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Abstract

A family of economic and demographic models governed by linear delay differential equations is considered. They can be expressed as optimal control problems subject to delay differential equations (DDEs) characterized by some non-trivial mathematical difficulties: state/control constraints and delay in the control. The study is carried out rewriting the problem as an (equivalent) optimal control problem in infinite dimensions and then using the dynamic programming approach (DPA).

Similar problems have been studied in the literature using classical and strong (approximating) solutions of the Hamilton-Jacobi-Bellman (HJB) equation. Here a more general formulation is treated thanks to the use of viscosity solutions approach. Indeed a general current objective function is considered and the concavity of the Hamiltonian is not required. It is shown that the value function is a viscosity solution of the HJB equation and a verification theorem in the framework of viscosity solutions is proved.

Key words: viscosity solutions, delay differential equation, vintage models.

1 Introduction

The present work can be considered as a continuation of the studies presented by Fabbri et al. (2006). We treat a class of economic and demographic problems, written as optimal control problems with delay state equation. We use an equivalent formulation of the delay problem introducing a suitable Hilbert space and re-writing the state equation as a suitable Ordinary Differential Equation[1] (ODE) in the Hilbert space.

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[1]The method we use is due to Vinter and Kwong (1981) and Delfour (1986; 1980; 1984). In the paper we will refer to the book of Bensoussan et al. (1992) that give a precise systematization of the argument.

The family of models we study arises in particular in the demographic and economic literature. The references for models for epidemiology and dynamic population governed by linear delay differential equations, to which an abstract formulation in Hilbert spaces is possible, are presented in Section 2. We will then recall a demographic model with an explicit age structure by Boucekkine et al. (2002) (Subsection 2.2), an AK model with vintage capital by Boucekkine et al. (2005) (briefly described in Subsubsection 2.1.1)[2], a AK model for obsolescence and depreciation by Boucekkine et al. (2004) (Subsubsection 2.1.3) and an advertising model with delay effects by Gozzi and Marinelli (2004); Gozzi et al. (2006); Faggian and Gozzi (2004) (Subsubsection 2.1.2). Some of them are described in Fabbri et al. (2006) in more details.

We use the dynamic programming approach (DPA). We briefly recall[3] that the DPA consists of four main steps: (i) Write the dynamic programming principle for the value function and its infinitesimal version, the HJB equation, (ii) Solve the HJB equation and prove that the solution is the value function, (iii) Prove a verification theorem (which can involve the value function) that gives the optimal control as function of the state finding the closed loop relation, (iv) Solve, if possible, the closed loop equation, obtained inserting the closed loop relation in the state equation.

The main difference between Fabbri et al. (2006) and the present work is the different method used to study the HJB equation. In Fabbri et al. (2006) we studied the HJB equation using an approximation method with techniques similar to the ones used by Faggian (2005*b*;*a*); Faggian and Gozzi (2004) for other classes of problems. Here we treat a more general case, studying the existence of viscosity solutions for the HJB equation. Indeed, as we also remarked in the introduction of Fabbri et al. (2006), the use of viscosity solutions in the study of HJB equation allows to avoid the concavity assumption for the Hamiltonian and for the target functional of the problem. In this way problems with multiple optimal solutions[4], where the value function is not everywhere differentiable, are also tractable. Moreover, using viscosity solution approach, we do not require that the control and the state are de-coupled in the objective function (see Subsection 3.2 and in particular Remark 3.5). We prove that the value function is a viscosity solution of the HJB equation (Theorem 5.9) and then we give a verification theorem (Theorem 6.4). A verification result represents a key step in the dynamic programming approach to optimal control problems, indeed it verifies whether a given admissible control is optimal and, more importantly,

[2]The model by Boucekkine et al. (2005) was also studied by Fabbri and Gozzi (2006) using the dynamic programming approach.

[3]A more detailed description of the method can be found for example in Fabbri et al. (2006).

[4]We refer to Deissenberg et al. (2004) for a bibliographical account of such problems arising in economics

suggests a way of constructing optimal feedback control. We are not able at the moment to give a uniqueness result for the viscosity solution of the HJB equation. It will be an issue for future work.

On viscosity solution approach We have already recalled that a crucial step in the the DPA to optimal control problems is solving the associated HJB equation. Such a solution can be used to find optimal controls in a closed-loop form. There are many possible definitions of solutions of a PDE and in particular of the HJB equation related to optimal control problems. Which shall we choose? In the classical works (Fleming and Rishel, 1975) the authors use a regular solution approach: the solution of the HJB equation is a regular (C^1) function that satisfies pointwise, with its derivatives, the equation. However in many cases, interesting from an applied point of view, the solution of the HJB equation is neither C^1 nor differentiable. Crandall and Lions (1983) introduced the definition of viscosity solution for the HJB equation related to optimal control problem in finite dimensions. In general the idea is that the solution can be less regular, for example continuous, and the solution is defined using either sub and super differential or using test functions. The notion of viscosity solution is a generalization of the notion of regular solution in the sense that every regular solution of the HJB equation is also a viscosity solution. Moreover there are many examples of HJB equations that admit viscosity solutions but do not have classical solutions. Under quite general hypotheses, in the finite dimensional case, it can be proved that the HJB equation related to an optimal control problem admits a unique viscosity solution and that such a solution is exactly the value function of the problem. Viscosity solutions can be used to find verification results and to solve optimal control problems that cannot be treated with classical solutions. In the infinite dimensional case the things are quite more complex and the literature is smaller. It remains true that viscosity solutions are an extension of classical solutions and can be used to treat a greater number of problems.

A brief summary on the literature The viscosity method, introduced in the study of the finite dimensional HJ equation by Crandall and Lions (1983) was extended to the infinite dimensional case by the same authors in a series of works (Crandall and Lions, 1985; 1986*a*;*b*; 1990; 1991; 1994*a*;*b*). New variants of the notion of viscosity solutions of HJB equations in Hilbert spaces are given by Ishii (1993) and by Tataru (1992*a*;*b*; 1994).

The study of viscosity solution for HJB equations in Hilbert spaces arising from optimal control problems of systems modeled by PDE *with boundary control term* is more recent. In this research field there is not a complete theory but some works on specific PDE that adapt the ideas and the techniques of viscosity solutions to special cases. For the first order HJB equations see the works by Cannarsa et al. (1991; 1993); Cannarsa and Tessitore (1994;

1996*a*;*b*); Gozzi et al. (2002); Fabbri (2006*b*). It must be noted that most of these works treats the case in which the generator of the semigroup that appears in the ODE is selfadjoint.

Infinite dimensional HJB equations arising from DDEs *with delay in the control* present an unbounded term similar to the one arising in boundary control problems. To our knowledge such HJB equations have been studied only by Fabbri and Gozzi (2006); Fabbri et al. (2006) using classical and strong[5] solution. The existing papers do not cover the case studied in the present work.

The abstract method and the applications As we have already stressed, in this paper we use an abstract formulation for linear delay differential equation. In particular we write the DDEs as an equivalent ODE in an Hilbert space and then we study the infinite dimensional HJB equation related to such a formulation.

This kind of abstract approach is not only a mathematical study but it is useful to obtaining applied results. In Fabbri and Gozzi (2006) such kind of method was used to study an AK growth model with vintage capital (the same recalled in Subsubsection 2.1.1 with the Constant Relative Risk Aversion (CRRA) functional) finding more precise results with respect to the existing literature that studied the problem using maximum principle, a tool that seems more "applied". One of the improvements was in finding, for example, the long run behavior of the system and various constants of the model in explicit form. Analogous results can probably obtained using the same tools in the model for obsolescence and depreciation presented by Boucekkine et al. (2006) and in the time-to-build model by Asea and Zak (1999) (see also Bambi, 2006). Indeed such models present a CRRA functional and are governed by a delay differential equation of the form required by Fabbri (2006*a*).

In the present paper we study a general case, with a generic functional, and then an explicit solution of the HJB equation is not available but the DPA is a useful tool. It allows to obtain a verification result in the general case that can be exploited in the cases in which the value function is given (possibly only numerically).

The plan of the work In Section 2 we recall some demographic and economic models that use linear delay differential equation and in particular three key models (in which an optimal control problem appears) that we will use to formulate our general problem. In Section 3 we describe the optimal control problem in delay formulation with some remarks on the difficulties we encountered (in Subsection 3.4 we explain why it cannot be treated with standard techniques). Then (Section 4) we briefly recall the equivalence of the optimal control problem subject to DDE and a suitable optimal control problem subject to ODE in an specific Hilbert space. In

[5]A strong solution is a suitable limit of classical solutions of approximating problems.

Section 5 we present the definition of viscosity solution of the HJB equation (Definition 5.2, Definition 5.3, Definition 5.4) and we prove (Theorem 5.9) that the value function of the problem is a viscosity solution of the HJB equation. In Section 6 we give a verification result (Theorem 6.4) using some techniques that will be better develop in Swiech et al. (2006).

