_s^1+r_s\right)ds\right\}\cdot Z_T\middle|\mathcal{G}_t\right]\\
=&\ 1_{\{\tau_1>t,\tau_2\leqslant t\}}\cdot E^{\mathbb{Q}_1}\left[\exp\left\{-\int_t^T\left(\alpha_s^1+\beta_s^1+r_s\right)ds\right\}\cdot Z_T\middle|\mathcal{G}_t,\tau_1>t,\tau_2\leqslant t\right]\\
=&\ 1_{\{\tau_1>t,\tau_2\leqslant t\}}\cdot E^{\mathbb{Q}_1}\left[\exp\left\{-\int_t^T\left(\alpha_s^1+\beta_s^1+r_s\right)ds\right\}\cdot Z_T\middle|\mathcal{F}_t,\tau_1>t,\tau_2\leqslant t\right]\\
=&\ 1_{\{\tau_1>t,\tau_2\leqslant t\}}\cdot\frac{E^{\mathbb{Q}_1}\left[1_{\{\tau_1>t,\tau_2\leqslant t\}}\cdot\exp\left\{-\int_t^T\left(\alpha_s^1+\beta_s^1+r_s\right)ds\right\}\cdot Z_T\middle|\mathcal{F}_t\right]}{\mathbb{Q}_1\left[\tau_1>t,\tau_2\leqslant t\right|\mathcal{F}_t]}\\
=&\ 1_{\{\tau_1>t,\tau_2\leqslant t\}}\cdot\frac{E^{\mathbb{Q}_1}\left[E^{\mathbb{Q}_1}\left[1_{\{\tau_2\leqslant t\}}\right|\mathcal{F}_T]\cdot\exp\left\{-\int_t^T\left(\alpha_s^1+\beta_s^1+r_s\right)ds\right\}\cdot Z_T\middle|\mathcal{F}_t\right]}{\mathbb{Q}_1\left[\tau_2\leqslant t\right|\mathcal{F}_t]}\\
=&\ 1_{\{\tau_1>t,\tau_2\leqslant t\}}\cdot E^{\mathbb{Q}_1}\left[\exp\left\{-\int_t^T\left(\alpha_s^1+\beta_s^1+r_s\right)ds\right\}\cdot Z_T\middle|\mathcal{F}_t\right]\\
=&\ 1_{\{\tau_1>t,\tau_2\leqslant t\}}\cdot E^{\mathbb{Q}}\left[\exp\left\{-\int_t^T\left(\alpha_s^1+\beta_s^1+r_s\right)ds\right\}\cdot Z_T\middle|\mathcal{F}_t\right]
\end{aligned}
$$

where the second last equality holds for the reason that τ_2 has intensity α_t^2 under survival measure $\mathbb{Q}_1$, and $\mathbb{Q}_1\left[\tau_2\leqslant t\right|\mathcal{F}_t]=\mathbb{Q}_1\left[\tau_2\leqslant t\right|\mathcal{F}_T]=exp\left\{-\int_0^t\alpha_s^2ds\right\}$, which is $\mathcal{F}_t$-measurable. The last equality holds because distributions of α_t^i, β_t^i and r_t remain the same when changing measures from $\mathbb{Q}$ to $\mathbb{Q}_1$.

(2).As λ_t^1 is intensity of τ_1 under $\mathbb{Q}$ in the HBPR framework, $M_t^1=H_t^1-\int_0^t 1_{\{\tau_1>s\}}\cdot\lambda_s^1ds$

is $(\mathbb{G},\mathbb{Q})$-martingale. Therefore,

$$
\begin{aligned}
& 1_{\{\tau_1>t,\tau_2\leqslant t\}}\cdot E^{\mathbb{Q}}\left[\int_t^T D(t,s)\,dH_s^1\middle|\mathcal{G}_t\right]\\
=\ & 1_{\{\tau_1>t,\tau_2\leqslant t\}}\cdot E^{\mathbb{Q}}\left[\int_t^T D(t,s)\left(dH_s^1-1_{\{\tau_1>s\}}\lambda_s^1ds\right)+\int_t^T D(t,s)\,1_{\{\tau_1>s\}}\lambda_s^1ds\middle|\mathcal{G}_t\right]\\
=\ & 1_{\{\tau_1>t,\tau_2\leqslant t\}}\cdot E^{\mathbb{Q}}\left[\int_t^T D(t,s)\,1_{\{\tau_1>s\}}\lambda_s^1ds\middle|\mathcal{G}_t\right]\\
=\ & 1_{\{\tau_1>t,\tau_2\leqslant t\}}\cdot\int_t^T E^{\mathbb{Q}}\left[D(t,s)\,1_{\{\tau_1>s\}}\left(\alpha_s^1+\beta_s^1\right)\middle|\mathcal{G}_t\right]ds\\
=\ & 1_{\{\tau_1>t,\tau_2\leqslant t\}}\cdot\int_t^T E^{\mathbb{Q}}\left[\exp\left\{-\int_t^s\left(\alpha_u^1+\beta_u^1+r_u\right)du\right\}\left(\alpha_s^1+\beta_s^1\right)\middle|\mathcal{F}_t\right]ds
\end{aligned}
$$

where the second equality holds because of martingale property of M_t^1. The last equality is direct application of formula (3.10).

(3).To calculate conditional probability of τ_2, we decompose the survival event $\{\tau_2>s\}$ into two parts, $\{\tau_2>s,\tau_1>s\}$ and $\{\tau_2>s,\tau_1\leq s\}$. Changing measure from $\mathbb{Q}$ to $\mathbb{Q}_2$, then $\forall s\in(0,T]$ and $Z_T\in\mathcal{F}_T$, we get

$$
\begin{aligned}
E^{\mathbb{Q}}\left\{1_{\{\tau_2>s,\tau_1>s\}}\cdot Z_T\right\} &= E^{\mathbb{Q}_2}\left[1_{\{\tau_1>s\}}\exp\left\{-\int_0^s\lambda_u^2du\right\}\cdot Z_T\right]\\
&= E^{\mathbb{Q}_2}\left[E^{\mathbb{Q}_2}\left(1_{\{\tau_1>s\}}\middle|\mathcal{F}_T\right)\exp\left\{-\int_0^s\alpha_u^2du\right\}\cdot Z_T\right]\\
&= E^{\mathbb{Q}}\left[\exp\left\{-\int_0^s\left[\alpha_u^1+\alpha_u^2\right]du\right\}\cdot Z_T\right] \qquad (3.16)
\end{aligned}
$$

where the last equality holds for the reason that τ_1 has default intensity α_t^1 under survival measure $\mathbb{Q}_2$ and distributions of α_t^i's remain the same when changing measures from $\mathbb{Q}$ to $\mathbb{Q}_2$.

