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1 Introduction

The asymptotic behavior of expressions of the form $\sum_{t=1}^{n} f(r_n x_t)$ where x_t is an integrated process, r_n is a sequence of norming constants, and f is a measurable function has been the subject of a number of articles in recent years. We mention Borodin and Ibragimov (1995), Park and Phillips (1999), de Jong (2003), Jeganathan (2004), Pötscher (2004), de Jong and Whang (2005), Berkes and Horvath (2006), and Christopeit (2009) which study weak convergence results for such expressions under various conditions on x_t and the function f. Of course, these results also provide information on the order of magnitude of $\sum_{t=1}^{n} f(r_n x_t)$. However, to the best of our knowledge no result is available for the case where f is non-integrable with respect to Lebesgue-measure in a neighborhood of a given point, say $x = 0$. In this paper we are interested in bounds on the order of magnitude of $\sum_{t=1}^{n} |x_t|^{-\alpha}$ when $\alpha \geq 1$, a case where the implied function f is not integrable in any neighborhood of zero. As a by-product, we shall also obtain bounds on the order of magnitude for $\sum_{t=1}^{n} w_{t+1}^{k} |x_t|^{-\alpha}$ where w_t denotes the increment of x_t and $k = 1$ or 2. While the emphasis in this paper is on negative powers that are non-integrable in any neighborhood of zero (i.e., $\alpha \geq 1$), we present the results for general $\alpha \in \mathbb{R}$. We do not care to improve the results in case $\alpha < 1$, but we shall occasionally mention better results available in this case (or in subcases thereof) without attempting to be complete in the coverage of such (better) results specific to the case $\alpha < 1$.

*I would like to thank Kalidas Jana for inquiring about the order of magnitude of some of the quantities now treated in the paper. I am indebted to Robert de Jong for comments on an earlier version that have led to an improvement in Theorem 1.

2 Results

Consider an integrated process

$$x_t = x_{t-1} + w_t$$

for integer $t \geq 1$, with the initial real-valued random variable x_0 being independent of the process $(w_t)_{t\geq 1}$ which is assumed to be given by

$$w_t = \sum_{j=0}^{\infty} \phi_j \varepsilon_{t-j}.$$

Here $(\varepsilon_i)_{i\in\mathbb{Z}}$ are independent and identically distributed real-valued random variables that have mean 0 and a finite variance, which – without loss of generality – is set equal to 1. The coefficients ϕ_j are assumed to satisfy $\sum_{j=0}^{\infty} \left|\phi_j\right| < \infty$ and $\sum_{j=0}^{\infty} \phi_j \neq 0$. Furthermore, ε_i is supposed to have a density q with respect to (w.r.t.) Lebesgue-measure. We note that under these assumptions x_t possesses a density w.r.t. Lebesgue-measure for every $t \geq 1$, and the same is true for w_t; cf. Section 3.1 in Pötscher (2004). Furthermore, the characteristic function ψ of ε_i is assumed to satisfy

$$\int_{-\infty}^{\infty} |\psi(s)|^{\nu}\, ds < \infty$$

for some $1 \leq \nu < \infty$. *These assumptions will be maintained throughout the paper.* They have been used in various forms, e.g., in Park and Phillips (1999), de Jong (2004), de Jong and Whang (2005), and Pötscher (2004). We recall from Lemma 3.1 in Pötscher (2004) that under these conditions densities h_t of $t^{-1/2}x_t$ exist such that for a suitable integer $t_* \geq 1$

$$\sup_{t\geq t_*} \|h_t\|_{\infty} < \infty \tag{1}$$

is satisfied. In the following we set $\kappa = \sup_{t\geq t_*} \|h_t\|_{\infty}$.

2.1 Bounds on the Order of Magnitude of $\sum_{t=1}^{n} |x_t|^{-\alpha}$

We first consider the behavior of $\sum_{t=1}^{n} |x_t|^{-\alpha}$. Note that under our assumptions this quantity is almost surely well-defined and finite for every $\alpha \in \mathbb{R}$.[1] Recall that we are mainly interested in the case $\alpha \geq 1$.