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2 Demographic and economic models

The Hilbert setting we describe[6] can be used to express in abstract form linear delay differential equations (LDDEs). LDDEs are used to model a large variety of phenomena. Systems of such equations, possibly combined with other types of functional equation arise for example in modelling the dynamics of epidemics (Hethcote and van den Driessche, 1995; 2000; Smith, 1983; Waltman, 1974) and in biomedical models (Bachar and Dorfmayr, 2004; Culshaw and Ruan, 2000) (see also Luzyanina et al., 2004, for a numerical approach). A review on the use of DDEs (linear and nonlinear) in biosciences, in particular in population dynamics, ecology, epidemiology, immunology and physiology can be found in Bocharova and Rihanb (2000) and Baker et al. (1999).

The Hilbert setting we describe can be also used to treat multidimensional linear delay differential systems and in particular the linearizations of models governed by DDEs near equilibrium points (Li and Ma, 2004, page 1234).

2.1 Three main examples

In this subsection we briefly recall three economic models. They are our *main* examples because we will use them to understand which can be the “right” assumption in the formulation of the general case. As seen in Subsection 2.2 they are formally very similar to some dynamic population models. The first is an AK-model with vintage capital introduced by Boucekkine et al. (2005), the second is an advertising model with delay effects by Gozzi and Marinelli (2004); Gozzi et al. (2006); Faggian and Gozzi (2004) and the third is an AK model for obsolescence and depreciation by Boucekkine et al. (2006).

[6]As we already recalled it is due in particular to to Vinter and Kwong (1981) and Delfour (1986; 1980; 1984).

2.1.1 An AK model with vintage capital

The AK-growth model with vintage capital presented by Boucekkine et al. (2005) is based on the following accumulation law for capital goods

$$k(s) = \int_{s-R}^{s} i(\tau)\mathrm{d}\tau$$

where $i(\tau)$ is the investment at time τ. That is, capital goods are accumulated for the length of time R (scrapping time) and then dismissed. It is to note that such an approach introduces a differentiation in investments that depends on their age. It is assumed a linear production function, that is

$$y(s) = ak(s)$$

for some constant $a > 0$ where $y(s)$ is the output at time s. We assume that at every time s the planner chooses how to split the production into consumption $c(s)$ and investment in new capital $i(s)$:

$$y(s) = c(s) + i(s),$$

then the state equation may be written into infinitesimal terms as follows

$$\dot{k}(s) = i(s) - i(s-R), \quad s \in [0, +\infty)$$

that is, as a LDDE. The social planner has to maximize the following functional

$$\int_0^{+\infty} e^{-\rho s}\frac{c(s)^{1-\sigma}}{1-\sigma}\mathrm{d}s = \int_0^{+\infty} e^{-\rho s}\frac{(ak(s)-i(s))^{1-\sigma}}{1-\sigma}\mathrm{d}s \tag{1}$$

We assume that the investment at time s and the consumption at time s cannot be negative:

$$i(s) \geq 0, \qquad c(s) \geq 0, \quad \forall s \in [t, T] \tag{2}$$

So the admissible set has the form:

$$\mathcal{A} \stackrel{def}{=} \{i(\cdot) \in L^2_{loc}([0,+\infty), \mathbb{R}) \ : \ 0 \leq i(s) \leq ak(s) \ a.e. \ in \ [0, +\infty)\}.$$

2.1.2 An advertising model with delay effects

Consider the following dynamic advertising model presented in the stochastic case by Gozzi et al. (2006) and by Gozzi and Marinelli (2004), and, in deterministic one, by Faggian and Gozzi (2004) (see also Feichtinger et al. (1994) and the references therein for related models)

Let $t \geq 0$ be an initial time, and $T > t$ a terminal time ($T < +\infty$ here). Moreover let $\gamma(s)$, with $0 \leq t \leq s \leq T$, represent the stock of advertising goodwill of the product to be launched. Then the model for the dynamics is given by the following controlled Delay Differential Equation (DDE) with delay $R > 0$ where z models the intensity of advertising spending:

$$\begin{cases} \dot{\gamma}(s) = a_0\gamma(s) + \int_{-R}^{0} \gamma(s+\xi)da_1(\xi) + b_0 z(s) + \int_{-R}^{0} z(s+\xi)db_1(\xi) \quad s \in [t,T] \\ \gamma(t) = x; \quad \gamma(\xi) = \theta(\xi),\ z(\xi) = \delta(\xi)\ \ \forall \xi \in [t-R,t], \end{cases} \tag{3}$$

with the following assumptions:

- a_0 is a constant factor of image deterioration in absence of advertising, $a_0 \leq 0$;
- $a_1(\cdot)$ is the distribution of the forgetting time, $a_1(\cdot) \in L^2([-R,0];\mathbb{R})$;
- b_0 is a constant advertising effectiveness factor, $b_0 \geq 0$;
- $b_1(\cdot)$ is the density function of the time lag between the advertising expenditure z and the corresponding effect on the goodwill level, $b_1(\cdot) \in L^2([-R,0];\mathbb{R}_+)$;
- x is the level of goodwill at the beginning of the advertising campaign, $x \geq 0$;
- $\theta(\cdot)$ and $\delta(\cdot)$ are respectively the goodwill and the spending rate before the beginning, $\theta(\cdot) \geq 0$, with $\theta(0) = x$, and $\delta(\cdot) \geq 0$.

Finally, we define the objective functional as

$$J(t,x;z(\cdot)) = \varphi_0(\gamma(T)) + \int_t^T h_0(z(s))\,ds, \tag{4}$$

2.1.3 A model for obsolescence and depreciation

Boucekkine et al. (2006) presented an AK model for obsolescence and depreciation that allows to disentangle obsolescence and physical depreciation. The state variable is the production net of the maintenance and repair costs. It satisfies the DDE:

$$y(t) = \int_{t-R}^{t} (\Omega e^{-\delta(t-s)} - \eta) i(s) \mathrm{d}s \tag{5}$$

where Ω, η and δ are real positive constants and $\eta = e^{-\delta T}\Omega$. The control variable is given by the investment $i(s)$ that has to satisfy the constraint $0 \leq i(s) \leq y(s)$. The planner has to maximize the functional

$$\int_0^{+\infty} e^{-\rho s} \frac{(y(s)-i(s))^{1-\sigma}}{1-\sigma} \mathrm{d}s \tag{6}$$

for some positive constant σ and some discount factor ρ.

Remark 2.1. *Boucekkine et al. (1997; 2001) use a numerical method to approach similar problems.*

2.2 Demographic applications

In the sequel we will focus our attentions mainly on the three economic examples we have described but, as seen for example in Boucekkine et al. (2004), the formalism of such models are very similar to the one used in some models that describe demographic evolutions. They consider a demographic models with an explicit age structure. At any time t, denote by $h(v)$ the human capital of the cohort (or generation) born at v, $v \leq t$. $T(t)$ is the time spent at school by all individuals so $t - T(t)$ is the last generation that entered the job market at t. $A(t)$ is the maximal age attainable, so $t - A(t)$ is the last generation still at work so the aggregate stock of human capital available at time t is:

$$H(t) = \int_{t-A(t)}^{t-T(t)} h(v) e^{nv} m(t-v) \mathrm{d}v$$

where: n is the growth rate of population, e^{nv} is size of the cohort born at v, and $m(t-v)$ is the probability for an individual born at v to be still alive at t. In Boucekkine et al. (2002) the authors study a case in which $A(t)$ and $T(t)$ are found to be constant and the model is exactly of the family we are studying.

3 The Problem

3.1 The delay state equation

From now on we consider a fixed delay $R > 0$. With notation similar to that of the book by Bensoussan et al. (1992) and the same used in Fabbri et al. (2006), given $T > t \geq 0$ and $z \in L^2([t-R, T], \mathbb{R})$ for every $s \in [t, T]$ we call $z_s \in L^2([-R, 0]; \mathbb{R})$ the function

$$\begin{cases} z_s : [-R, 0] \to \mathbb{R} \\ z_s(r) \stackrel{def}{=} z(s+r) \end{cases} . \tag{7}$$

Given an admissible control $u(\cdot) \in L^2(t, T)$, we consider the the following delay differential equation:

$$\begin{cases} \dot{y}(s) = N(y_s) + B(u_s) + f(s) \quad \mathit{for}\ s \in [t, T] \\ (y(t), y_t, u_t) = (\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2([-R, 0]; \mathbb{R}) \times L^2([-R, 0]; \mathbb{R}) \end{cases} \tag{8}$$

where y_t and u_t are interpreted by means of the definition above and

$$N, B : C([-R, 0], \mathbb{R}) \to \mathbb{R}, \tag{9}$$

in particular:

Hypothesis 3.1. $N, B\colon C([-R,0],\mathbb{R}) \to \mathbb{R}$ *are continuous linear functionals.*

In the delay setting the initial data are a triple (ϕ^0, ϕ^1, ω) whose first component is the state at the initial time t, the second and third are respectively the history of the state and the history of the control up to time t (more precisely, on the interval $[t-R, t]$). In the following we will consider the case $f \equiv 0$.