Similarly, $\forall s\in(0,T]$, $\eta_1\neq 1$ and $Z_T\in\mathcal{F}_T$, changing measure from $\mathbb{Q}$ to $\mathbb{Q}_2$, we get

$$
\begin{aligned}
& E^{\mathbb{Q}}\left\{1_{\{\tau_2>s,\tau_1\leqslant s\}}\cdot Z_T\right\}\\
=\ & E^{\mathbb{Q}_2}\left[1_{\{\tau_1\leqslant s\}}\exp\left\{-\int_0^s\left[\alpha_u^2+\beta_u^2\cdot 1_{\{\tau_1\leqslant u\}}\right]du\right\}\cdot Z_T\right]\\
=\ & E^{\mathbb{Q}_2}\left[\exp\left\{-\int_0^s\alpha_u^2du\right\}E^{\mathbb{Q}_2}\left(1_{\{\tau_1\leqslant s\}}\exp\left\{-\int_{\tau_1}^s\beta_u^2du\right\}\middle|\mathcal{F}_T\right)\cdot Z_T\right]\\
=\ & E^{\mathbb{Q}_2}\left[\exp\left\{-\int_0^s\alpha_u^2du\right\}\int_0^s\exp\left\{-\int_v^s\beta_u^2du\right\}\alpha_v^1\exp\left\{-\int_0^v\alpha_u^1du\right\}dv\cdot Z_T\right]\\
=\ & E^{\mathbb{Q}_2}\left[\exp\left\{-\int_0^s\left[\alpha_u^1+\alpha_u^2\right]du\right\}\int_0^s\exp\left\{\int_v^s\left[1-\eta_1\right]\alpha_u^1du\right\}\alpha_v^1dv\cdot Z_T\right]\\
=\ & \frac{1}{1-\eta_1}E^{\mathbb{Q}_2}\left[\exp\left\{-\int_0^s\left[\alpha_u^1+\alpha_u^2\right]du\right\}\left(\exp\left\{(1-\eta_1)\int_0^s\alpha_u^1du\right\}-1\right)\cdot Z_T\right]\\
=\ & \frac{1}{1-\eta_1}E^{\mathbb{Q}}\left[\left(\exp\left\{-\int_0^s\left[\alpha_u^2+\beta_u^2\right]du\right\}-\exp\left\{-\int_0^s\left[\alpha_u^1+\alpha_u^2\right]du\right\}\right)\cdot Z_T\right] \qquad (3.17)
\end{aligned}
$$

where the third equality holds for the reason that τ_1 has default intensity α_t^1 under survival measure $\mathbb{Q}_2$. The fourth equality is direct consequence of construction of β_t^2 as proportion of

α_t^1, which allows explicit formula for the integral that is solved in the fifth equality.

Put equations (3.16) and (3.17) together, we get

$$\begin{aligned} & E^{\mathbb{Q}}\left\{1_{\{\tau_2>s\}}\cdot Z_T\right\} \\ = & \frac{1}{1-\eta_1}E^{\mathbb{Q}}\left[\left(\exp\left\{-\int_0^s\left[\alpha_u^2+\beta_u^2\right]du\right\}-\eta_1\exp\left\{-\int_0^s\left[\alpha_u^1+\alpha_u^2\right]du\right\}\right)\cdot Z_T\right] \end{aligned} \tag{3.18}$$

for $\eta_1 \neq 1$ and $\forall s \in (0,T]$ and $\forall Z_T \in \mathcal{F}_T$.

Meanwhile, it is obvious that the following equation holds $\forall s \in (0,T]$ and $\forall Z_T \in \mathcal{F}_T$,

$$E^{\mathbb{Q}}\left\{1_{\{\tau_2>s\}}\cdot Z_T\right\} = E^{\mathbb{Q}}\left[E^{\mathbb{Q}}\left(1_{\{\tau_2>s\}}\middle|\,\mathcal{F}_T\right)\cdot Z_T\right] \tag{3.19}$$

Consequently, one can easily get formula (3.13) for $\eta_1 \neq 1$ by comparing equations (3.18) and (3.19).

When $\eta_1 = 1$, $\mathbb{Q}\left(\tau_2 > s\middle|\,\mathcal{F}_T\right)$ is defined as limit of expression (3.13) as $\eta_1 \to 1$, which can easily attained through *L'Hospital's Rule*.

(4).First, we calculate joint survival probability $\mathbb{Q}\left\{\tau_1 > T, \tau_2 > S\middle|\,\mathcal{F}_T\right\}$ for $S < T$. Change measure from $\mathbb{Q}$ to $\mathbb{Q}_1$, then for any $Z_T \in F_T$ we get

$$\begin{aligned} & E^{\mathbb{Q}}\left\{1_{\{\tau_1>T,\tau_2>S\}}\cdot Z_T\right\} = E^{\mathbb{Q}_1}\left[1_{\{\tau_2>S\}}\exp\left\{-\int_0^T\lambda_s^1 ds\right\}\cdot Z_T\right] \\ = & E^{\mathbb{Q}_1}\left[1_{\{\tau_2>S\}}\exp\left\{-\int_0^T\alpha_s^1 ds\right\}\exp\left\{-\int_S^T\beta_s^1 1_{\{\tau_2\leqslant s\}}ds\right\}\cdot Z_T\right] \\ = & E^{\mathbb{Q}_1}\left[E^{\mathbb{Q}_1}\left[1_{\{\tau_2>S\}}\exp\left\{-\int_S^T\beta_s^1 1_{\{\tau_2\leqslant s\}}ds\right\}\middle|\,\mathcal{F}_T\right]\exp\left\{-\int_0^T\alpha_s^1 ds\right\}\cdot Z_T\right] \\ = & \frac{1}{1-\eta_2}E^{\mathbb{Q}_1}\left[\exp\left\{-\int_0^T\left[\alpha_s^1+\alpha_s^2\right]ds\right\}\left[\exp\left\{(1-\eta_2)\int_S^T\alpha_u^2 du\right\}-\eta_2\right]\cdot Z_T\right] \end{aligned} \tag{3.20}$$

for $\eta_2 \neq 1$. Meanwhile, it is obvious that the following equation holds for any $Z_T \in \mathcal{F}_T$,

$$E^{\mathbb{Q}}\left\{1_{\{\tau_1>T,\tau_2>S\}}\cdot Z_T\right\} = E^{\mathbb{Q}}\left\{\mathbb{Q}\left\{\tau_1 > T, \tau_2 > S\middle|\,\mathcal{F}_T\right\}\cdot Z_T\right\} \tag{3.21}$$

