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Abstract

As it is known, there is no rule satisfying *Additivity* in the complete domain of bankruptcy problems. This paper proposes a notion of *partial Additivity* in this context, to be called μ*-additivity*. We find that μ*-Additivity*, together with two quite compelling axioms, *Anonymity* and *Continuity*, identify the *Minimal Overlap* rule, introduced by O'Neill (1982)

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1. Introduction

There is empirical evidence pointing out the large number of firms involved in some *merging process*. As a recent example, let us mention that,

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in Spain, it is expected that the number of (regional) *saving banks* will be drastically reduced from 45 in 2009 to 20 after the restructuring process in which the financial sector is involved. The reasons justifying why firms merge include the effort to gain market power, tax advantages of taking over a loss making firm, efficiency, increasing market share and diversification among other factors. Simultaneously to this merging wave, we found that the number of firms going bankrupt has also grown. This fact can be partially explained by the global financial crisis.

What this paper tries to explain is how the debtors of a bankrupted firm should be reimbursed. Taking into account that they could not avoid, nor promote, any merging process, the procedure employed to solve our problem should be *neutral* to the presence of any merging process. By neutrality we mean that what each creditor would lose, due to the bankruptcy situation, should not be affected by any entrepreneurial alliance. The question is quite simple but, as this paper will point out, its analysis is not trivial.

The particular way that we will model this economic problem allows it to be also applied to some other distribution situations like how to share out a deceased estate among its inheritors, the design of (multi-issues) tax or tariff systems, etc. Therefore, we will follow the approach introduced by O'Neill (1982), when formulating *solutions for bankruptcy problems*. The analysis of bankruptcy problems can be considered a simple and robust tool to model how agents should be rationed. In these situations, each agent in a group asks for a quantity of a perfectly divisible good, but the available amount is not enough to satisfy all the agents' demands.

The literature proposes mainly two approaches to provide *satisfactory* solutions for these problems. The first one, the *axiomatic* analysis, proposes particular solutions arising from the assumption that some '*Equity Principles*' should prevail. The study by Thomson (2003) provides a nice overview of the main results following this methodology. The second approach is based on a interpretation of bankruptcy problems as (transferable utility) cooperative games, *TU-games* henceforth. This formulation, introduced by O'Neill (1982), has been employed to argue in favor of some particular rules. Regarding this, see the papers by O'Neill (1982) and Aumann and Maschler (1985).

This close relationship between bankruptcy situations and cooperative games might well lead to analyze how strong the liaison between both frameworks is. When concentrating on solutions for TU-games, and related to properties reflecting *neutrality* on the distributive process, *additivity* is one of the most extensively imposed requirements. In fact, the fulfillment of this property has been extensively demanded in a huge family of allocation problems analyzed from a co-operative perspective. Just as an illustrative example, Moretti and Patrone (2008) refers to the *Shapley value* application to cost allocation, social networks, water issues, biology, reliability theory and belief formation. It is well-known that, in his seminal paper, Shapley (1953) pointed out the additivity of the value he proposed.

When concentrating on bankruptcy situations, *additivity* of a solution imposes, as in TU-games frameworks, neutrality on the distributive process. Just to illustrate it, let us consider a creditor lending some funds to two firms, say A and B. After a merging process, involving both firms the new corporation, to be called C, bankrupts. When reimbursing this corporation's creditors, one can propose two ways to proceed. The first one is considering the problem of (partially) reimbursing C's creditors; whereas the second one lies in solving the 'primitive' problems related to A and B with respect to their creditors. What additivity should require is that both processes yield the same outcome.

As far as we know, the paper by Bergantiños and Méndez-Naya (2001) is the only one which explores additivity in Bankruptcy frameworks. Their conclusions in that matter can be summarized as follows:

(1) There is no bankruptcy rule satisfying additivity; and

(2) If we concentrate on a (very restrictive) family of bankruptcy problems, the Ibn Ezra's rule is the only one which conciliates additivity and equal treatment.