Remark 3.2. *The optimal control problem that we need to study (the one to which our delay examples apply) has initial time $t = 0$. Nevertheless the DPA require to embed the problem in a family of problems obtained varying the initial time t (besides the initial state) in the interval $[0, T]$. The viscosity solution of the HJB equation (38) will be defined (see Definition 5.4) on the whole interval $[0, T]$ and will give information on all the problems of the family, in particular on the original one with $t = 0$.*

The equation (8) is a general form that includes our three main examples. Namely:

- In Boucekkine et al. (2005); Fabbri and Gozzi (2006) (see Subsubsection 2.1.1) we have $N = 0$ and $B = \delta_0 - \delta_R$ so the state equation is
$$k(s) = \int_{s-R}^{s} i(r)\mathrm{d}r \tag{10}$$
- In Gozzi et al. (2006); Gozzi and Marinelli (2004) (see Subsubsection 2.1.2) the definitions of N and B are respectively
$$\begin{array}{l} N\colon C([-R,0]) \to \mathbb{R} \\ N\colon \gamma \mapsto a_0\gamma(0) + \int_{-R}^{0} \gamma(r) da_1(r) \end{array} \tag{11}$$
$$\begin{array}{l} B\colon C([-R,0]) \to \mathbb{R} \\ B\colon \gamma \mapsto b_0\gamma(0) + \int_{-R}^{0} \gamma(r) db_1(r) \end{array} \tag{12}$$
- In Boucekkine et al. (2006) (see Subsubsection 2.1.3) $N = 0$ and
$$\begin{array}{l} B\colon C([-R,0]) \to \mathbb{R} \\ B\colon \gamma \mapsto (\Omega - \eta)\gamma(0) - \delta\Omega \int_{-R}^{0} e^{\delta r}\gamma(r)\mathrm{d}r \end{array} \tag{13}$$

Proposition 3.3. *Given an initial condition $(\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2(-R,0) \times L^2(-R,0)$, a control $u \in L^2_{loc}[0,+\infty)$ and a $f \in L^2([0,T]\mathbb{R})$ there exists a unique solution $y(\cdot)$ of (8) in $H^1_{loc}[0,\infty)$. Moreover for all $T > 0$ there exists a constant $c(T)$ depending only on $R, T, \|N\|$ and $\|B\|$ such that*

$$|y|_{H^1(0,T)} \le c(T)\Big(|\phi^0| + |\phi^1|_{L^2(-R,0)} + |\omega|_{L^2(-R,0)} + |u|_{L^2(0,T)} + |f|_{L^2(0,T)}\Big) \tag{14}$$

Proof. See Bensoussan et al. (1992) Theorem 3.3 page 217 for the first part and Theorem 3.3 page 217, Theorem 4.1 page. 222 and page 255 for the second statement. □

3.2 The target functional

We consider a target functional to be maximized of the form

$$\int_t^T L_0(s, y(s), u(s))\mathrm{d}s + h_0(y(T)) \tag{15}$$

where

$$\begin{aligned} &L_0 \colon [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ &h_0 \colon \mathbb{R} \to \mathbb{R} \end{aligned} \tag{16}$$

are continuous functions.

Remark 3.4. *In our main examples the functional are the following*

- *In Boucekkine et al. (2005); Fabbri and Gozzi (2006) (see Subsubsection 2.1.1) the horizon is infinite and the objective functional was CRRA:*

$$\int_0^{+\infty} \frac{(Ak(s) - i(s))^{1-\sigma}}{1-\sigma}\mathrm{d}s \tag{17}$$

- *In Boucekkine et al. (2006) (see Subsubsection 2.1.3) the functional is CRRA:*

$$\int_0^{+\infty} \frac{(y(s) - i(s))^{1-\sigma}}{1-\sigma}\mathrm{d}s. \tag{18}$$

- *In Faggian and Gozzi (2004) the functional is concave and of the form:*

$$\int_t^T l_0(s, c(s)) + n_0(s, y(s))\mathrm{d}s + m_0(y(T)) \tag{19}$$

Remark 3.5. *The generality of the objective functional is one of the improvements due to the viscosity solutions approach, indeed in Fabbri et al. (2006) the authors considered only objective functionals of the form*

$$\int_t^T e^{-\rho s} l_0(c(s))\mathrm{d}s + m_0(y(T)) \tag{20}$$

where l_0 and m_0 are concave functions, and the utility function l_0 depends only on the consumption (that is the control) c.

Remark 3.6. *We consider here finite horizon problem but similar results can be obtained in the infinite horizon case.*

3.3 The constraints

The last thing to choose to define the optimization problem is the set of the admissible trajectories. In our main examples a lower bound on the control variable is assumed. In Boucekkine et al. (2005); Fabbri and Gozzi (2006) (Subsubsection 2.1.1) the constraint $u \geq 0$ is assumed and the same is done in Boucekkine et al. (2006) (Subsubsection 2.1.3). Here we assume a more general constraint:

$$u \geq \Gamma_{-}(y) \tag{21}$$

where $\Gamma_{-} \colon \mathbb{R} \to (-\infty, 0]$ is a continuous function (see Hypothesis 4.3 for other assumptions on Γ_{-}).

Moreover we assume another state-control constraint that is a generalization of the constraints imposed in Boucekkine et al. (2005); Fabbri and Gozzi (2006); Boucekkine et al. (2006): the control cannot be greater than some number depending on the state. For example in Boucekkine et al. (2005); Fabbri and Gozzi (2006) the investment i cannot be greater then the production $ak(t)$, in Boucekkine et al. (2006) we have $i \leq y$. Here we impose

$$u \leq \Gamma_{+}(y) \tag{22}$$

where $\Gamma_{+} \colon \mathbb{R} \to [0, +\infty)$ is a continuous function. (In Boucekkine et al. (2005); Fabbri and Gozzi (2006) (Subsubsection 2.1.1) $\Gamma_{+}(y) = Ay$, in Boucekkine et al. (2006) (see Subsubsection 2.1.3) $\Gamma_{+}(y) = y$)

3.4 The main technical difficulties of the problem

The three main components of an optimal control problem are the state equation, the target functional and the constraints. Here all the components present some non-trivial difficulties with respect to the well established theory:

- The state equation: we consider a general homogeneous linear DDE, in which the derivative of the state y depends both on the history of the state y_s (the notation $_s$ was introduced in (7)) and on the history of the control u_s. The presence of the delay in the control yields a unbounded term. There are similar terms in the papers Cannarsa et al. (1993); Cannarsa and Tessitore (1994; 1996*a*;*b*); Gozzi et al. (2002); Fabbri (2006*b*) that study viscosity solution for HJB equation related to optimal control problems governed by specific PDEs and whose results do not apply to our case. Moreover in our state equation as reformulated in M^2 (see below) a non-analytic semigroup appears. The only work, as far we know, that treats viscosity solution of HJB equation with boundary term and with non-analytic semigroup is Fabbri (2006*b*), but only a very specific transport PDE is treated there.

- The constraints: we consider both state-control constraints (see Hypothesis 4.3 for a precise definition).

- The target functional: we consider a functional of the form

$$\int_t^T L_0(s, y(s), u(s))\mathrm{d}s + h_0(y(T)) \tag{23}$$

  where we assume L_0 and h_0 merely continuous. In Boucekkine et al. (2005); Fabbri and Gozzi (2006); Fabbri (2006*a*) a CRRA utility function is considered and in Fabbri et al. (2006) a concave utility function is used.

4 The problem in Hilbert spaces

In this section we remind how to rewrite the state equations of a control problem subject to a DDE as a control problem subject to an ODE in a suitable Hilbert space. The reader is referred to Fabbri et al. (2006) or to the 4th Chapter of the book Bensoussan et al. (1992) for details.

Notation 4.1. *In the text we will always follow these notations:*

- $y(\cdot)$ *is the solution of the DDE (8),*

- (ϕ^0, ϕ^1, ω) *is the initial datum in the DDE (8)*

- $x(\cdot)$ *is the state in the Hilbert space* $M^2 = \mathbb{R} \times L^2[-R, 0]$ *and solves the differential equation (28). Note that* $x^0(\cdot) = y(\cdot)$

- $\langle a, b\rangle_{\mathbb{R}} = ab$ *is the product in* $\mathbb{R}$ *of two real number* $a, b \in \mathbb{R}$

- $\langle \cdot, \cdot\rangle_{L^2}$ *will indicate the scalar product in* $L^2(-R, 0)$*: if* $\phi^1 \in L^2$ *and* $\psi^1 \in L^2$ *the scalar product is defined as*

$$\left\langle \phi^1, \psi^1 \right\rangle_{L^2} = \int_{-R}^0 \phi^1(r)\psi^1(s)\mathrm{d}s \tag{24}$$

- *The brackets* $\langle \cdot, \cdot\rangle$ *without index will indicate the scalar product in* M^2*: if* $\phi = (\phi^0, \phi^1) \in M^2$ *and* $\psi = (\psi^0, \psi^1) \in M^2$ *the scalar product is defined as*

$$\langle \phi, \psi\rangle = \phi^0\psi^0 + \left\langle \phi^1, \psi^1 \right\rangle_{L^2} \tag{25}$$

- *The brackets* $\langle \cdot, \cdot\rangle_{X\times X'}$ *is the duality pairing between a space* X *and the dual* X'.