Theorem 1

$$\sum_{t=1}^{n} |x_t|^{-\alpha} = \begin{cases} O_p(n^{\alpha/2}) & \textit{if} \quad \alpha > 1 \\ O_p(n^{1/2}\log n) & \textit{if} \quad \alpha = 1 \\ O_p(n^{1-\alpha/2}) & \textit{if} \quad \alpha < 1 \end{cases}$$

[1] In particular, how, and if, we assign a value in the extended real line to $|x_t|^{-\alpha}$ on the event $\{x_t = 0\}$ has no consequence for the results.

Proof. Since $\sum_{t=1}^{t_*-1} |x_t|^{-\alpha}$ is almost surely real-valued it suffices to prove the result for $\sum_{t=t_*}^{n} |x_t|^{-\alpha}$. We first consider the case $\alpha \geq 1$: For $0 < \delta < 1$ we have almost surely

$$\begin{aligned}\sum_{t=t_*}^{n} |x_t|^{-\alpha} &= \sum_{t=t_*}^{n} |x_t|^{-\alpha} \mathbf{1}\left(\left|t^{-1/2}x_t\right| > \delta/n\right) + \sum_{t=t_*}^{n} |x_t|^{-\alpha} \mathbf{1}\left(\left|t^{-1/2}x_t\right| \leq \delta/n\right) \\ &= A_n(\delta) + B_n(\delta)\end{aligned}$$

where t_* is as in (1) and $n \geq t_*$. First consider $B_n(\delta)$: Set

$$D_n(\delta) = \bigcup_{t=t_*}^{n} \left\{\left|t^{-1/2}x_t\right| \leq \delta/n\right\}.$$

Observe that $\{B_n(\delta) > 0\} = D_n(\delta)$ up to null-sets and

$$\begin{aligned}\Pr\left(B_n(\delta) > 0\right) &= \Pr\left(D_n(\delta)\right) \leq \sum_{t=t_*}^{n} \Pr\left(\left|t^{-1/2}x_t\right| \leq \delta/n\right) \\ &= \sum_{t=t_*}^{n} \int_{-\delta/n}^{\delta/n} h_t(z)dz \leq 2\kappa\delta\end{aligned}$$

holds for all $n \geq t_*$ in view of (1). Next we bound $A_n(\delta)$: Observe that for $t \geq t_*$

$$\begin{aligned}& E\left(\left|t^{-1/2}x_t\right|^{-\alpha} \mathbf{1}(\left|t^{-1/2}x_t\right| > \delta/n)\right) \\ = \; & E\left(\left|t^{-1/2}x_t\right|^{-\alpha} \mathbf{1}(1 > \left|t^{-1/2}x_t\right| > \delta/n)\right) + E\left(\left|t^{-1/2}x_t\right|^{-\alpha} \mathbf{1}(\left|t^{-1/2}x_t\right| \geq 1)\right) \\ \leq \; & \int_{\delta/n<|z|<1} |z|^{-\alpha} h_t(z)dz + 1 \leq 2\kappa \int_{\delta/n}^{1} z^{-\alpha}dz + 1 \\ \leq \; & \begin{cases} 1 + 2\kappa(\alpha-1)^{-1}\delta^{1-\alpha}n^{\alpha-1} & \text{if } \alpha > 1 \\ 1 + 2\kappa\log\left(\delta^{-1}\right) + 2\kappa\log n & \text{if } \alpha = 1. \end{cases}\end{aligned}$$

Consequently, for $n \geq \max(t_*, 3)$ we have

$$\begin{aligned}E(A_n(\delta)) &= \sum_{t=t_*}^{n} t^{-\alpha/2} E\left(\left|t^{-1/2}x_t\right|^{-\alpha} \mathbf{1}(\left|t^{-1/2}x_t\right| > \delta/n)\right) \\ &\leq \begin{cases} \left(1 + 2\kappa(\alpha-1)^{-1}\delta^{1-\alpha}\right) n^{\alpha/2} & \text{if } \alpha > 1 \\ \left(1 + 2\kappa + 2\kappa\log\left(\delta^{-1}\right)\right) n^{1/2}\log n & \text{if } \alpha = 1. \end{cases}\end{aligned}$$