Finally, we get

$$\begin{aligned} & \mathbb{Q}\left\{\tau_1 > T, \tau_2 > S\middle|\,\mathcal{F}_T\right\} \\ = & \frac{1}{1-\eta_2}\exp\left\{-\int_0^T\left[\alpha_s^1+\alpha_s^2\right]ds\right\}\left[\exp\left\{\int_S^T(1-\eta_2)\,\alpha_u^2 du\right\}-\eta_2\right] \end{aligned} \tag{3.22}$$

for $\eta_2 \neq 1$. When $\eta_2 = 1$, $\mathbb{Q}\left\{\tau_1 > T, \tau_2 > S\middle|\,\mathcal{F}_T\right\}$ is defined as limit of expression (3.22) as $\eta_2 \to 1$, which can easily be attained by *L'Hospital's Rule*

$$\mathbb{Q}\left\{\tau_1 > T, \tau_2 > S\middle|\,\mathcal{F}_T\right\} = \exp\left\{-\int_0^T\left[\alpha_s^1+\alpha_s^2\right]ds\right\}\left[\int_S^T\alpha_u^2 du + 1\right] \tag{3.23}$$

Therefore, joint survival probability $\mathbb{Q}\left\{\tau_1 > T, \tau_2 > T\middle|\,\mathcal{F}_T\right\}$ can be expressed as follows

$$\mathbb{Q}\left\{\tau_1 > T, \tau_2 > T\middle|\,\mathcal{F}_T\right\} = \exp\left\{-\int_0^T\left[\alpha_s^1+\alpha_s^2\right]ds\right\}, \forall\eta_2 > 0 \tag{3.24}$$

Consequently, the first equality in formula (3.15) can be concluded from equations (3.22-3.24), and the approximation holds when $T - S$ is small enough.

□

4 Unilateral CVA for Risky CDS

This section uses the explicit and semi-explicit formulas in Lemma 2 to derive a semi-analytical solution for unilateral CVA for risky CDS at time 0. Equation (2.4) implies that we have to discount positive MtM from time τ_2 to present. Therefore, all we care about is MtM value of the short CDS at default time τ_2 of counterparty, not before τ_2. Thus we have to compute $MtM_{Seller}^{CDS}(t,T)$ on the event $\{\tau_1 > t, \tau_2 \leqslant t\}$.

Theorem 1. *Under the contagion model with stochastic intensities (3.6-3.7), given a stochastic interest rate r_t, mark-to-market value of a short CDS at time t with market spread S is explicitly expressed as*

$$\begin{aligned} 1_{\{\tau_1>t,\tau_2\leqslant t\}} MtM_{Seller}^{CDS}(t,T) \;&=\; 1_{\{\tau_1>t,\tau_2\leqslant t\}} \cdot \left\{ \int_t^T \left[S\cdot \tilde{P}(t,s) - L_1 \cdot \bar{P}(t,s) \right] ds \right\} \\ &\equiv\; 1_{\{\tau_1>t,\tau_2\leqslant t\}} \cdot \overline{MtM}(t,T) \end{aligned} \tag{4.25}$$

on event $\{\tau_1 > t, \tau_2 \leqslant t\}$. $\tilde{P}(t,s)$ and $\bar{P}(t,s)$ are defined in Lemma 2, and explicit expressions of them will be available if α_t^i's and r_t are set to be affine processes. $\overline{MtM}(t,T)$ is called pre-default value of $MtM_{Seller}^{CDS}(t,T)$ which is $\mathcal{F}_t$-adapted.

On event $\{\tau_1 \leq t\}$, we have

$$1_{\{\tau_1\leq t\}} MtM_{Seller}^{CDS}(t,T) = 1_{\{\tau_1\leq t\}} E\left[\Pi_{Seller}^{CDS}(t,T) \,|\mathcal{G}_t \right] = 0 \tag{4.26}$$

Proof. On the event $\{\tau_1 > t, \tau_2 \leqslant t\}$, explicit formulaes for unit survival claim and unit recovery value are displayed in Lemma 2, thus

$$\begin{aligned} & 1_{\{\tau_1>t,\tau_2\leqslant t\}} MtM_{Seller}^{CDS}(t,T) \\ =\; & 1_{\{\tau_1>t,\tau_2\leqslant t\}} E^{\mathbb{Q}} \left[S\cdot \int_t^T D(t,s)\, 1_{\{\tau_1>s\}} ds - L_1 \cdot \int_t^T D(t,s)\, dH_s^1 \middle| \mathcal{G}_t \right] \\ =\; & 1_{\{\tau_1>t,\tau_2\leqslant t\}} S\cdot \int_t^T E^{\mathbb{Q}} \left[\exp\left\{ -\int_t^s \left(\alpha_u^1 + \beta_u^1 + r_u\right) du \right\} \middle| \mathcal{F}_t \right] ds \\ & -1_{\{\tau_1>t,\tau_2\leqslant t\}} \mathrm{L}_1 \cdot \int_t^T E^{\mathbb{Q}} \left[\exp\left\{ -\int_t^s \left(\alpha_u^1 + \beta_u^1 + r_u\right) du \right\} \cdot \left(\alpha_s^1 + \beta_s^1\right) \middle| \mathcal{F}_t \right] ds \\ =\; & 1_{\{\tau_1>t,\tau_2\leqslant t\}} \cdot \left\{ \int_t^T \left[S\cdot \tilde{P}(t,s) - L_1 \cdot \bar{P}(t,s) \right] ds \right\} \end{aligned}$$

On the event $\{\tau_1 \leq t\}$, cashflow of CDS has already been truncated thus formula (4.26) is obvious.