- *The symbol* $|y|_X$ *means the norm of the element* y *in the Banach space* X

- *The symbol $\|T\|$ is the operator norm of the operator T.*

- *$C^1([0,T]\times M^2)$ is the set of the functions $\varphi\colon [0,T]\times M^2 \to \mathbb{R}$ that are continuously differentiable.*

- *If $\varphi \in C^1([0,T]\times M^2)$ we call $\partial_t\varphi(t,x)$ the partial derivative along the variable t and $\nabla\varphi(t,x)$ the differential with respect to the state variable $x \in M^2$*

Consider L the linear operator defined in Subsection 8. Thanks to Hypothesis 3.1 we can state that

Proposition 4.2. *The operator A^* defined as:*

$$\begin{cases} D(A^*) = \left\{(\phi^0,\phi^1) \in M^2 \ : \ \phi^1 \in W^{1,2}(-R,0) \text{ and } \phi^0 = \phi^1(0)\right\} \\ A^*(\phi^0,\phi^1) = (L\phi^1, D\phi^1) \end{cases} \tag{26}$$

is the generator of a C_0 semigroup on the Hilbert space $M^2 \stackrel{def}{=} \mathbb{R}\times L^2([-R,0];\mathbb{R})$

Proof. See Bensoussan et al. (1992) Chapter 4. □

In view of the form of $D(A^*)$ the operator B can be seen, abusing somehow of the notation, as the linear continuous functional

$$\begin{cases} B\colon D(A^*) \to \mathbb{R} \\ B\colon (\varphi^0,\varphi^1) \mapsto B(\varphi^1) \end{cases} \tag{27}$$

where $D(A^*)$ is endowed with the graph norm [7]. In the following we will consider B in this second definition. We consider then the adjoints of A^* and B called respectively A and B^*.

The DDE (8) is included, in the sense specified below, into the following ODE in the Hilbert space M^2

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}x(s) = Ax(s) + B^*z(s) \\ x(t) = x. \end{cases} \tag{28}$$

indeed (28) admits a unique solution $x(\cdot)$ over a suitable subset of $C([0,T];M^2)$. Such a solution is a couple $x(s) = (x^0(s), x^1(s)) \in \mathbb{R}\times L^2(-R,0)$[8] where $x^0(s)$ is the unique absolutely continuous solution $y(s)$ of

[7] For $x \in D(A^*)$ the graph norm $|x|_{D(A^*)}$ is defined as

$$|x|_{D(A^*)} = |x|_{M^2} + |A^*x|_{M^2}.$$

[8] We will write

$$x(s)_{u(\cdot),t,x} = (x^0_{u(\cdot),t,x}(s), x^1_{u(\cdot),t,x}(s))$$

when we want to emphasize the dependence on the control and on the initial data.

(8) and x^1 a suitable transformation of the histories of the state y and of the control u. See Fabbri et al. (2006) and Appendix A for a more precise description of such a transformation in the pilot-example and Bensoussan et al. (1992) for a more general situation.

We need now to translate the constraints and the target functional in abstract terms. In the next hypothesis we formalize the state-control constraint described above as $u \in [\Gamma_-(y), \Gamma_+(y)]$:

Hypothesis 4.3. *If we consider a control $u(\cdot)$ and the related state trajectory $x(\cdot) = (x^0(\cdot), x^1(\cdot))$ we impose the state-control constraint*

$$\Gamma_-(x^0(s)) \le u(s) \le \Gamma_+(x^0(s)) \quad \forall s \in [t, T] \tag{29}$$

where Γ_- and Γ_+ are locally Lipschitz continuous functions

$$\begin{aligned} \Gamma_+ &\colon \mathbb{R} \to [0, +\infty) \\ \Gamma_- &\colon \mathbb{R} \to (-\infty, 0] \end{aligned} \tag{30}$$

and such that $|\Gamma_-(t)| \le a + b|t|$ and $|\Gamma_+(t)| \le a + b|t|$ for two positive constant a and b.

The set of admissible controls is

$$\mathcal{U}_{t,x} \stackrel{def}{=} \{u(\cdot) \in L^2(t, T) \ : \ \Gamma_-(x^0_{u(\cdot),t,x}(s)) \le u(s) \le \Gamma_+(x^0_{u(\cdot),t,x}(s))\} \tag{31}$$

The target functional (15) written in the new variables is

$$\int_t^T L_0(s, x^0(s), u(s))\mathrm{d}s + h_0(x^0(T)).$$

So we rewrite it as follows

$$J(t, x, u(\cdot)) = \int_t^T L(s, x(s), u(s))\mathrm{d}s + h(x(T)) \tag{32}$$

where

$$\begin{cases} L\colon [0, T] \times M^2 \times \mathbb{R} \to \mathbb{R} \\ L\colon (s, x, u) \mapsto L_0(s, x^0, u) \end{cases} \tag{33}$$

$$\begin{cases} h\colon M^2 \to \mathbb{R} \\ h\colon x \mapsto h_0(x^0) \end{cases} \tag{34}$$

and so L and h are continuous functions. Moreover we ask that

Hypothesis 4.4. *L and h are uniformly continuous and*

$$|L(s, x, u) - L(s, y, u)| \le \sigma(|x - y|) \quad \textit{for all } (s, u) \in [0, T] \times \mathbb{R} \tag{35}$$

where σ is a modulus of continuity[9].

[9]That is, a continuous positive function such that $\sigma(r) \to 0$ for $r \to 0^+$.

The original optimization problem is equivalent to the optimal control problem in M^2 with state equation (28) and target functional given by (32).

Lemma 4.5. *Assuming Hypothesis 4.3 and given an initial datum* $(\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2(-R, 0) \times L^2(-R, 0)$ *then equation (8) has a unique solution* $y(\cdot)$ *in* $H^1(t, T)$*. It is bounded in the interval* $[t, T]$ *uniformly in the control* $u(\cdot) \in \mathcal{U}_{t,x}$ *and in the initial time* $t \in [0, T)$*. We call* K *a constant such that* $|y(s)| \leq K$ *for any* $t \in [0, T)$*, any control* $u(\cdot) \in \mathcal{U}_{t,x}$ *and any* $s \in [t, T]$*.*

Proof. See Appendix A. □

Remark 4.6. *Using the Hypothesis 4.3 such a result implies* $u(s) \leq a + bK$ *for all the controls in* $\mathcal{U}_{t,x}$*.*

Lemma 4.7. *If Hypothesis 4.3 holds, calling* $x(s)$ *the solution of (28),*

$$|x(s) - x|_{M^2} \xrightarrow{s \to t^+} 0 \tag{36}$$

uniformly in (t, x) *and in the control* $u(\cdot) \in \mathcal{U}_{t,x}$

Proof. See Appendix A. □

The value function of the problem is defined as

$$V(t, x) = \sup_{u(\cdot) \in \mathcal{U}_{t,x}} J(t, x, u(\cdot)) \tag{37}$$

Proposition 4.8. *The value function* $V : [0, T] \times M^2 \to \mathbb{R}$ *is continuous*

Proof. See Appendix A. □

5 Viscosity solutions for HJB equation

The HJB equation of the system is defined as

$$\begin{cases} \partial_t w(t, x) + \langle \nabla w(t, x), Ax \rangle + H(t, x, \nabla w(t, x)) = 0 \\ w(T, x) = h(x) \end{cases} \tag{38}$$

where H is defined as follows

$$\begin{cases} H : [0, T] \times D(A^*) \to \mathbb{R} \\ H(t, x, p) \stackrel{def}{=} \sup_{u \in [\Gamma_-(x^0), \Gamma_+(x^0)]} \{uB(p) + L(t, x, u)\} \end{cases} \tag{39}$$

We refer to H as to the *Hamiltonian* of the system

5.1 Definition and preliminary lemma

Definition 5.1. *We say that a function* $\varphi \in C^1([0,T]\times M^2)$ *is a test function and we will write* $\varphi \in \textsc{Test}$ *if* $\nabla\varphi(s,x) \in D(A^*)$ *for all* $(s,x) \in [0,T] \times M^2$ *and* $A^*\nabla\varphi \colon [0,T]\times M^2 \to \mathbb{R}$ *is continuous. This means that* $\nabla\varphi \in C([0,T]\times M^2; D(A^*))$ *where* $D(A^*)$ *is endowed with the graph norm.*

Definition 5.2. $w \in C([0,T] \times M^2)$ *is a viscosity subsolution of the HJB equation (or simply a "subsolution") if* $w(T,x) \le h(x)$ *for all* $x \in M^2$ *and for every* $\varphi \in \textsc{Test}$ *and every local minimum point* (t,x) *of* $w - \varphi$ *we have*

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + H(t,x,\nabla\varphi(t,x)) \le 0 \tag{40}$$

Definition 5.3. $w \in C([0,T] \times M^2)$ *is a viscosity supersolution of the HJB equation (or simply a "supersolution") if* $w(T,x) \ge h(x)$ *for all* $x \in M^2$ *and for every* $\varphi \in \textsc{Test}$ *and every local maximum point* (t,x) *of* $w - \varphi$ *we have*

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + H(t,x,\nabla\varphi(t,x)) \ge 0 \tag{41}$$