Now, for arbitrary $\varepsilon > 0$ define $\delta(\varepsilon) = \varepsilon/(4\kappa)$ and choose $M(\varepsilon) > 0$ large enough to satisfy

$$M(\varepsilon) > \begin{cases} 4\varepsilon^{-1}\left(1 + 2\kappa(\alpha-1)^{-1}\delta(\varepsilon)^{1-\alpha}\right) & \text{if } \alpha > 1 \\ 4\varepsilon^{-1}\left(1 + 2\kappa + 2\kappa\log\left(\delta(\varepsilon)^{-1}\right)\right) & \text{if } \alpha = 1. \end{cases}$$

Then, with $d_n = n^{\alpha/2}$ in case $\alpha > 1$ and $d_n = n^{1/2}\log n$ in case $\alpha = 1$, we obtain using Markov's inequality

$$\begin{aligned}
&\Pr\left(d_n^{-1}\sum_{t=t_*}^{n}|x_t|^{-\alpha} > M(\varepsilon)\right)\\
\leq\;& \Pr\left(d_n^{-1}A_n(\delta(\varepsilon)) > M(\varepsilon)/2\right) \;+\; \Pr\left(d_n^{-1}B_n(\delta(\varepsilon)) > M(\varepsilon)/2\right)\\
\leq\;& 2d_n^{-1}E(A_n(\delta(\varepsilon)))/M(\varepsilon) \;+\; \Pr\left(B_n(\delta(\varepsilon)) > 0\right) \;<\; \varepsilon
\end{aligned}$$

for all $n \geq \max(t_*, 3)$. Since $\sum_{t=t_*}^{n}|x_t|^{-\alpha}$ is almost surely real-valued for all $n \geq t_*$, this completes the proof in case $\alpha \geq 1$. In case $\alpha < 1$ observe that for $n \geq t_*$

$$\begin{aligned}
E\left(\sum_{t=t_*}^{n}|x_t|^{-\alpha}\right) &= \sum_{t=t_*}^{n} t^{-\alpha/2}E\left(\left|t^{-1/2}x_t\right|^{-\alpha}\mathbf{1}\left(\left|t^{-1/2}x_t\right| < 1\right)\right)\\
&\quad + \sum_{t=t_*}^{n} E\left(|x_t|^{-\alpha}\mathbf{1}\left(\left|t^{-1/2}x_t\right| \geq 1\right)\right)\\
&\leq \sum_{t=t_*}^{n} t^{-\alpha/2}\int_{|z|<1}|z|^{-\alpha}h_t(z)dz \;+\; \sum_{t=t_*}^{n} t^{-\alpha/2}\\
&\leq \left(2(1-\alpha)^{-1}\kappa + 1\right)\sum_{t=1}^{n} t^{-\alpha/2} = O(n^{1-\alpha/2}).
\end{aligned}$$

Hence, by Markov's inequality $\sum_{t=t_*}^{n}|x_{t-1}|^{-\alpha} = O_p(n^{1-\alpha/2})$. ■

Remark 2 (i) For values of α such that $x^{-\alpha}$ is well-defined for every x except possibly $x = 0$, the quantity $\sum_{t=1}^{n} x_t^{-\alpha}$ is almost surely well-defined and real-valued. By the triangle inequality, Theorem 1 applies also to $\sum_{t=1}^{n} x_t^{-\alpha}$.

(ii) Not surprisingly, the expectation of $\sum_{t=1}^{n}|x_t|^{-\alpha}$ will typically be infinite in the case $\alpha \geq 1$ (e.g., if the density of x_t is bounded from below in a neighborhood of zero as is the case if x_t is Gaussian). The expectation can, however, also be infinite in other case (e.g., if $\alpha < -2$ and moments of x_t of order $-\alpha$ do not exist).