□

Equation (4.26) implies that $MtM_{Seller}^{CDS}(t,T)$ concentrates all quality on the event $\{\tau_1 > t\}$. Thus we have

$$MtM_{Seller}^{CDS}(t,T) = 1_{\{\tau_1>t,\tau_2\leqslant t\}} MtM_{Seller}^{CDS}(t,T) + 1_{\{\tau_1>t,\tau_2>t\}} MtM_{Seller}^{CDS}(t,T)$$

Consequently, MtM value of the CDS short position upon default of counterparty has the following expression

$$MtM_{Seller}^{CDS}(\tau_2,T) \;=\; 1_{\{\tau_1>\tau_2,\tau_2\leqslant\tau_2\}} \cdot MtM_{Seller}^{CDS}(\tau_2,T)$$

$$= 1_{\{\tau_1>\tau_2\}} \cdot \overline{MtM}\left(\tau_2, T\right) \tag{4.27}$$

Put this expression into equation (2.4), we get

Theorem 2. *Under the contagion model with stochastic intensities (3.6-3.7), given a stochastic interest rate r_t, unilateral credit valuation adjustment of a short CDS at time 0 with market spread S is semi-explicitly expressed as*

$$CVA(0,T) \approx L_2 E^{\mathbb{Q}}\left[\sum_{j=1}^{N} \Delta T_j \exp\left\{-\int_0^{T_j}\left[\alpha_s^1+\alpha_s^2+r_s\right]ds\right\}\alpha_{T_j}^2\left(\overline{MtM}\left(T_j,T\right)\right)^+\right] \tag{4.28}$$

where $T_j, j=1,...,N$ are discrete payment days of CDS, with $T_N = T$ and $\Delta T_j = T_j - T_{j-1}$. $\square$.

Proof: Put equation (4.27) into equation (2.4), we get

$$\begin{aligned}
CVA(0,T) &= L_2 \cdot E^{\mathbb{Q}}\left[1_{\{\tau_2\leqslant T\}} D(0,\tau_2)\left(MtM_{Seller}^{CDS}(\tau_2,T)\right)^+\right] \\
&= L_2 \cdot E^{\mathbb{Q}}\left[1_{\{\tau_2\leqslant T\}} D(0,\tau_2) 1_{\{\tau_1>\tau_2\}}\left(\overline{MtM}\left(\tau_2,T\right)\right)^+\right] \\
&= L_2 \cdot E^{\mathbb{Q}}\left[\sum_{j=1}^{N} 1_{\{T_{j-1}<\tau_2\leqslant T_j\}} D(0,\tau_2) 1_{\{\tau_1>\tau_2\}}\left(\overline{MtM}\left(\tau_2,T\right)\right)^+\right] \qquad (4.29)
\end{aligned}$$

Although joint distribution of τ_1 and τ_2, conditional on $\mathcal{F}_T$ if necessary, is available in our contagion model with stochastic intensities, see Bao et al. [1] for details, the expectation in the above equation could not be solved analytically even if affine factors are supposed to assure $\overline{MtM}\left(\tau_2, T\right)$ is in explicit form. Therefore, we adopt the "*default bucketing*" technique used in Brigo et al. [3] to defer defaultable payment $\left(MtM_{Seller}^{CDS}(\tau_2,T)\right)^+$ in the interval $T_{j-1} < \tau_2 \le T_j$ to the next payment date T_j. Consequently, we get

$$\begin{aligned}
CVA(0,T) &\approx L_2 \cdot E^{\mathbb{Q}}\left[\sum_{j=1}^{N} 1_{\{T_{j-1}<\tau_2\leqslant T_j\}} \cdot D(0,T_j)\left(MtM_{Seller}^{CDS}(T_j,T)\right)^+\right] \\
&= L_2 \cdot E^{\mathbb{Q}}\left[\sum_{j=1}^{N} 1_{\{T_{j-1}<\tau_2\leqslant T_j\}} \cdot D(0,T_j) 1_{\{\tau_1>T_j\}}\left(\overline{MtM}\left(T_j,T\right)\right)^+\right] \\
&= L_2 \cdot E^{\mathbb{Q}}\left[\sum_{j=1}^{N} \mathbb{Q}\left(\tau_1>T_j, T_{j-1}<\tau_2\le T_j \middle| \mathcal{F}_{T_j}\right) \cdot D(0,T_j)\left(\overline{MtM}\left(T_j,T\right)\right)^+\right] \\
&\approx L_2 \cdot E^{\mathbb{Q}}\left[\sum_{j=1}^{N} \Delta T_j \exp\left\{-\int_0^{T_j}\left[\alpha_s^1+\alpha_s^2\right]ds\right\}\alpha_{T_j}^2 \cdot D(0,T_j)\left(\overline{MtM}\left(T_j,T\right)\right)^+\right] \\
&= L_2 \cdot E^{\mathbb{Q}}\left[\sum_{j=1}^{N} \Delta T_j \exp\left\{-\int_0^{T_j}\left[\alpha_s^1+\alpha_s^2+r_s\right]ds\right\}\alpha_{T_j}^2 \cdot \left(\overline{MtM}\left(T_j,T\right)\right)^+\right] \qquad (4.30)
\end{aligned}$$

where the second approximation holds for the reason that equation (3.15) is used to simplify the expression.

$\square$

Formula (4.28) gives a semi-analytical solution of CVA for a CDS short position. Usually, this equation is not in closed form even if α_t^i's and r_t are affine specified and $\overline{MtM}(t,T)$ is affine or exponential affine function of factors. Therefore, Monte Carlo simulation is necessary to implement this formula, just as what Brigo and Crépey did in their papers. The most important procedure in this implementation is simulating pathes of α_t^i's and r_t. The procedure will be clear once dynamics of α_t^i's and r_t are specified, especially if they are affine specified.

The impact of L_2 on CVA is obvious in formula (4.28). However, the impact of pre-default intensity volatility, interest rate and contagion is somehow implicit. Section 6 will perform some numerical test to analyze influence of contagion, volatility and correlation on CVA.

5 Affine Specification of Intensities and Interest Rate

To account for volatilities of pre-default intensities α_t^i's, we specify them to follow mutually independent CIR processes, that is $\alpha_t^1 = x_t$ and $\alpha_t^2 = z_t$, where dynamics of the two CIR factors are expressed as

$$\begin{cases} dx_t = k_x\left[\theta_x - x_t\right]dt + \sigma_x\sqrt{x_t}dW_t^x \\ dz_t = k_z\left[\theta_z - z_t\right]dt + \sigma_z\sqrt{z_t}dW_t^z \end{cases}, \text{ with } dW_t^x \perp dW_t^z \tag{5.31}$$

under martingale measure $\mathbb{Q}$. Parameters $\kappa_x, \kappa_z, \theta_x, \theta_z, \sigma_x$ and σ_z are supposed to satisfy $2k_x\theta_x > \sigma_x^2$ and $2k_z\theta_z > \sigma_z^2$ so that 0 is unattainable for x_t and z_t. We assume in this paper that pre-default intensities are mutually independent and suppose default dependence are fully characterized by contagion. This is only for illustration convenience and the affine specification of α_t^1 and α_t^2 can easily be extended to allow commonly dependence on a group of mutually independent CIR variables but weighting differently on them to incorporates non-trivial correlation of α_t^1 and α_t^2, while still remaining analytically solvable. This is straightforward from equations (3.11), (3.12), (4.25) and (4.28), because all relevant quantities are in the form that affine specification could conclude explicit expressions.