Therefore, what Bergantiños and Méndez-Naya (2001) points out is that *additivity* is a very demanding property in our framework.[4] Moreover, if we want to explore bankruptcy rules that, being well-defined for any problem, satisfy some weak version of *additivity*, we must limit ourselves to considering rules that coincide with the Ibn Ezra's proposal in the framework in which it is defined.

Taking into account the above restrictions, this paper proposes a weak notion of additivity, that we call μ-*Additivity*, and a study of the bankruptcy rules satisfying it. What we find is that the only rule for which anonymity,

[4]Section 3 discusses the rationale of such a fact both from an economical and a mathematical point of view.

continuity and μ-*Additivity* are compatible is the *Minimal Overlap Rule*, introduced by O'Neill (1982).

Anonymity and Continuity are two properties which have been widely justified in the literature for Bankruptcy problems. What μ-additivity would suggest is that additivity should be a requirement for comparable problems, from the creditors' point of view. Following this interpretation, and trying to be precise in describing when two bankruptcy problems are comparable, we consider three elements:

(1) For any two agents, their relative credits are similar, i.e. the agent conceding the highest credit is the same for both problems;

(2) For each agent, her credit position, related to the debtor's assets is similar, i.e. her credits exceed the creditor's assets in a problem, this situation should not be reversed in the other; and

(3) The sacrifice that each agent would impose on her 'rivals', if her credit is the first to be paid, should always (or never) be lower than such a credit.

The rest of the paper is organized as follows. Section 2 introduces the model and the main definitions related to bankruptcy problems. Section 3 discusses the notion of *Additivity* from a bankruptcy perspective and, based on some impossibility results, introduces the notion of '*Partial Additivity*'. Section 4 provides our main result, consisting on an axiomatization of the *Minimal Overlap Rule* based on the weak additivity property, discussed in the previous section. Technical proofs are relegated to the Appendix.

2. Bankruptcy. The Framework and Main Definitions.

Let us consider an individual, the debtor, having some debts. Let $\mathcal{N} = \{1, \ldots, i, \ldots, n\}$ denote the set of her creditors, that will be considered fixed throughout the paper. $E \geq 0$ will denote the valuation of the debtor's assets, and will be called the *Estate*. For any fixed creditor, say i, $c_i \geq 0$ will denote her credit, i.e. the quantity that the debtor owes to her. Vector $c = (c_1, \ldots, c_i, \ldots, c_n)$ summarizes creditors' claims. We say that the debtor goes bankrupt if she has not enough assets to reimburse her debts.

Following the above, a bankruptcy problem can be fully described by a vector $(E, c) \in \mathbb{R}_+ \times \mathbb{R}^n_+$ such that

$$E \leq \sum_{i=1}^{n} c_i \tag{1}$$

Note that Condition (1) reflects that reimbursing creditors' debts might be incompatible. Therefore, these agents' aspirations (on recovering their debts) would be rationed. Let $\mathcal{B}$ denote the family of all the bankruptcy problems. For notational convenience, we describe the set of bankruptcy problems having a 'super-creditor', i.e. an individual whose credit is not lower than the estate, as

$$\mathcal{B}_S = \left\{ (E, c) \in \mathcal{B} \colon E \leq \max_{i \in \mathcal{N}} c_i \right\} \tag{2}$$

and, for any family of bankruptcy problems, say $\mathcal{B}_\star \subseteq \mathcal{B}$, $\mathcal{B}^O_\star$ will denote the subclass of problems with increasingly ordered claims

$$\mathcal{B}^O_\star = \{(E, c) \in \mathcal{B}_\star \colon c_i \leq c_j \text{ whenever } \ i \leq j\} \tag{3}$$

Definition 1. A **Bankruptcy rule** is a function $\varphi \colon \mathcal{B} \to \mathbb{R}^n_+$, such that for each problem $(E, c) \in \mathcal{B}$, (*a*) $\sum_{i=1}^{n} \varphi_i(E,c) = E$; and

(*b*) $0 \leq \varphi_i(E,c) \leq c_i$ for each creditor i.