Definition 5.4. $w \in C([0,T] \times M^2)$ *is a viscosity solution of the HJB equation if it is, at the same time, a supersolution and a subsolution.*

Proposition 5.5. *Given* $(t,x) \in [0,T] \times M^2$ *and* $\varphi \in \textsc{Test}$ *there exists a real continuous function* $O(s)$ *such that* $O(s) \xrightarrow{s\to t^+} 0$ *and such that for every admissible control* $u(\cdot) \in \mathcal{U}_{t,x}$ *we have that*

$$\left| \frac{\varphi(s,x(s)) - \varphi(t,x)}{s-t} - \partial_t\varphi(t,x) - \langle A^*\nabla\varphi(t,x), x\rangle - \right. \\ \left. - \frac{\int_t^s \langle B(\nabla\varphi(t,x)), u(r)\rangle_{\mathbb{R}}\, \mathrm{d}r}{s-t} \right| \le O(s) \tag{42}$$

(where we called $x(s)$ *the trajectory that starts at time* t *from* x *and subject to the control* $u(\cdot)$*).*

Moreover if $u(\cdot) \in \mathcal{U}_{t,x}$ *is continuous in* t *we have that*

$$\frac{\varphi(s,x(s)) - \varphi(t,x)}{s-t} \xrightarrow{s\to t^+} \\ \xrightarrow{s\to t^+} \partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + \langle B(\nabla\varphi(t,x)), u(t)\rangle_{\mathbb{R}} \tag{43}$$

Proof. See Appendix A. □

Remark 5.6. *We want to emphasize that* $O(s)$ *is independent of the control and that this fact will be crucial when we prove that the value function is a viscosity supersolution of the HJB equation.*

Corollary 5.7. *Given $(t, x) \in [0, T] \times M^2$ and $\varphi \in \textsc{Test}$ and an admissible control $u(\cdot) \in \mathcal{U}_{t,x}$ we have that*

$$\begin{aligned} &\varphi(s, x(s)) - \varphi(t, x) = \\ &= \int_t^s \partial_t \varphi(r, x(r)) + \langle A^* \nabla \varphi(r, x(r)), x(r) \rangle + \langle B(\nabla \varphi(r, x(r))), u(r) \rangle_{\mathbb{R}} \, \mathrm{d}r \end{aligned} \tag{44}$$

(where we called $x(s)$ the trajectory that starts at time t from x and subject to the control $u(\cdot)$).

5.2 The value function as viscosity solution of HJB equation

Proposition 5.8. (Bellman's optimality principle) *The value function V, defined in (37) satisfies for all $s > t$:*

$$V(t, x) = \sup_{u(\cdot) \in \mathcal{U}_{t,x}} \left(V(s, x(s)) + \int_t^s L(r, x(r), u(r)) \mathrm{d}r \right) \tag{45}$$

where $x(s)$ is the trajectory at time s starting from x subject to control $u(\cdot) \in \mathcal{U}_{t,x}$.

Proof. It can be done using standard arguments. See for example Li and Yong (1995) Chapter 6. □

We can now prove that the value function is a viscosity solution of the HJB equation.

Theorem 5.9. *The value function V is a viscosity solution of the HJB equation.*

Proof. See appendix A. □

Remark 5.10. *We are not able at the moment to give a uniqueness result for the viscosity solution of the HJB equation. It will be an issue for future work.*

6 A verification result

We use the following lemma

Lemma 6.1. *Let $f \in C([0, T])$. Extend f to a g on $(-\infty, +\infty)$ with $g(t) = g(T)$ for $t > T$ and $g(t) = g(0)$ for $t < 0$. Suppose there is a $\rho \in L^1(0, T; \mathbb{R})$ such that*

$$\left| \liminf_{h \to 0} \frac{g(t + h) - g(t)}{h} \right| \le \rho(t) \quad a.e.\ t \in [0, T] \tag{46}$$

Then

$$g(\beta) - g(\alpha) \geq \int_\alpha^\beta \liminf_{h\to 0} \frac{g(t+h) - g(t)}{h} \mathrm{d}t \quad \forall\, 0 \leq \alpha \leq \beta \leq T \tag{47}$$

Proof. The proof can be found in Yong and Zhou (1999) page 270. □

We first introduce a set related with a subset of the subdifferential of a function in $C([0,T] \times M^2)$. Its definition is suggested by the definition of sub/super solution. We define

Definition 6.2. *Given $v \in C([0,T] \times M^2)$ and $(t,x) \in [0,T] \times M^2$ we define $Ev(t,x)$ as*

$$\begin{aligned} Ev(t,x) = \{(q,p) \in \mathbb{R} \times D(A^*) \ : \ & \exists \varphi \in \textsc{Test}, s.t. \\ & v - \varphi \textit{ attains a loc. min. in } (t,x), \\ & \partial_t \varphi(t,x) = q, \ \nabla\varphi(t,x) = p, \\ & \textit{and } v(t,x) = \varphi(t,x)\} \end{aligned} \tag{48}$$

Remark 6.3. *$Ev(t,x)$ is a subset of the subdifferential of v.*

We can now pass to formulating and proving a verification theorem:

Theorem 6.4. *Let $(t,x) \in [0,T] \times M^2$ be an initial datum ($x(t) = x$). Let $u(\cdot) \in \mathcal{U}_{t,x}$ and $x(\cdot)$ be the relate trajectory. Let $q \in L^1(t,T;\mathbb{R})$, $p \in L^1(t,T;D(A^*))$ be such that*

$$(q(s), p(s)) \in EV(t, x_{t,y}(s)) \quad \textit{for almost all } s \in (t,T) \tag{49}$$

Moreover if $u(\cdot)$ satisfies

$$\int_t^T \langle A^* p(s), x(s)\rangle_{M^2} + \langle Bp(s), u(s)\rangle_{\mathbb{R}} + q(s)\, \mathrm{d}s \geq \\ \geq \int_t^T -L(s, x(s), u(s))\, \mathrm{d}s, \tag{50}$$

then $u(\cdot)$ is an optimal control at (t,x).

Proof. See Appendix A. □

A Appendix: Proofs

In this appendix we often refer to the book Bensoussan et al. (1992): for a deeper description of the delay differential equations and their equivalent formulation in an Hilbert space, the reader is referred to the 4th Chapter of such a book.

In this section we will use the following notation (the same of Bensoussan et al. (1992) and also of Fabbri et al. (2006)):

Given N, B two continuous linear functionals

$$N, B\colon C([-R,0]) \to \mathbb{R}$$

with norm respectively $\|N\|$ and $\|B\|$ (as in Hypothesis 3.1), we define $\mathcal{N}$ and $\mathcal{B}$ be the following applications

$$\begin{aligned} \mathcal{N},\mathcal{B}\colon C_c((-R,T);\mathbb{R}) &\to L^2(0,T) \\ \mathcal{N}(\phi)\colon t &\mapsto N(\phi_t) \\ \mathcal{B}(\phi)\colon t &\mapsto B(\phi_t) \end{aligned} \tag{51}$$

where ϕ_t has the meaning of equation (7).

Theorem A.1. *$\mathcal{N},\mathcal{B}\colon C_c((-R,T);\mathbb{R}) \to L^2(0,T)$ have continuous linear extensions $L^2(-R,T) \to L^2(0,T)$ with norm $\leq \|N\|$ and $\leq \|B\|$.*

Proof. See Bensoussan et al. (1992) Theorem 3.3 page. 217. □

Definition A.2. *Let a and b, $a < b$, two real number. Let $\mathcal{F}(a,b)$ be a set of functions from $[a,b]$ to $\mathbb{R}$. For each u in $\mathcal{F}(a,b)$ and all $s \in [a,b]$, define the functions $e^s_- u$ and $e^s_+ u$ as follows*

$$e^s_- u\colon [a,+\infty) \to \mathbb{R}, \quad e^s_- u(t) = \begin{cases} u(t) & t \in [a,s] \\ 0 & t \in (s,+\infty) \end{cases}$$
$$e^s_+ u\colon (-\infty,b) \to \mathbb{R}, \quad e^s_+ u(t) = \begin{cases} 0 & t \in (-\infty,s] \\ u(t) & t \in (s,b] \end{cases}$$

Using the $\mathcal{N}$ and $\mathcal{B}$ notation we can rewrite the (8) as

$$\begin{cases} \dot{y}(t) = \mathcal{N}y + \mathcal{B}u + f \\ (y(0), y_0, u_0) = (\phi^0, \phi^1, \omega) \in \mathbb{R} \times L^2(-R,0) \times L^2(-R,0) \end{cases} \tag{52}$$

Using e^s_- and $e_+ s$ we can decompose $y(\cdot)$ and $u(\cdot)$ as $y = e^0_+ y + e^0_+ \phi^1$ and $u = e^0_+ u + e^0_+ \omega$. So we can separate the solution $y(t)$, $t \geq 0$ and the control $u(t)$, $t \geq 0$ from the initial functions ϕ^1 and ω:

$$\begin{cases} \dot{y}(t) = \mathcal{N}e^0_+ y + \mathcal{B}e^0_+ u + \mathcal{N}e^0_- \phi^1 + \mathcal{B}e^0_- \omega + f \\ y(0) = \phi^0 \in \mathbb{R} \end{cases} \tag{53}$$