This paper also incorporates stochastic interest rate into modeling. We assume r_t is dependent on the affine factors x_t and z_t as

$$r_t = \kappa_x \cdot x_t + \kappa_z \cdot z_t \tag{5.32}$$

where κ_x and κ_z are positive constants to account for positive correlations with credit spreads. The explanation of κ_x and κ_z being correlation with spreads may be straightforward but not convincing. We show this by deriving exact instantaneous correlations between pre-default intensities and interest rate.

First, we note that κ_x and κ_z are not two free parameters, but subject to one constraint condition $r_0 = \kappa_x \cdot x_0 + \kappa_z \cdot z_0$, where r_0, $x_0 = \alpha_0^1$ and $z_0 = \alpha_0^2$ are model inputs that are previously given. Based on this equality, we get

$$\begin{aligned} Corr\left(d\alpha_t^1, dr_t\right) &= \frac{d\alpha_t^1 \cdot dr_t}{\sqrt{d\alpha_t^1 \cdot d\alpha_t^1}\sqrt{dr_t \cdot dr_t}} \\ &= \frac{1}{\sqrt{1 + \frac{r_0^2}{z_0^2}\left(\frac{1}{\kappa_x} - \frac{x_0}{r_0}\right)^2 \sigma_z^2\alpha_t^2/\sigma_x^2\alpha_t^1}} \end{aligned} \tag{5.33}$$

Therefore, $Corr\left(d\alpha_t^1, dr_t\right)$ is strictly increasing with respect to κ_x if and only if $\frac{1}{\kappa_x} > \frac{x_0}{r_0}$, or equivalently $\kappa_x \cdot \frac{x_0}{r_0} < 1$. Similarly, we have $Corr\left(d\alpha_t^2, dr_t\right)$ being strictly increasing with respect to κ_z if and only if $\kappa_z \cdot \frac{z_0}{r_0} < 1$. From $\kappa_x \cdot \frac{x_0}{r_0} + \kappa_z \cdot \frac{z_0}{r_0} = 1$ and positivity constraint on

κ_x and κ_z we conclude that the two equalities always hold in our model. This implies that κ_x and κ_z represent the relative levels of instantaneous correlations of α_t^1 and α_t^2 with r_t.

The following two propositions give some basic formulas in affine models.

Proposition 2. *Assume X_t is an $\mathcal{F}_t$-adapted affine process, more specifically a CIR process*

$$dX_t = k\left[\theta - X_t\right]dt + \sigma\sqrt{X_t}dW_t \sim \mathbb{Q} \tag{5.34}$$

Then valuation of $E^{\mathbb{Q}}\left[\exp\left\{-\int_t^T X_s ds\right\}\middle|\mathcal{F}_t\right]$ and $E^{\mathbb{Q}}\left[\exp\left\{-\int_t^T X_s ds\right\}X_T\middle|\mathcal{F}_t\right]$ can be expressed as the following affine forms,

$$P_X(t,T) = E\left[\exp\left\{-\int_t^T X_s ds\right\}\middle|\mathcal{F}_t\right] = A_X(t,T)e^{-B_X(t,T)X_t} \tag{5.35}$$

where

$$\begin{cases} B_X(t,T) = \dfrac{2\left[e^{(T-t)h}-1\right]}{2h+(k+h)\left[e^{(T-t)h}-1\right]} \\ A_X(t,T) = \left[\dfrac{2he^{(T-t)(k+h)/2}}{2h+(k+h)\left[e^{(T-t)h}-1\right]}\right]^{\frac{2k\theta}{\sigma^2}} \end{cases}, \text{ with } h = \sqrt{k^2+2\sigma^2} \tag{5.36}$$

and

$$Q_X(t,T) = E^{\mathbb{Q}}\left[\exp\left\{-\int_t^T X_s ds\right\}X_T\middle|\mathcal{F}_t\right] = \bar{A}_X(t,T,X_t)P_X(t,T) \tag{5.37}$$

where

$$\begin{aligned} \bar{A}_X(t,T,X_t) &= -\frac{1}{A_X(t,T)}\frac{\partial A_X(t,T)}{\partial T} + \frac{\partial B_X(t,T)}{\partial T}X_t \\ &= \frac{2k\theta\left[e^{(T-t)h}-1\right]}{2h+(k+h)\left[e^{(T-t)h}-1\right]} + \frac{4h^2e^{(T-t)h}}{\left(2h+(k+h)\left[e^{(T-t)h}-1\right]\right)^2}X_t \end{aligned} \tag{5.38}$$

□.

Proof: As well known, price of "zero-coupon bond" $P_X(t,T)$ with one affine factor X_t can be expressed in the affine form (5.35) with coefficient functions $A_X(t,T)$ and $B_X(t,T)$ expressed in equation (5.36).

As for price of "default-free" claim $E^{\mathbb{Q}}\left[\exp\left\{-\int_t^T X_s ds\right\}X_T\middle|\mathcal{F}_t\right]$, it is conclusion of formula (5.35), because

$$\begin{aligned} Q_X(t,T) &= E\left[\exp\left\{-\int_t^T X_s ds\right\}X_T\middle|\mathcal{F}_t\right] = -\frac{\partial P_X(t,T)}{\partial T} \\ &= \left(-\frac{1}{A_X(t,T)}\frac{\partial A_X(t,T)}{\partial T} + \frac{\partial B_X(t,T)}{\partial T}X_t\right)A_X(t,T)e^{-B_X(t,T)X_t} \\ &= \bar{A}_X(t,T,X_t)P_X(t,T) \end{aligned}$$