Among the many rules that have been explored in the literature, we will introduce those two that will be most useful in our analysis.[5] The first one, to be called **Ibn Ezra**'s rule, is a 'semi-solution' in the sense that it is not defined for every problem. The second one, known as the **Minimal Overlap** rule, was proposed by O'Neill (1982) as a possible extension of Ibn Ezra's rule to be defined for any problem.

In order to properly define the above rules, let us consider a problem $(E,c) \in \mathcal{B}^O$.[6] We say that it is an *Ibn Ezra's problem* if and only if it has a *super-creditor*.

Definition 2. Ibn Ezra's rule is the function $\varphi^{IE}\colon \mathcal{B}_S \to \mathbb{R}^n_+$, associating to each problem $(E,c) \in \mathcal{B}^O_S$, and creditor i, the amount

$$\varphi_i^{IE}(E,c) = \sum_{j=1}^{i} \frac{\min\{E,c_j\} - \min\{E,c_{j-1}\}}{n-j+1} \tag{4}$$

where $c_0 = 0$.

Chun and Thomson (2005) proposes a formal description for the Minimal Overlap rule. What these authors suggest is to proceed as follows. Let us

[5]As we have already mentioned, the reader can find a nice and complete overview of the most relevant Bankruptcy rules in Thomson (2003).

[6]For expository simplicity and technical convenience, we assume that creditors' claims are increasingly ordered. Otherwise, we can re-arrange the creditors' labels to reach this objective. Therefore, even though this assumption is proposed, the Ibn Ezra's and Minimal Overlap rules are also defined when the agents' claims are not increasingly ordered.

consider a bankruptcy problem $(E, c) \in \mathcal{B}^O$, then what each creditor recovers is described as follows:

(a) if $(E, c) \in \mathcal{B}_S$ then

$$\varphi^{MO}(E, c) = \varphi^{IE}(E, c)\,; \text{ or} \tag{5}$$

(b) if $(E, c) \notin \mathcal{B}_S$ then there is a unique $t^\star \geq 0$ such that

$$t^\star = E - \sum_{i=1}^{n} \max\{c_i - t^\star, 0\}\,. \tag{6}$$

In such a case, the Minimal Overlap rule associates to creditor i the amount

$$\varphi_i^{MO}(E, c) = \varphi_i^{IE}(E - t^\star, c) + \max\{c_i - t^\star, 0\}\,. \tag{7}$$

Recently, Alcalde et al. (2008) found an alternative expression for the Minimal Overlap rule, which is equivalent to the one introduced by Chun and Thomson (2005), but helps to provide a (direct) intuition on what O'Neill (1982) could have had in mind about how to extend the proposal by Ibn Ezra.

Definition 3. The Minimal Overlap rule is the function $\varphi^{MO}\colon \mathcal{B} \to \mathbb{R}_+^n$, that associates, to each problem (E, c), the vector

$$\varphi^{MO}(E, c) = \varphi^{IE}(\min\{E, c_n\}, c) + \varphi^{cel}(E^r, c^r) \tag{8}$$

where φ^{cel} stands for the Constrained Equal Losses rule and the 'residual' bankruptcy problem, to which the Ibn Ezra's rule was not applied, is described by $E^r = \max\{E - c_n, 0\}$, and $c^r = c - \varphi^{IE}(\min\{E, c_n\}, c)$.

3. Additivity and Bankruptcy Rules

The aim of this section is to introduce a discussion on the notion of additivity in the framework of bankruptcy problems. Just to illustrate it, let us consider the following example. A creditor, say i, loans some quantity to two firms, say F and G. Let c_i^F and c_i^G denote these quantities. After a merging process, firm H emerges as the fusion of F and G and, unfortunately H goes bankrupt. If we denote by E^F and E^G the valuations of firms F and G respectively; and their respective debts vectors are denoted by c^F and c^G, we can have that[7]

(a) $E^H = E^F + E^G$, and

(b) $c^H = c^F + c^G$

What creditor i would expect to obtain at the division process, for any Bankruptcy rule, say φ is

$$\varphi_i \left(E^H, c^H\right) \geq \varphi_i \left(E^F, c^F\right) + \varphi_i \left(E^G, c^G\right)$$

Note that, otherwise, i could claim that she has been 'punished' due to the merging process, and she would not have the ability to object against such a decision made by the two firms. If such an argument is extended to all the creditors we have that the above inequality must become an equality. This is the essence of the additivity notion.