Now we are ready to describe the key-step in order to obtain $\mathbb{R} \times L^2(-R,0)$ as state space. The system (53) does not directly use the initial function ϕ^1 and ω but only the sum of their images $\mathcal{N}e^0_- \phi^1 + \mathcal{B}e^0_- \omega$. We need a last step before we can write the delay equation in Hilbert space. We introduce two operators

$$\begin{cases} \overline{N}\colon L^2(-R,0) \to L^2(-R,0) \\ (\overline{N}\phi^1)(\alpha) \overset{def}{=} (\mathcal{N}e^0_- \phi^1)(-\alpha) \qquad \alpha \in (-R,0) \end{cases}$$

and

$$\begin{cases} \overline{B}\colon L^2(-R,0) \to L^2(-R,0) \\ (\overline{B}\omega)(\alpha) \stackrel{def}{=} (\mathcal{B}e^0_-\omega)(-\alpha) \qquad \alpha \in (-R,0) \end{cases}$$

The operators $\overline{N}$ and $\overline{B}$ are continuous (see Bensoussan et al., 1992, page 235). We note that

$$\mathcal{N}e^0_+\phi^1(t) + \mathcal{B}e^0_+\omega(t) = (e^{-R}_+(\overline{N}\phi^1 + \overline{B}\omega))(-t) \quad \textit{for } t \geq 0$$

So, if we call

$$\xi^1 = (\overline{N}\phi^1 + \overline{B}\omega) \tag{54}$$

and $\xi^0 = \phi^0$, we can rewrite the (53) (and then the (8) as

$$\begin{cases} \dot{y}(t) = (\mathcal{N}e^0_+ y)(t) + (\mathcal{B}e^0_+ u)(t) + (e^{-R}_+\xi^1)(-t) + f(t) \\ y(0) = \xi^0 \in \mathbb{R} \end{cases} \tag{55}$$

where $\mathbb{R} \times L^2(-R,0) \ni \xi \stackrel{def}{=} (\xi^0, \xi^1)$. The (55) makes sense for all $\xi \in \mathbb{R} \times L^2(-R,0)$ also when ξ^1 is not of the form (54). So we have embedded the original system (8) in a family of systems of the form (55).

We consider from now on the case $f = 0$.

Using such notations we can also write in a more precise way the relation between the solution of equation (28) and the initial delay differential equation: we call the solution $x(t)$ of (28) *structural state*. The expression of the structural state $x(\cdot)$ at time $t \geq 0$ is

Definition A.3. *The* structural state $x(t)$ *at time* $t \geq 0$ *is defined by*

$$x(t) \stackrel{def}{=} (y(t), \overline{N}(e^0_+ y)_t + \overline{B}(e^0_+ u)_t + \Xi(t)\xi^1) \tag{56}$$

where $\Xi(t)$ *is the right translation operator defined as*

$$(\Xi(t)\xi^1)(r) = (e^{-R}_+\xi^1)(r-t) \qquad r \in [-R,0] \tag{57}$$

Proof of Lemma 4.5:

Proof. The existence of the solution follows from Proposition 3.3. It can be proved (see (55)) that the solution of (8) is also the solution of the equation

$$\begin{cases} \dot{y}(s) = N(e^t_+ y)_s + B(e^t_+ u)_s + (e^{-R}_+\xi^1)(-t) \quad \textit{for } s \geq t \\ y(t) = \phi^0 \in \mathbb{R} \end{cases} \tag{58}$$

where $\xi^1 = (\overline{N}\phi^1 + \overline{B}\omega)$. So, using Hypothesis 4.3 we can state that, for every control $u(\cdot) \in \mathcal{U}_{t,x}$ and related trajectory $y(\cdot)$, the solution y_M of the following ODE satisfies $|y(s)| \leq |y_M(s-t)|$ for all $s \in [t,T]$:

$$\begin{cases} \dot{y}_M(s) = \|N\| y_M(s) + \|B\|(a + b y_M(s)) + (e^{-R}_+\xi^1)(-t) \quad \textit{for } s \geq 0 \\ y_M(0) = |\phi^0| \in \mathbb{R} \end{cases} \tag{59}$$

and y_M is bounded on $[0,T]$ and this complete the proof. □

Proof of Lemma 4.7:

Proof. We have to prove that $|x(s) - x|_{M^2} \xrightarrow{s \to t^+} 0$ uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$, so it is enough to show that $|x^0(s) - x^0|_{\mathbb{R}} \xrightarrow{s \to t^+} 0$ uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$ and that $|x^1(s) - x^1|_{L^2} \xrightarrow{s \to t^+} 0$ uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$. The first fact is a corollary of the proof of Lemma 4.5 (because $|x^0(s)-x^0| \leq y_M(s-t)$ defined in (59), for the second, using the expression (56):

$$\begin{aligned}\left|x^1(s) - x^1\right|_{L^2} &\leq \left|\Xi(s)x^1 - x^1\right|_{L^2} + \left|\overline{N}(e_+^0 y)_s\right|_{L^2} + \left|\overline{B}(e_+^0 u)_s\right|_{L^2} \leq \\ &\leq \left|\Xi(s)x^1 - x^1\right|_{L^2} + \|\overline{N}\|(s-t)^{\frac{1}{2}}K + \|\overline{B}\|(s-t)^{\frac{1}{2}}(a+Kb) \end{aligned} \quad (60)$$

where a e b are the constants of Hypothesis 4.3, K the constant of Lemma 4.5 and Remark 4.6 and $\Xi(t)$ is the right translation operator defined in (57 as

Now we observe $\left|\Xi(s)x^1 - x^1\right|_{L^2} \xrightarrow{s \to 0} 0$ for the continuity of the translation with respect to the L^2 norm and such a limit does not depend on the control, the other two term are given by a constant multiplied by $(s-t)^{1/2}$ and so they go to zero uniformly in the control. ☐

Proof of Proposition 4.8:

Proof. We consider $[0,T] \times M^2 \ni (t_n, x_n) \xrightarrow[\mathbb{R}\times M^2]{n \to \infty} (t,x)$. We have to estimate the terms

$$|V(t,x) - V(t,x_n)| \quad \textit{and} \quad |V(t_n,x_n) - V(t_n,x)| \quad (61)$$

the difficulties are similar, we analyze the term $|V(t,x) - V(t,y)|$, the other can be treated using similar steps. Using arguments similar to the ones of Lemma 4.5[10] we can state that there exists a $M > 0$ such that, for every admissible control,

$$|x_n(s)| \leq M \quad \text{for every } s \in [t_n, T],\ n \in \mathbb{N}$$

in particular $|x_n^0(s)| \leq M$. In view of Hypothesis 4.3 the restrictions of Γ_+ and Γ_- in $[-M, M]$ are Lipschitz continuous for some Lipschitz constant Z. Suppose that $V(t,x) \geq V(t,x_n)$, then we take an ε-optimal control $u^\varepsilon(\cdot)$ for $V(t,x)$. The problem is that $u^\varepsilon(\cdot)$ could not be in the set $\mathcal{U}_{t,x_n}$. So we consider the approximating control given in feedback form:

$$u_n^\varepsilon(s) \stackrel{def}{=} \begin{cases} u^\varepsilon(s) & \text{if } u^\varepsilon(s) \in [\Gamma_-(x_{n\varepsilon}(s)), \Gamma_+(x_{n\varepsilon}(s))] \\ \Gamma_-(x_{n\varepsilon}(s)) & \text{if } u^\varepsilon(s) \in [\Gamma_-(x_n(s)), \Gamma_-(x_{n\varepsilon}(s))] \\ \Gamma_+(x_{n\varepsilon}(s)) & \text{if } u^\varepsilon(s) \in [\Gamma_+(x_{n\varepsilon}(s)), \Gamma_+(x_n(s))] \end{cases} \quad (62)$$

[10]Using that $(e_+^{-R}\overline{N}\phi^1 + \overline{B}\omega)(\cdot)$ is continuous with respect to the initial data.

where $x_{n\varepsilon}(\cdot)$ the solution of

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s} x_{n\varepsilon}(s) = A x_{n\varepsilon}(s) + B^* u_n^\varepsilon(s) \\ x_{n\varepsilon}(t) = x_n. \end{cases} \tag{63}$$

The definition of $u^\varepsilon(s)$ implies that it is bounded, measurable, and then $L^2[0,T]$. We call $x_\varepsilon(\cdot)$ the solution of

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s} x_\varepsilon(s) = A x_\varepsilon(s) + B^* u^\varepsilon(s) \\ x_\varepsilon(t) = x. \end{cases} \tag{64}$$

and we call $y(\cdot) \stackrel{def}{=} x_\varepsilon(\cdot) - x_{n\varepsilon}(\cdot)$, By definition of $u_n^\varepsilon(\cdot)$ we know that

$$|u^\varepsilon(s) - u_n^\varepsilon(s)| \leq Z|y^0(s)| \tag{65}$$

where $y^0(s)$ is the first component of $y(s)$. Moreover $y^0(\cdot)$ solves the following DDE (using the notation of (55):

$$\begin{cases} \dot{y}^0(s) = (\mathcal{N} e_+^0 y^0)(s) + (\mathcal{B} e_+^0 (u^\varepsilon(s) - u_n^\varepsilon))(s) + e_+^{-R}(x^1 - x_n^1)(-s) \\ y^0(t) = x^0 - x_n^0 \end{cases}$$