Remark 3 Suppose the stronger summability condition $\sum_{j=0}^{\infty} j^{1/2}\left|\phi_j\right| < \infty$ is satisfied. Under this additional assumption, more is known in case $\alpha < 1$ than just the upper bound on the order of magnitude of $\sum_{t=1}^{n}|x_t|^{-\alpha}$ given by Theorem 1: If $\alpha < 1$ then

$$n^{\alpha/2-1}\sum_{t=1}^{n}|x_t|^{-\alpha} \xrightarrow{d} |\sigma|^{-\alpha}\int_0^1 |W(s)|^{-\alpha}\,ds \qquad (2)$$

for $n \to \infty$, with the limiting variable being positive with probability one; as a consequence, $n^{1-\alpha/2}$ is the exact order of magnitude in probability of $\sum_{t=1}^{n}|x_t|^{-\alpha}$. Here W is standard Brownian motion and $\sigma = \sum_{j=0}^{\infty}\phi_j$, which is non-zero by assumption. Relation (2) follows from the first claim in Corollary 3.3 in Pötscher (2004), applied to the function T given by $T(x) = |x|^{-\alpha}$ for $x \neq 0$ and $T(0) = 0$, and from the observation that $n^{\alpha/2-1} \sum_{t=1}^{b} |x_t|^{-\alpha} \to 0$ as $n \to \infty$ for every fixed integer b. Note that T is locally integrable since $\alpha < 1$ and that T satisfies $T(\lambda x) = |\lambda|^{-\alpha} T(x)$ for all $x \in \mathbb{R}$ and all $\lambda \neq 0$. Also note that the integral in (2) is almost surely well-defined and finite (independently of how one interprets $|W(s)|^{-\alpha}$ for $W(s) = 0$ in case $\alpha > 0$), cf. (2.4) and Remark 2.1 in Pötscher (2004). [In the case $\alpha \leq 0$, it is well-known that (2) holds even under much weaker conditions than used here, cf. Lemma A.1 in Pötscher (2004). Since the emphasis in this paper is on positive α, we make no attempt to spell out these sharper and well-known results for $\alpha \leq 0$.]

Remark 4 Suppose $\sum_{j=0}^{\infty} j^{1/2} \left|\phi_j\right| < \infty$ is satisfied. In case $\alpha \geq 1$ a crude lower bound for the order of magnitude in probability of $\sum_{t=1}^{n} |x_t|^{-\alpha}$ is given by $n^{1-\alpha/2}$, in the sense that

$$\lim_{n\to\infty} \Pr\left(n^{\alpha/2-1} \sum_{t=1}^{n} |x_t|^{-\alpha} > M\right) = 1$$

holds for every M, i.e., $n^{\alpha/2-1} \sum_{t=1}^{n} |x_t|^{-\alpha} \to \infty$ in probability. This can be seen as follows: Let $T_k(x) = \min(k, |x|^{-\alpha})$ for $k \in \mathbb{N}$. Then we have almost surely

$$n^{\alpha/2-1} \sum_{t=1}^{n} |x_t|^{-\alpha} = n^{-1} \sum_{t=1}^{n} \left|n^{-1/2} x_t\right|^{-\alpha} \geq n^{-1} \sum_{t=1}^{n} T_k(n^{-1/2} x_t)$$

for every $k \in \mathbb{N}$. Furthermore, $n^{-1} \sum_{t=1}^{n} T_k(n^{-1/2} x_t)$ converges in distribution to $\int_0^1 T_k(\sigma W(s)) ds$ by Corollary 3.4 in Pötscher (2004). Now, by Corollary 7.4 in Chung and Williams (1990) and the monotone convergence theorem we have almost surely

$$\int_0^1 T_k(\sigma W(s)) ds = \int_{-\infty}^{\infty} T_k(\sigma x) L(1,x) dx \to |\sigma|^{-\alpha} \int_{-\infty}^{\infty} |x|^{-\alpha} L(1,x) dx = \infty$$

for $k \to \infty$. The last equality in the above display follows since $L(1,0) > 0$ almost surely and $L(1,x)$ having almost surely continuous sample path together imply that there exists a neighborhood U of zero (that may depend on the realization of $L(1,\cdot)$) such that $\inf_{x\in U} L(1,x) > 0$ holds almost surely. In case $\alpha = 1$, inspection of this lower bound and the upper bound given by Theorem 1 now shows that these bounds agree up to a logarithmic term and in this sense are close to being sharp (under the stricter summability condition on ϕ_j imposed here). For $\alpha > 1$, however, there is a substantial gap between the lower and upper bound. [The method leading to the lower bound seems to be too crude to provide a tight bound. We also do not know if the upper bound is tight.]