[7]We are implicitly assuming that $\mathcal{N}$, the set of creditors, is the same for both firms, and that there are no intra-group debts, i.e. F is not a G's creditor or a debtor.

Definition 4. Let φ be a Bankruptcy rule. We say that it satisfies **additivity** if for any two problems, (E^1, c^1) and (E^2, c^2) we have that

$$\varphi\left(E^1, c^1\right) + \varphi\left(E^2, c^2\right) = \varphi\left(E^1 + E^2, c^1 + c^2\right). \tag{9}$$

Bergantiños and Méndez-Naya (2001) provided an example pointing out that additivity is a very demanding property for Bankruptcy rules. This is why they could show that there is no rule satisfying additivity.

Some reasons justifying this fact can be found. The first one lies in the relationship between bankruptcy problems and the TU-games, as suggested by O'Neill (1982). This author proposed to associate to each bankruptcy problem (E, c) the TU-game $(\mathcal{N}, V)$, where the characteristic function $V\colon 2^{\mathcal{N}} \to \mathbb{R}_+$ assigns to each coalition $S \subseteq \mathcal{N}$ the amount

$$V(S) = \max\left\{E - \sum_{i \notin S} c_i, 0\right\}. \tag{10}$$

In this sense, Example 1 points out that additivity of bankruptcy problems might not induce additivity of the respective TU-games.

Example 1. Let us consider the following three-agent bankruptcy problems. $(E, c) = (9, (8, 8, 8))$, and $(E', c') = (31, (4, 12, 22))$. Therefore, the aggregate bankruptcy problem is $(E + E', c + c') = (40, (12, 20, 30))$. Let V (resp. V', V'') denote the characteristic function relative to the problem (E, c) (resp. (E', c'), $(E + E', c + c')$). Following Equation (10) we have that

S	$V(S)$	$V'(S)$	$V''(S)$	$V(S)+V'(S)$
$\{1\}$	0	0	0	0
$\{2\}$	0	5	**0**	**5**
$\{3\}$	0	15	**8**	**15**
$\{1,2\}$	1	9	10	10
$\{1,3\}$	1	19	20	20
$\{2,3\}$	1	27	28	28
$\{1,2,3\}$	9	31	40	40

Therefore, the TU-game induced by adding the two bankruptcy problems differs from the addition of the TU-games induced by both problems.

The second reason which explains that additivity is a strong requirement, comes from an economic perspective. Let us consider a *company* that can be seen as the result of a merging process involving some firms. When the company as a whole, goes bankrupt, the **degree of bankruptcy**,[8] is not usually homogeneous considering the firms that configure the company. This will justify the fact that not all the creditors should be rationed, taking into account the company's financial situation, but that of the firms receiving their credits. Therefore, what this situation suggests is that additivity, as introduced by Definition 4, might not be a reasonable property for Bankruptcy rules, unless the problems to be added share at least, a '*similar bankruptcy*

[8]Given a bankruptcy problem (E, c), we can define its *degree of bankruptcy* as the expression

$$\mathcal{D}(E,c) = 1 - \frac{E}{\sum_{i=1}^{n} c_i}.$$

degree'.

4. μ-Additivity and the Minimal Overlap Rule

The aim of this section is to describe *reasonable conditions* under which additivity is both satisfied by some bankruptcy rules and justified from an economic point of view. The main idea for our requirements starts from considering not only the 'degree of bankruptcy' as a comparison measure, but also what each creditor would impose on her 'rivals' as a sacrifice when her debts were completely reimbursed.