Arguing as in the proof of Lemma 4.5 and using (65) we can state that $|y^0(s)| \leq y_M(s)|$ where y_M is the solution of the ODE

$$\begin{cases} \dot{y}_M(s) = \|N\| y_M(s) + \|B\| y_M(s) + e_+^{-R}|x^1 - x_n^1|(-s) \\ y_M(t) = |x^0 - x_n^0| \end{cases}.$$

We have

$$y_M(s) = |x^0 - x_n^0| e^{(\|N\|+\|B\|)(s-t)} + \int_s^t e^{(\|N\|+\|B\|)(s-\tau)} e_+^{-R} |x^1 - x_n^1|(-\tau) \mathrm{d}\tau \leq \\ \leq C\|x - x_n\|_{M^2} \tag{66}$$

for all $s \in [t,T]$ so,

$$|x_\varepsilon^0(s) - x_{n\varepsilon}^0(s)| \leq C\|x - x_n\|_{M^2} \quad \text{for all } s \in [t,T]$$

and

$$|u^\varepsilon(s) - u_n^\varepsilon(s)| \leq ZC\|x - x_n\|_{M^2} \quad \text{for all } s \in [t,T]$$

So, by the uniform continuity of the L we can conclude that

$$|L(s, x_\varepsilon^0(s), u^\varepsilon(s)) - L(s, x_{n\varepsilon}^0(s), u_n^\varepsilon(s)) \leq \sigma(\|x - x_n\|_{M^2}) \quad \text{for all } s \in [t,T]$$

So, for the continuity of h we have (using $\sigma(\cdot)$ for a generic modulus),

$$J(t,x,u^\varepsilon(\cdot)) - J(t, x_n, u_n^\varepsilon(\cdot)) \leq \sigma(\|x - x_n\|_{M^2})$$

and then

$$|V(t,x) - V(t,x_n)| = V(t,x) - V(t,x_n) \leq \varepsilon + \sigma(\|x - x_n\|_{M^2})$$

We conclude for the arbitrariness of ε. □

Proof of Proposition 5.5:

Proof. We write

$$\frac{\varphi(s,x(s))-\varphi(t,x)}{s-t} = I_t + I_0 + I_1 \stackrel{def}{=} \partial_t\varphi(\xi^t(s),\xi^x(s)) +$$
$$+\left\langle \nabla\varphi(t,x), \frac{x(s)-x}{s-t}\right\rangle + \left\langle \nabla\varphi(\xi^t(s),\xi^x(s)) - \nabla\varphi(t,x), \frac{x(s)-x}{s-t}\right\rangle \tag{67}$$

where $[t,T]\times M^2 \ni \xi(s) = (\xi^t(s),\xi^x(s))$ is a point of the line segment connecting (t,x) and $(s,x(s))$. In view of Lemma 4.7, $|x(s)-x|_{M^2} \xrightarrow{s\to t^+} 0$ uniformly in $u(\cdot)\in\mathcal{U}_{t,x}$, so $|\xi(s)-(t,x)|_{\mathbb{R}\times M^2} \xrightarrow{s\to t^+} 0$ uniformly in $u(\cdot)\in\mathcal{U}_{t,x}$ and in particular

$$|\xi^x(s)-x|_{M^2} \xrightarrow{s\to t^+} 0 \text{ uniformly in } u(\cdot)\in\mathcal{U}_{t,x} \tag{68}$$

and then

$$|\xi(s)-(t,x)|_{[t,T]\times M^2} \le |s-t| + |\xi^x(s)-x|_{M^2} \xrightarrow{s\to t^+} 0$$
$$\text{uniformly in } u(\cdot)\in\mathcal{U}_{t,x}. \tag{69}$$

By definition of test function we have that

$$\nabla\varphi\colon [0,T]\times M^2 \to D(A^*) \quad \textit{and it is continuous.} \tag{70}$$

Then

$$|\nabla\varphi(\xi^t(s),\xi^x(s)) - \nabla\varphi(t,x)|_{D(A^*)} \xrightarrow{s\to t^+} 0 \tag{71}$$

uniformly in $u(\cdot)\in\mathcal{U}_{t,x}$.

As observed in Faggian (2001/2002) page 59 the state equation (28) may be extended to an equation in $D(A^*)'$ of the form

$$\begin{cases} \dot{x}(s) = A^{(E)}x(s) + B^*u(s) \\ x(t) = x \end{cases} \tag{72}$$

where $A^{(E)}$ is an extension of A and, in view of Lemma 4.5 and Remark 4.6, $|B^*u(s)|_{D(A^*)'} \le |B|_{D(A^*)'}|a+bK|$, where a and b. The solution of (72) in $D(A^*)'$ can be expressed in mild form Pazy (1983) as:

$$x(s) = e^{(s-t)A^{(E)}}x + \int_t^s e^{(s-r)A^{(E)}}B^*u(r)\mathrm{d}r \tag{73}$$

So, since $x\in X\subseteq D(A^{(E)})$ we can choose a constant C that depends on x such that, for all admissible controls and all $s\in[t,T]$,

$$\frac{|x(s)-x|_{D(A^*)'}}{s-t} \le C \tag{74}$$

So by (71) and (74), we can say that $|I_1| \xrightarrow{s\to t^+} 0$ uniform in $u(\cdot) \in \mathcal{U}_{t,x}$. Thanks to the uniformly (in $u(\cdot) \in \mathcal{U}_{t,x}$) convergence $\xi(s) \to (t,x)$ we can also state that $I_t = \partial_t\varphi(\xi^t(s), \xi^x(s)) \xrightarrow{s\to t^+} \partial_t\varphi(t,x)$ uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$. So to prove the thesis it remains to show that

$$\left| \frac{\langle \nabla\varphi(t,x), x(s)-x\rangle}{s-t} - \langle A^*\nabla\varphi(t,x), x\rangle - \right.$$
$$\left. - \frac{\int_t^s \langle B(\nabla\varphi(t,x)), u(r)\rangle_{\mathbb{R}}\, \mathrm{d}r}{s-t} \right| =$$
$$= \left| \left\langle \nabla\varphi(t,x), \left(\frac{x(s)-x}{s-t} - A^{(E)}x - \frac{\int_t^s B^*u(r)\mathrm{d}r}{s-t} \right) \right\rangle_{D(A^*)\times D(A^*)'} \right| \le O(s) \tag{75}$$

uniformly in $u(\cdot) \in \mathcal{U}_{t,x}$.

We can use (73) and write down explicitly the expression $\frac{x(s)-x}{s-t}$ in $D(A^*)'$:

$$\frac{x(s)-x}{s-t} = \frac{(e^{(s-t)A^{(E)}} - \mathbb{1})x}{s-t} + \frac{\int_t^s e^{(s-r)A^{(E)}} B^*u(r)\mathrm{d}r}{s-t} \tag{76}$$

So we need to estimate:

$$\left| \frac{x(s)-x}{s-t} - A^{(E)}(x) - \frac{\int_t^s B^*u(r)\mathrm{d}r}{s-t} \right|_{D(A^*)'} =$$
$$= \left| \frac{(e^{sA^{(E)}} - \mathbb{1})x}{s-t} - A^{(E)}(x) + \frac{\int_t^s \left(e^{(s-r)A^{(E)}} - \mathbb{1}\right) B^*u(r)\mathrm{d}r}{s-t} \right|_{D(A^*)'} \tag{77}$$

where the term $\frac{(e^{sA}-\mathbb{1})x}{s-t} - A^{(E)}(x) \xrightarrow[D(A^*)']{s\to t^+} 0$ because $x \in M^2 \in D(A^{(E)})$ (the convergence is uniform in $u(\cdot) \in \mathcal{U}_{t,x}$ because it does not depend on $u(\cdot)$) and the second term can be estimated, using Lemma 4.5 and Remark 4.6, with

$$\frac{\int_t^s |u(r)| \left| \left(e^{(s-r)A^{(E)}} - \mathbb{1}\right) B\right|_{D(A^*)'} \mathrm{d}r}{s-t} \le (aK+b) \sup_{r\in[t,s]} \left| \left(e^{(s-r)A^{(E)}} - \mathbb{1}\right) B\right|_{D(A^*)'} \tag{78}$$

that goes to zero (the estimate is uniform in the control). Then since $\nabla\varphi(t,x) \in D(A^*)$, the proof is complete.