Remark 5 (i) All results above for $\sum_{t=1}^{n}|x_t|^{-\alpha}$ apply analogously to sums of the form $\sum_{t=a}^{n}|x_t|^{-\alpha}$ for any integer $a>1$. [This follows, except for Remark 4, since $\sum_{t=1}^{a-1}|x_t|^{-\alpha}$ is almost surely finite; for Remark 4 note that the lower bound $n^{-1}\sum_{t=a}^{n}T(n^{-1/2}x_t)$ differs from $n^{-1}\sum_{t=1}^{n}T(n^{-1/2}x_t)$ only by a term that is $o_p(1)$ since T is bounded.]

(ii) For $\alpha\leq 0$ all results above for $\sum_{t=1}^{n}|x_t|^{-\alpha}$ carry over to $\sum_{t=0}^{n}|x_t|^{-\alpha}$. For $\alpha>0$ this is again so, provided the distribution of x_0 does not assign positive mass to the point 0; otherwise, $\sum_{t=0}^{n}|x_t|^{-\alpha}$ is undefined on the event where $x_0=0$; if one chooses to define $|x_0|^{-\alpha}=\infty$ on this event, then the above results clearly do not apply (except for the lower bound given in Remark 4 which holds then a fortiori).

Remark 6 Suppose that the assumptions of Theorem 5 in de Jong and Whang (2005) are satisfied. Note that these assumptions imply the maintained assumptions of the present paper. Let $\alpha>1$ be an integer. For every $\beta>0$ we clearly have that almost surely

$$\begin{aligned}
& n^{-(\beta(\alpha-1)+1+\alpha/2)}\sum_{t=1}^{n}|x_t|^{-\alpha} \\
\geq\;& n^{-\beta(\alpha-1)}n^{-1}|\sigma|^{\alpha}\sum_{t=1}^{n}\left|\sigma n^{-1/2}x_t\right|^{-\alpha}\mathbf{1}\left(\left|\sigma n^{-1/2}x_t\right|>n^{-\beta}\right)
\end{aligned}$$

where σ is as above. The statement in Theorem 5 of de Jong and Whang (2005) implies that there is a $\beta>0$, where β depends on the distribution of ε_t, such that the right-hand side in the above display converges in distribution to a non-degenerate random variable. But this contradicts the upper bound we have obtained in Theorem 1, casting doubt on Theorem 5 (and the closely related Theorem 4) in de Jong and Whang (2005).

2.2 Bounds on the Order of Magnitude of $\sum_{t=1}^{n}w_{t+1}^{k}|x_t|^{-\alpha}$

We next illustrate how the above results can be used to derive upper bounds on the order of magnitude of $\sum_{t=1}^{n}w_{t+1}^{k}|x_t|^{-\alpha}$. Note that this expression is almost surely well-defined and finite for every $\alpha\in\mathbb{R}$ and $k\in\mathbb{N}$.[2] Applying the Cauchy-Schwarz inequality gives almost surely

$$\left|\sum_{t=1}^{n}w_{t+1}^{k}|x_t|^{-\alpha}\right|\leq\left(\sum_{t=1}^{n}w_{t+1}^{2k}\right)^{1/2}\left(\sum_{t=1}^{n}|x_t|^{-2\alpha}\right)^{1/2}.$$

[2] In particular, how, and if, we assign a value in the extended real line to $w_{t+1}^{k}|x_t|^{-\alpha}$ on the event $\{x_t=0\}$ has no consequence for the results.