Just to formalize the above idea, let us consider the following scenario. Given a bankruptcy problem (E, c), and an agent $i \in \mathcal{N}$, $E_i = \max\{0, E - c_i\}$ denotes the amount that agents other than i would distribute after fully reimbursing (if possible) i's credits. We will denote by $\mu_i(E, c)$ the (constrained egalitarian) loss in which agents, other than i, incur when i's credits have been, as much as possible, reimbursed; i.e. $\mu_i(E, c)$ is the unique solution to

$$\sum_{j \neq i} \max\{0, c_j - \mu_i(E, c)\} = E_i.$$

A notion for *level of imposed sacrifice*, by an agent to her rivals, would lie in how the above idea is related to such an agent's claim.

Definition 5. Given a bankruptcy problem (E, c), we say that agent i's claim is **under-transferred** if, and only if, $\mu_i(E, c) < c_i$.

We now introduce a notion of *partial additivity* which is satisfied by some Bankruptcy rules.

Axiom 1. We say that a Bankruptcy rule, say φ, satisfies μ**-Additivity** if

$$\varphi\left(E,c\right)+\varphi\left(E',c'\right)=\varphi\left(E+E',c+c'\right).$$

for any two problems (E,c) and (E',c') such that

(a) $(c_i-c_j)\left(c_i^{'}-c_j^{'}\right)\geq 0$ for each i and j in $\mathcal{N}$;

(b) $(E-c_i)\left(E-c_i^{'}\right)\geq 0$ for each i in $\mathcal{N}$; and

(c) Each agent's claim is under-transferred in (E,c) if, and only if, it is under-transferred in $\left(E',c'\right)$,

Note that what Axiom 1 suggests is that additivity should be preserved in problems sharing some similarities related to their internal structure:

(a) In both problems the agents' claims should be ordered in a similar way, i.e. if i's claim is greater that j's claim in a problem, it should not be the case that j's claim is greater that i's claim in the other.

(b) In both problems, each agent's claim should have the same position relative to the estate, i.e. if some agent's claim is lower than the estate in a problem, it should not be the case that, for the other problem, her claim exceeds the estate. And,

(c) In both problems the level of imposed sacrifice by each agent, should have the same position, related to her claim.

In order to present our main result, we need to introduce two axioms that are usually employed in Bankruptcy Theory. The first one, *anonymity*, establishes that what each creditor recovers does not depend on her name, but on the internal structure of the problem. The second one is the classical requirement of *continuity*.

Axiom 2. We say that a Bankruptcy rule, say φ, satisfies **Anonymity**, or is anonymous, if for each problem (E, c) and any permutation[9] π,

$$\pi\left[\varphi\left(E, c\right)\right] = \varphi\left(E, \pi\left(c\right)\right).$$

Axiom 3. We say that Bankruptcy rule φ satisfies **Continuity**, or is continuous, if for each sequence of bankruptcy problems $\{E^{\nu}, c^{\nu}\}_{\nu \in \mathbb{N}}$, if

$$\lim_{\nu \to \infty} \{E^{\nu}, c^{\nu}\} = (E, c) \in \mathcal{B}.$$

then

$$\lim_{\nu \to \infty} \varphi\left(E^{\nu}, c^{\nu}\right) = \varphi\left(E, c\right).$$

We can now establish the following result, whose proof is relegated to the Appendix.

Theorem 1. *Let φ be a Bankruptcy rule. φ satisfies* **Anonymity***,* **Continuity** *and* μ-**Additivity** *if, and only if, φ is the* **Minimal Overlap rule***.*

By way of conclusion, let us mention that Theorem 1 is tight in the sense that the uniqueness result requires all the three axioms, and no axiom is implied by the other two. Just to clarify that, let us note that:

[9] A permutation π is a bijection applying $\mathcal{N}$ onto itself. In this paper, and abusing notation, $\pi\left(c\right)$ will denote the claims vector obtained by applying permutation π to its components, i.e. the i-th component for $\pi\left(c\right)$ is c_j whenever $j = \pi\left(i\right)$. Similar reasoning considerations apply for $\pi\left[\varphi\left(E, c\right)\right]$.