The (43), with $u(\cdot)$ continuous, is a simple corollary of the proof of the first part. Indeed if $u(\cdot)$ is continuous we have that

$$\frac{\int_t^s \langle B(\nabla\varphi(t,x)), u(r)\rangle_{\mathbb{R}}\, \mathrm{d}r}{s-t} \to \langle B(\nabla\varphi(t,x)), u(t)\rangle_{\mathbb{R}} \tag{79}$$

□

Proof of Theorem 5.9:

Proof. **Subsolution:**
Let (t,x) be a local minimum of $V-\varphi$ for $\varphi \in \textsc{Test}$. We can assume that $(V-\varphi)(t,x)=0$. We choose $u \in [\Gamma_-(x^0), \Gamma_+(x^0)]$. We consider a continuous control $u(\cdot) \in \mathcal{U}_{t,x}$ such that $u(t)=u$[11]. We call $x(s)$ the trajectory starting from (t,x) and subject to $u(\cdot) \in \mathcal{U}_{t,x}$. Then for $s>t$ with $s-t$ small enough we have

$$V(s,x(s)) - \varphi(s,x(s)) \geq V(t,x) - \varphi(t,x) \tag{80}$$

and thanks to the Bellman principle of optimality we know that

$$V(t,x) \geq V(s,x(s)) + \int_t^s L(r,x(r),u(r))\mathrm{d}r. \tag{81}$$

Then

$$\varphi(s,x(s)) - \varphi(t,x) \leq V(s,x(s)) - V(t,x) \leq -\int_t^s L(r,x(r),u(r))\mathrm{d}r, \tag{82}$$

which implies, dividing by $(s-t)$,

$$\frac{\varphi(s,x(s)) - \varphi(t,x)}{s-t} \leq -\frac{\int_t^s L(r,x(r),u(r))\mathrm{d}r}{s-t}. \tag{83}$$

Using Proposition 5.5 we pass to the limit as $s \to t^+$ and obtain

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + \langle B(\nabla\varphi(t,x)), u(t)\rangle_{\mathbb{R}} \leq -L(t,x,u) \tag{84}$$

so

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + (\langle B(\nabla\varphi(t,x)), u\rangle_{\mathbb{R}} + L(t,x,u)) \leq 0 \tag{85}$$

Taking the $\sup_{u\in[\Gamma_-(x^0),\Gamma_+(x^0)]}$ we obtain the subsolution inequality:

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + H(t,x,\nabla\varphi(t,x)) \leq 0 \tag{86}$$

Supersolution:
Let (t,x) be a maximum for $V-\varphi$ and such that $(V-\varphi)(t,x)=0$. For $\varepsilon>0$ we take $u(\cdot) \in \mathcal{U}_{t,x}$ an ε^2-optimal strategy[12]. We call $x(s)$ the trajectory starting from (t,x) and subject to $u(\cdot) \in \mathcal{U}_{t,x}$. Now for $(s-t)$ small enough

$$V(t,x) - V(s,x(s)) \geq \varphi(t,x) - \varphi(s,x(s)) \tag{87}$$

[11] It exists: for example if $u>0$ the control $u(s) = \frac{u}{\Gamma_+(x^0)}\Gamma_+(x^0(s))$ until $\Gamma_+(x^0(s)>0$ and then equal to 0: since Γ_+ is locally Lipschitz and sublinear all works.

[12] ε^2-optimal means that $J(t,x,u(\cdot)) \geq V(t,x) - \varepsilon^2$.

and from ε^2 optimality we know that

$$V(t,x) - V(s,x(s)) \le \varepsilon^2 + \int_t^s L(r,x(r),u(r))\mathrm{d}r \tag{88}$$

so

$$\frac{\varphi(s,x(s)) - \varphi(t,x)}{s-t} \ge \frac{-\varepsilon^2 - \int_t^s L(r,x(r),u(r))\mathrm{d}r}{s-t} \tag{89}$$

We take $(s-t) = \varepsilon$ so that

$$\frac{\varphi(t+\varepsilon, x(t+\varepsilon)) - \varphi(t,x)}{\varepsilon} \ge -\varepsilon - \frac{\int_t^{t+\varepsilon} -L(r,x(r),u(r))\mathrm{d}r}{\varepsilon} \tag{90}$$

and in view of Proposition 5.5 we can choose, independently on the control $u(\cdot) \in \mathcal{U}_{t,x}$, a $O(\varepsilon)$ with $O(\varepsilon) \xrightarrow{\varepsilon \to 0} 0$ such that:

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + \\ + \frac{\int_t^{t+\varepsilon} \langle B(\nabla\varphi(t,x)), u(r)\rangle_{\mathbb{R}} + L(r,x(r),u(r))\mathrm{d}r}{\varepsilon} \ge -\varepsilon + O(\varepsilon). \tag{91}$$

We now take the supremum over u inside the integral and let $\varepsilon \to 0$ and obtain that

$$\partial_t\varphi(t,x) + \langle A^*\nabla\varphi(t,x), x\rangle + H(t,x,\nabla\varphi(t,x)) \ge 0 \tag{92}$$

Then V is a supersolution of the HJB equation. So V is both a viscosity supersolution and a viscosity subsolution of the HJB equation and then, by definition, it is a viscosity solution of the HJB equation. □

<u>Proof of Theorem 6.4:</u>

Proof. The function

$$\begin{cases} \Psi\colon [t,T] \to \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \\ \Psi\colon s \mapsto \left(\langle A^*p(s), x(s)\rangle_{M^2}, \langle Bp(s), u(s)\rangle_{\mathbb{R}}, q(s), L(s,x(s),u(s))\right) \end{cases} \tag{93}$$

in view of Lemma 4.5 is in $L^1(t,T;\mathbb{R}^4)$. So the set of the right-Lebesgue point is of full measure. We choose $\bar{s}$ a point in such a set. We can continue to choose $\bar{s}$ in a full measure set if we assume that (49) is satisfied at $\bar{s}$. We set $\bar{x} := x(\bar{s})$ and we consider a functions $\varphi \equiv \varphi^{\bar{s},\bar{x}} \in \textsc{Test}$ such that $V \ge \varphi$ in a neighborhood of $(\bar{s},\bar{x})$, $V(\bar{s},\bar{x}) - \varphi(\bar{s},\bar{x}) = 0$ and $(\partial_t)(\varphi)(\bar{s},\bar{x})) = q(\bar{s})$, $\nabla\varphi(\bar{s},\bar{x}) = p(\bar{s})$. Then for $\tau \in (\bar{s},T]$ and $(\tau - \bar{s})$ small enough we have

$$\frac{V(\tau,x(\tau)) - V(\bar{s},\bar{x})}{\tau - \bar{s}} \ge \frac{\varphi(\tau,x(\tau)) - \varphi(\bar{s},\bar{x})}{\tau - \bar{s}} \ge \tag{94}$$

for Proposition 5.5

$$\geq \partial_t \varphi(\bar{s}, \bar{x}) + \frac{\int_{\bar{s}}^{\tau} \langle B\nabla\varphi(\bar{s}, \bar{x}), u(r)\rangle_{\mathbb{R}} \, \mathrm{d}r}{\tau - \bar{s}} + \langle A^* \nabla\varphi(\bar{s}, \bar{x}), x\rangle + O(\tau - \bar{s}) \tag{95}$$

In view of the choice of $\bar{s}$ we know that

$$\frac{\int_{\bar{s}}^{\tau} \langle B\nabla\varphi(\bar{s}, \bar{x}), u(r)\rangle_{\mathbb{R}} \, \mathrm{d}r}{\tau - \bar{s}} \xrightarrow{\tau \to \bar{s}^+} \langle B\nabla\varphi(\bar{s}, \bar{x}), u(\bar{s})\rangle_{\mathbb{R}} \tag{96}$$

So that for almost every $\bar{s}$ in $[t, T]$ we have

$$\begin{aligned} \liminf_{\tau \downarrow \bar{s}} \frac{V(\tau, x(\tau)) - V(\bar{s}, x(\bar{s}))}{\tau - \bar{s}} \geq \\ \geq \langle B\nabla\varphi(\bar{s}, x(\bar{s})), u(\bar{s})\rangle_{\mathbb{R}} + \\ + \partial_t\varphi(\bar{s}, x(\bar{s})) + \langle A^*\nabla\varphi(\bar{s}, x(\bar{s})), x(\bar{s})\rangle = \\ = \langle Bp(\bar{s}), u(\bar{s})\rangle_{\mathbb{R}} + q(\bar{s}) + \langle A^*\nabla p(\bar{s}), x(\bar{s})\rangle \end{aligned} \tag{97}$$

then we can use Lemma 6.1 and find that

$$\begin{aligned} V(T, x(T)) - V(t, x) \geq \\ \geq \int_t^T \langle Bp(\bar{s}), u(\bar{s}))\rangle_{\mathbb{R}} + q(\bar{s}) + \langle A^*\nabla p(\bar{s}), x(\bar{s})\rangle \, \mathrm{d}\bar{s} \geq \end{aligned} \tag{98}$$

using (50)

$$\geq \int_t^T -L(r, x(r), u(r))\mathrm{d}r \tag{99}$$

Hence

$$\begin{aligned} V(t, x) \leq V(T, x(T)) + \int_t^T L(r, x(r), u(r))\mathrm{d}r = \\ = h(x(T)) + \int_t^T L(r, x(r), u(r))\mathrm{d}r \end{aligned} \tag{100}$$

and then $(x(\cdot), u(\cdot))$ is an optimal pair. □

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