Hence, if $Ew_t^{2k} < \infty$ holds, we obtain from the ergodic theorem (applied to w_{t+1}^{2k}) and Theorem 1

$$\sum_{t=1}^{n} w_{t+1}^{k} |x_t|^{-\alpha} = \begin{cases} O_p(n^{(\alpha+1)/2}) & \text{if} \quad \alpha > 1/2 \\ O_p(n^{3/4} (\log n)^{1/2}) & \text{if} \quad \alpha = 1/2 \\ O_p(n^{1-\alpha/2}) & \text{if} \quad \alpha < 1/2 \end{cases} . \tag{3}$$

Obviously, the same bound holds more generally if w_{t+1}^k is replaced by $f(w_{t+1})$ with $Ef^2(w_{t+1}) < \infty$ (or even by some arbitrary process v_t, as long as it is defined on the same probability space and v_t^2 satisfies a law of large numbers). Variations of this bound can obviously be obtained by using Hölder's inequality.

Remark 7 (i) In the trivial case $\alpha = 0$ we can appeal to the law of large numbers for w_{t+1}^k (or to the central limit theorem in case $k = 1$) and directly obtain the bound $O_p(n)$ (or $O_p(n^{1/2})$).

(ii) For $\alpha \leq -2$ distributional convergence of $n^{(\alpha-1)/2} \sum_{t=1}^{n} w_{t+1} |x_t|^{-\alpha}$ can be obtained from Theorem 3.1 in Ibragimov and Phillips (2008). This theorem makes assumptions on the process x_t that are stronger in some dimensions (e.g., higher moment assumptions) but are weaker in other respects (e.g., no assumption about existence of a density). Not surprisingly, for this range of values of α, the resulting bound on the order of magnitude of $\sum_{t=1}^{n} w_{t+1} |x_t|^{-\alpha}$ is better than the simple-minded bound (3). However, for $\alpha > -2$ (which includes the case of negative powers of interest here) the results in Ibragimov and Phillips (2008) do not apply.

In the important special case where $w_t = \varepsilon_t$ (i.e., $\phi_j = 0$ for all $j > 0$) and $k = 1$ or $= 2$, bounds better than (3) can be obtained by observing that the sequence $\sum_{t=1}^{n} w_{t+1} |x_t|^{-\alpha}$ is then a martingale transform and by combining Theorem 1 with results in Lai and Wei (1982). [Note that $\sum_{t=1}^{n} w_{t+1} |x_t|^{-\alpha}$ will typically not be a martingale as the first moment will in general not exist, cf. Remark 2(ii); hence, martingale central limit theorems are not applicable.]

Proposition 8 *Suppose that in addition to the maintained assumptions also $\phi_j = 0$ for all $j > 0$ holds. Then*

$$\sum_{t=1}^{n} w_{t+1} |x_t|^{-\alpha} = \begin{cases} o_p(n^{\alpha/2} (\log n)^{1/2+\tau}) & \text{if} \quad \alpha > 1/2 \\ o_p(n^{1/4} (\log n)^{1+\tau}) & \text{if} \quad \alpha = 1/2 \\ o_p(n^{(1-\alpha)/2} (\log n)^{1/2+\tau}) & \text{if} \quad \alpha < 1/2 \end{cases}$$

and

$$\sum_{t=1}^{n} w_{t+1}^2 |x_t|^{-\alpha} = \begin{cases} o_p\left(n^{\alpha/2+\tau}\right) & \text{if} \quad \alpha \geq 1 \\ o_p\left(n^{1-\alpha/2+\tau}\right) & \text{if} \quad \alpha < 1 \end{cases}$$

hold for every $\tau > 0$.

Proof. In case $\alpha = 0$ the result is trivially true (with more precise information on the order of magnitude following immediately from the central limit theorem and the law of large numbers, respectively). Hence assume $\alpha \neq 0$. Since