□

To simplify illustration here and in the next section, we extend the coefficient functions in equation (5.36) to include a parameter α, with

$$\begin{cases} B(t,T;k,\theta,\sigma,\alpha) = \dfrac{2\left[e^{(T-t)h}-1\right]}{2h+(k+h)\left[e^{(T-t)h}-1\right]} \\ A(t,T;k,\theta,\sigma,\alpha) = \left[\dfrac{2he^{(T-t)(k+h)/2}}{2h+(k+h)\left[e^{(T-t)h}-1\right]}\right]^{\frac{2k\theta}{\sigma^2}} \end{cases}, \text{ with } h=\sqrt{k^2+2\alpha\sigma^2} \quad (5.39)$$

Functions in equation (5.36) are special cases of equation (5.39) with $\alpha = 1$. Actually, $B(t,T;k,\theta,\sigma,\alpha)$ and $A(t,T;k,\theta,\sigma,\alpha)$ are corresponding coefficient functions for "zero-coupon bond" with CIR discount rate $\alpha \cdot X_t$. This is straightforward to prove. Moreover, we define two coefficient functions as

$$\begin{cases} W(t,T;k,\theta,\sigma,\alpha) = \dfrac{2k\alpha\theta\left[e^{(T-t)h}-1\right]}{2h+(k+h)\left[e^{(T-t)h}-1\right]} \\ M(t,T;k,\theta,\sigma,\alpha) = \dfrac{4h^2e^{(T-t)h}}{\left(2h+(k+h)\left[e^{(T-t)h}-1\right]\right)^2} \end{cases} \quad \textit{with } h=\sqrt{k^2+2\alpha\sigma^2} \quad (5.40)$$

Then we define

$$\bar{A}_X(t,T,X_t;k,\theta,\sigma,\alpha) = W(t,T;k,\theta,\sigma,\alpha) + M(t,T;k,\theta,\sigma,\alpha)\cdot\alpha X_t \quad (5.41)$$

Therefore, $\bar{A}_X(t,T,X_t;k,\theta,\sigma)$ in equation (5.38) is a special case of this function with $\alpha = 1$.

Proposition 3. *Under affine specification of factors x_t and z_t in equation (5.31), we get*

$$\begin{aligned} P(t,T) &= E\left[\exp\left\{-\int_t^T [\alpha\cdot x_s+\beta\cdot z_s]\,ds\right\}\middle|\mathcal{F}_t\right] \\ &= A_x(t,T;\alpha)A_z(t,T;\beta)e^{-B_x(t,T;\alpha)\cdot\alpha x_t - B_z(t,T;\beta)\cdot\beta z_t} \end{aligned} \quad (5.42)$$

with

$$\begin{cases} B_x(t,T;\alpha) = B(t,T;k_x,\theta_x,\sigma_x,\alpha) \\ A_x(t,T;\alpha) = A(t,T;k_x,\theta_x,\sigma_x,\alpha) \\ B_z(t,T;\beta) = B(t,T;k_z,\theta_z,\sigma_z,\beta) \\ A_z(t,T;\beta) = A(t,T;k_z,\theta_z,\sigma_z,\beta) \end{cases}$$

Moreover, for $\alpha \neq 0$ and $\beta \neq 0$ we have

$$\begin{aligned} Q(t,T) &= E\left[\exp\left\{-\int_t^T [\alpha\cdot x_s+\beta\cdot z_s]\,ds\right\}[\mu\cdot x_T+\nu\cdot z_T]\middle|\mathcal{F}_t\right] \\ &= \left[\frac{\mu}{\alpha}\bar{A}_x(t,T,x_t;\alpha)+\frac{\nu}{\beta}\bar{A}_z(t,T,z_t;\beta)\right]P(t,T) \end{aligned} \quad (5.43)$$

with

$$\begin{cases} \bar{A}_x(t,T,x_t;\alpha) = \bar{A}_x(t,T,x_t;k_x,\theta_x,\sigma_x,\alpha) \\ \bar{A}_z(t,T,z_t;\beta) = \bar{A}_z(t,T,z_t;k_z,\theta_z,\sigma_z,\beta) \end{cases}$$

□.

Proof: First, note that if X_t is a CIR process with parameters (k,θ,σ), then $\alpha\cdot X_t$ is still a CIR process with parameters $(k,\alpha\theta,\sqrt{\alpha}\sigma)$. Thus formula (5.42) is direct consequence of formula (5.35) because of independence between x_t and z_t.

Second, $Q(t,T)$ can be reformulated as

$$Q(t,T) = E\left[\exp\left\{-\int_t^T[\alpha\cdot x_s+\beta\cdot z_s]\,ds\right\}\left(\frac{\mu}{\alpha}[\alpha\cdot x_T+\beta\cdot z_T]+\left(\nu-\frac{\mu\beta}{\alpha}\right)\cdot z_T\right)\middle|\mathcal{F}_t\right]$$

$$
\begin{aligned}
&= \frac{\mu}{\alpha} E\left[\exp\left\{-\int_t^T [\alpha \cdot x_s + \beta \cdot z_s]\, ds\right\}[\alpha \cdot x_T + \beta \cdot z_T]\middle| \mathcal{F}_t\right] \\
&\quad + \left(\nu - \frac{\mu\beta}{\alpha}\right) E\left[\exp\left\{-\int_t^T [\alpha \cdot x_s + \beta \cdot z_s]\, ds\right\} z_T \middle| \mathcal{F}_t\right] \\
&= \frac{\mu}{\alpha} E\left[\exp\left\{-\int_t^T [\alpha \cdot x_s + \beta \cdot z_s]\, ds\right\}[\alpha \cdot x_T + \beta \cdot z_T]\middle| \mathcal{F}_t\right] \\
&\quad + \left(\nu - \frac{\mu\beta}{\alpha}\right) E\left[\exp\left\{-\int_t^T \alpha \cdot x_s ds\right\}\middle| \mathcal{F}_t\right] E\left[\exp\left\{-\int_t^T \beta \cdot z_s ds\right\} z_T \middle| \mathcal{F}_t\right]
\end{aligned}
$$

where the first integral in the last equality of the above equation is given by

$$
\begin{aligned}
&E\left[\exp\left\{-\int_t^T [\alpha \cdot x_s + \beta \cdot z_s]\, ds\right\}[\alpha \cdot x_T + \beta \cdot z_T]\middle| \mathcal{F}_t\right] = -\frac{\partial}{\partial T} P(t,T) \\
&= \left[\bar{A}_x(t,T,x_t;\alpha) + \bar{A}_z(t,T,z_t;\beta)\right] P(t,T)
\end{aligned}
$$

and the second integral is "zero-coupon bond" with discount rate $\alpha \cdot x_t$, i.e.