$\sum_{s=1}^{t} w_s$ is a (nondegenerate) recurrent random walk under the assumptions of the proposition that is not of the lattice-type (as it has uncountably many possible values in the sense of Chung (2001, Section 8.3) by Lebesgue's differentiation theorem), it visits every interval infinitely often almost surely. From independence of x_0 and $(w_s)_{s\geq 1}$ we may conclude that almost surely $|x_t|$ falls into the interval $(1/2, 3/2)$ infinitely often. This shows that the sum $\sum_{t=1}^{n} |x_t|^{-\alpha}$ diverges almost surely for every value $\alpha \neq 0$. Now apply Lemma 2(iii) in Lai and Wei (1982) to conclude that

$$\sum_{t=1}^{n} w_{t+1} |x_t|^{-\alpha} = o\left(\left(\sum_{t=1}^{n} |x_t|^{-2\alpha}\right)^{1/2} \left(\log \sum_{t=1}^{n} |x_t|^{-2\alpha}\right)^{1/2+\theta}\right) \quad a.s.$$

and

$$\sum_{t=1}^{n} w_{t+1}^2 |x_t|^{-\alpha} = o\left(\left(\sum_{t=1}^{n} |x_t|^{-\alpha}\right)^{1+\theta}\right) \quad a.s.$$

for every $\theta > 0$. Apply Theorem 1 (applied to 2α and α, respectively) to complete the proof. ■

Remark 9 *(i) If a moment of w_t higher than the second moment exists, applying Corollary 2 in Lai and Wei (1982) yields the slightly better bound*

$$\sum_{t=1}^{n} w_{t+1} |x_t|^{-\alpha} = \begin{cases} O_p(n^{\alpha/2} (\log n)^{1/2}) & \text{if } \alpha > 1/2 \\ O_p(n^{1/4} (\log n)) & \text{if } \alpha = 1/2 \\ O_p(n^{(1-\alpha)/2} (\log n)^{1/2}) & \text{if } \alpha < 1/2 \end{cases} .$$

(ii) Under the assumptions of Proposition 8 the same bounds can be obtained for $\sum_{t=1}^{n} v_{t+1} |x_t|^{-\alpha}$ and $\sum_{t=1}^{n} v_{t+1}^2 |x_t|^{-\alpha}$, respectively, where (v_t) is an arbitrary martingale difference sequence w.r.t a filtration $\mathcal{F}_t$ (defined on the same probability space as x_t) satisfying the assumptions of Lemma 2(iii) (Corollary 2) in Lai and Wei (1982), provided x_t is measurable w.r.t. the σ-field $\mathcal{F}_t$.

References

Berkes, I. & L. Horvath (2006): "Convergence of Integral Functionals of Stochastic Processes. Econometric Theory 22, 304–322.

Borodin, A.N. & I.A. Ibragimov (1995): "Limit Theorems for Functionals of Random Walks". Proceedings of the Steklov Institute of Mathematics 195(2).

Christopeit, N. (2009): "Weak Convergence of Nonlinear Transformations of Integrated Processes: The Multivariate Case. Econometric Theory 25, 1180–1207.

Chung, K.L. (2001): A Course in Probability Theory. 3rd ed. Academic Press.

Chung, K.L. & R.J. Williams (1990): Introduction to Stochastic Integration. 2nd ed. Birkhäuser.

de Jong, R.M. (2004): "Addendum to 'Asymptotics for Nonlinear Transformations of Integrated Time Series'", Econometric Theory 20, 627-635.

de Jong, R.M. & C. Wang (2005): "Further Results on the Asymptotics for Nonlinear Transformations of Integrated Time Series", Econometric Theory 21, 413-430.

Jeganathan, P. (2004): "Convergence of Functionals of Sums of R.V.s to Local Times of Fractional Stable Motions." Annals of Probability 32, 1771–1795.

Lai, T.L. & C.Z. Wei (1982): "Least Squares Estimates in Stochastic Regression Models With Applications to Identification and Control of Dynamic Systems", Annals of Statistics 10, 154-166.

Ibragimov, R. & P.C.B. Phillips (2008): "Regression Asymptotics Using Martingale Convergence Methods", Econometric Theory 24, 888-947.

Park, J.Y. & P.C.B. Phillips (1999): "Asymptotics for Nonlinear Transformations of Integrated time Series", Econometric Theory 15, 269-298.

Pötscher, B.M. (2004): "Nonlinear Functionals and Convergence to Brownian Motion: Beyond the Continuous Mapping Theorem", Econometric Theory 20, 1-22.