$$
E\left[\exp\left\{-\int_t^T \alpha \cdot x_s ds\right\}\middle| \mathcal{F}_t\right] \equiv A_x(t,T;\alpha) e^{-B_x(t,T;\alpha)\cdot \alpha x_t} = P_x(t,T;\alpha)
$$

while the third integral is expressed as

$$
E\left[\exp\left\{-\int_t^T \beta \cdot z_s ds\right\} z_T \middle| \mathcal{F}_t\right] = \frac{1}{\beta} \bar{A}_z(t,T,z_t;\beta) P_z(t,T;\beta)
$$

Therefore,

$$
\begin{aligned}
Q(t,T) &= \frac{\mu}{\alpha}\left[\bar{A}_x(t,T,x_t;\alpha) + \bar{A}_z(t,T,z_t;\beta)\right] P(t,T) \\
&\quad + \left(\nu - \frac{\mu\beta}{\alpha}\right) P_x(t,T;\alpha) \frac{1}{\beta} \bar{A}_z(t,T,z_t;\beta) P_z(t,T;\beta) \\
&= \left[\frac{\mu}{\alpha}\bar{A}_x(t,T,x_t;\alpha) + \frac{\nu}{\beta}\bar{A}_z(t,T,z_t;\beta)\right] P(t,T)
\end{aligned}
$$

□

Theorem 3. *Under the affine specification of α_t^1, α_t^2 and r_t in equations (5.31) and (5.32), the pre-default price of risk-free CDS on event $\{\tau_1 > t, \tau_2 \le t\}$ is given by*

$$
\begin{aligned}
&\overline{MtM}(t,T) \\
&= \int_t^T \left[S - L_1\left[\frac{1}{1+\kappa_x}\bar{A}_x(t,s,x_t;1+\kappa_x) + \frac{\eta_2}{\eta_2+\kappa_z}\bar{A}_z(t,s,z_t;\eta_2+\kappa_z)\right]\right]\tilde{P}(t,s) ds \qquad (5.44)
\end{aligned}
$$

□.

Proof: As equation (4.25) indicated, we have to calculate $\tilde{P}(t,s)$ and $\bar{P}(t,s)$ in this affine environment, which are direct conclusions of the above lemma.

$$
\begin{aligned}
\tilde{P}(t,s) &= E^{\mathbb{Q}}\left[\exp\left\{-\int_t^s [(1+\kappa_x)\cdot x_u + (\eta_2+\kappa_z)\cdot z_u]\, du\right\}\middle| \mathcal{F}_t\right] \\
&= P_x(t,s;1+\kappa_x) P_z(t,s;\eta_2+\kappa_z)
\end{aligned}
$$

and

$$
\bar{P}(t,s) = E^{\mathbb{Q}}\left[\exp\left\{-\int_t^s [(1+\kappa_x)\cdot x_u + (\eta_2+\kappa_z)\cdot z_u]\, du\right\}(x_u + \eta_2 \cdot z_u)\middle| \mathcal{F}_t\right]
$$

$$= \left[\frac{1}{1+\kappa_x} \bar{A}_x(t,s,x_t;1+\kappa_x) + \frac{\eta_2}{\eta_2+\kappa_z} \bar{A}_z(t,s,z_t;\eta_2+\kappa_z) \right] \tilde{P}(t,s)$$

which conclude this proof.

□

6 Numerical Analysis

This section performs some numerical analysis of the semi-analytical and analytical expression in equations (4.28) and (5.44). To simulate sample pathes of α_t^i's and r_t, we need simulating two independent CIR process x_t and z_t. It is well known that the transition law of a CIR process X_t as in equation (5.34) given X_s can be expressed by

$$X_t = \frac{\sigma^2 \left(1 - e^{-k(t-s)}\right)}{4k} \cdot \chi_d^2 \left(\frac{4ke^{-k(t-s)}}{\sigma^2 \left(1 - e^{-k(t-s)}\right)} X_s \right)$$

where $d = \frac{4k\theta}{\sigma^2}$, and $\chi_d^2(v)$ represents a non-central chi-square random variable with d degree of freedom and v non centrality parameter. Once starts from an initial point X_0, we can simulate the process X_t exactly on a discrete time grid by sampling from the non-central chi-square distribution.

We consider CVA of a $T = 5$ year CDS in this numerical analysis. Typically, $\Delta T_j \equiv \Delta T$ is 0.25 of one year, i.e. 3 months, then discretizing one payment period into 3 grids is fine enough. Denote the numbers of entire payment periods and grids in each period by N and p, then $N = 20$ and $p = 3$ in this section. Thus the number of entire grids is $L = N \times p = 60$. δ is supposed to represent the fineness of the time grid in our valuation procedure, thus $\delta = 1/12$ in our case. We realize an entire number I of CVA samples, then divide sum of the samples by I to get a Monte Carlo valuation of CVA. Pseudo code of our numerical algorithm is illustrated in Table 3, where the procedure for calculating basic functions A, B, M and W is omitted.

Table 1: Benchmark parameters of CIR pre-default intensities

	X_0	k	θ	σ
Reference	0.03	0.50	0.05	0.50
Counterparty	0.01	0.80	0.02	0.20

We chose a group of reasonable parameters as benchmark case, then vary one of the parameters to analyze the impact of this parameter on CVA. The benchmark parameters for pre-default intensities are listed in Table 1, and the other benchmark parameters in our contagion model are listed in Table 2.

Table 2: The other benchmark parameters in contagion model

S	L_1	L_2	η_1	η_2	r_0	κ_x	κ_z
250 bp	60%	60%	0.1	0.25	5%	1	2

To model spread volatilities of reference firm and counterparty, two sources of volatilities are accounted in this papers, i.e. the volatilities of pre-default intensities from diffusion of Brownian

Table 3: Algorithm for valuing unilateral CVA in view of CDS seller by Monte Carlo

```
CUM_CVA = 0;
for i = 1 : I do
    V_0 = 0; D=1;
    for j = 1 : N do
        for l = [(j − 1)p + 1] : jp do
            generate x = x(l · δ) and z = z(l · δ);
            D = D · e^{−[(1+κ_x)x+(1+κ_z)z]·δ};
        end for
        MtM = 0;
        for l = (jp + 1) : L do
            a_x = A_x(jp · δ, l · δ; 1 + κ_x);
                ...
            m_z = M_z(jp · δ, l · δ; η_2 + κ_z);
            (we've got a_x, b_x, w_x, m_x, a_z, b_z, w_z, m_z here)
            mtm = {S − L_1 [( w_x/(1+κ_x) + η_2*w_z/(η_2+κ_z) ) + (m_x · x + η_2 * m_z · z)]}
                  *a_x * a_z * e^{(−b_x*(1+κ_x)*x+b_z*(η_2+κ_z)*z)} * δ;
            MtM = MtM + mtm;
        end for
        if MtM ≤ 0 then
          MtM = 0;
        end if
        V_0 = V_0 + D * z * MtM;
    end for
    CUM_CVA = CUM_CVA + V_0;
end for
CVA = [L_2 * p * δ * CUM_CVA] /I;
```

motions and contagion between the two firms. For illustration simplicity, this paper assumes that pre-default intensities are mutually independent and spread correlation is modeled solely through contagion upon the default of one firm to the survival one. From equations (4.28) and (5.44) it is obvious that CVA of the short CDS in the view of investor is irrelevant of contagion parameter η_1, that characterizes the extent to which contagion of default risk from reference firm to counterparty. This can be interpreted in the way that contagion from reference to counterparty happens only at the default time of reference firm, when the CDS will be unwound, thus CVA of this CDS is surely zero at that time. Experiment result of impact of contagion parameter η_2 on CVA is expressed in Table 4.

Table 4: Impact of η_2 on CVA

η_2	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
CVA	7.3	5.9	4.8	3.7	2.7	1.9	1.3	0.8	0.5

The trend of η_2's impact on CVA is quite clear from Table 4. The greater η_2 is, the smaller

CVA will be. This is reasonable in contagion model. Equation (2.4) asserts that CVA is discounted present value of positive MtM from the default time of counterparty. Greater η_2 implies greater default intensity of the survival reference firm after time τ_2. Therefore the unrealized cashflow after τ_2 will be less valued because of the greater discount rate.

Table 5: Impact of σ_x and σ_z on CVA when $\eta_2 = 0.25$

(σ_x, σ_z)	0.01	0.10	0.20	0.30	0.40
0.01	0.0	0.0	0.0	0.0	0.0
0.20	1.6	1.4	1.2	0.9	0.7
0.40	4.3	4.3	3.6	2.9	2.5
0.50	5.8	5.3	4.8	3.9	3.3
0.60	6.8	6.5	5.8	5.0	4.0
0.80	8.7	8.4	7.6	6.5	5.4

Volatilities of α_t^1 and α_t^2, as another source of spread volatilities in our contagion model, are also important parameters that have significant impact on CVA. Table 5 and Table 6 show the numerical results of CVA on varying σ_x and σ_z given $\eta_2 = 0.5$ and $\eta_2 = 0$, respectively. The pattern in Table 5 is pretty clear. CVA is increasingly dependent on volatility σ_x of reference firm's pre-default intensity while decreasingly dependent on volatility σ_z of counterparty's pre-default intensity. Moreover, Table 5 shows that CVA is much more sensitive on σ_x, which is volatility of pre-default intensity for reference entity. Comparing Table 4 and Table 5 we find that contagion is dominant in the two sources of spread volatilities, i.e. contagion from counterparty to reference has much greater impact on CVA than counterparty's own volatility of pre-default intensity. This confirms the importance of introducing contagion into default dependence while calculating CVA. Table 4, Table 5 and Table 6 jointly show that in the presence of contagion, i.e. $\eta_2 > 0$, contagion effect is dominant over volatility from counterparty's pre-default intensity, while reference's intensity volatility has significant impact on CVA.

Table 6: Impact of σ_x and σ_z on CVA when $\eta_2 = 0$

(σ_x, σ_z)	0.01	0.10	0.20	0.30	0.40
0.01	0.0	0.0	0.0	0.0	0.0
0.20	2.7	2.7	2.5	2.5	2.3
0.40	6.2	6.2	5.3	5.3	5.1
0.50	7.7	7.5	7.2	6.8	6.3
0.60	8.8	8.7	8.6	7.7	7.1
0.80	11.0	10.7	10.3	9.9	9.1

To check the impact of correlation between spread and interest on CVA, we just have to analyze the impact of κ_x or κ_z on CVA, as indicated in the beginning of Section 5. We chose κ_x as free parameter, then κ_z can be given as $\kappa_z = [r_0 - \kappa_x \cdot x_0]/z_0 = 5 - 3\kappa_x$ for the benchmark parameters. Moreover, we have $\kappa_x < r_0/x_0 = 5/3$. Thus we chose the 8 special values of κ_x in Table 7 for illustration. Table 7 shows a humped pattern of κ's impact on CVA in the situation of $\eta_2 = 0.5$. For the special cases of r_t being independent from one of the pre-default intensities, i.e. $\kappa_x = 0$ or $\kappa_z = 0$, CVA is significantly smaller than the modest cases. When increasing κ_x from 0 to its upper bound 5/3, CVA increases in the first phase until $\kappa_x = 1$. Then CVA decreases significantly as κ_x approximating its upper bound 5/3. For the situation of $\eta_2 = 0.25$, as the benchmark case in our numerical experiment, influence of contagion is weaker while impact of pre-default intensity correlation plays significant role. Table 7 shows that CVA is monotonic increasing with respect to κ_x, i.e. with correlation between α_t^1 and r_t.

Table 7: Impact of κ on CVA

κ_x	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.67
κ_z	5.00	4.25	3.50	2.75	2.00	1.25	0.50	0.00
CVA when $\eta_2 = 0.25$	3.7	4.0	4.1	4.4	4.8	5.0	5.4	5.6
CVA when $\eta_2 = 0.5$	3.6	4.0	4.3	4.6	4.6	4.5	3.6	2.1

7 Conclusion

This paper models volatility of MtM of a short CDS in contagion model with stochastic intensities and interest rate. Therefore, two sources of spread volatility is characterized in our model. Survival measure approach is adopted in this paper to calculate MtM of risk-free CDS as well as conditional distribution of counterparty's default time in defaultable environment, and semi-analytical solution for CVA is attained. Affine specification of intensities and interest rate concludes analytical expression for pre-default value of MtM. Pseudo code of the numerical algorithm is presented in this paper. Numerical analysis shows that contagion constitutes as the major source of volatility for MtM while calculating CVA. Strictly increasing and decreasing patterns of reference's and counterparty's volatilities on CVA is displayed in our experiment when contagion is present. A humped pattern of correlation between spread and interest rate is detected in the situation of relatively larger eta's while monotonic pattern is exhibited for the case of relatively smaller η's.

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