(a) The *Constrained Equal Awards* rule is both continuous and anonymous but fails to satisfy μ-additivity;

(b) Any asymmetric Bankruptcy rule belonging to the family that Alcalde et al. (2008) called the *Weighted Minimal Overlap rules* is both continuous and μ-additive, but does not satisfy anonymity, and

(c) Let us consider the Bankruptcy rule φ^c that suggests the Ibn Ezra's proposal for any problem (E, c) such that $E \leq \max_{i \in \mathcal{N}}$, and otherwise, if $\mathcal{P}$ denotes the set of agents whose claims are *under-transferred*,

$$\varphi_i^c(E, c) = \begin{cases} 0 & \text{if } \ i \notin \mathcal{P} \\ \varphi_i^{cel}\left(E, \left(\{0\}_{j \notin \mathcal{P}}, \{c\}_{j \in \mathcal{P}}\right)\right) & \text{if } \ i \in \mathcal{P} \end{cases}$$

where φ^{cel} stands for the *Constrained Equal Losses rule*. Note that this rule is anonymous and μ-additive but fails to be continuous.

References

Alcalde, J., Marco, M.C., Silva, J.A., 2008. The minimal overlap rule revisited. Social Choice and Welfare 31, 109–128.

Aumann, R.J., Maschler, M., 1985. Game theoretic analysis of a bankruptcy problem from the Talmud. Journal of Economic Theory 36, 195–213.

Bergantiños, G., Méndez-Naya, L., 2001. Additivity in bankruptcy problems and in allocation problems. Spanish Economic Review 3, 223–229.

Chun, Y., Thomson, W., 2005. Convergence under replication of rules to adjudicate conflicting claims. Games and Economic Behavior 50, 129–142.

Moretti, S., Patrone, F., 2008. Transversality of the Shapley value. TOP 16, 1–41.

O'Neill, B., 1982. A problem of rights arbitration from the Talmud. Mathematical Social Sciences 2, 345–371.

Shapley, L.S., 1953. A value for n-person games, in: Kuhn, H., Tucker, A. (Eds.), Contributions to the Theory of Games, Vol. II. Princeton University Press, Princeton, NJ. volume 28 of *Annals of Mathematics Studies*, pp. 307–317.

Thomson, W., 2003. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Mathematical Social Sciences 45, 249–297.

Appendix A. A Proof for Theorem 1

Throughout this Appendix we assume, without loss of generality, that c is increasingly ordered i.e., $c_i \leq c_j$ whenever $i < j$. For simplicity of exposition, for a given problem (E, c) we denote c^E the claims vector whose i-th component is $c_i^E = \min\{c_i, E\}$, for each agent $i \in \mathcal{N}$.

We first provide a result establishing that, under *Continuity*, *μ-Additivity* implies *Invariance under Claims Truncation.*

Proposition 1. *Let φ be a Bankruptcy rule satisfying μ-Additivity and Continuity. Then it satisfies Invariance under Claims Truncation, i.e. for each problem (E, c), $\varphi(E, c) = \varphi\left(E, c^E\right)$.*

Proof. Let φ be a rule satisfying *μ-Additivity* and *Continuity*. Let (E, c) be a problem where $0 < E < c_n = \max_{i\in\mathcal{N}}\{c_i\}$; and let $S \subset \mathcal{N}$ be the subset of agents claiming zero. Let us consider the following two cases.

Case 1: $c_n = c_{n-1}$ or $c_{n-1} < E$

Therefore, by *μ-Additivity*,

$$\varphi(E, c) = \varphi\left(E - \frac{1}{r}, \left((0)_{i\in S}, \left(c_i^E - \frac{1}{r}\right)_{i\in\mathcal{N}\setminus S}\right)\right) + \\ +\varphi\left(\left(\frac{1}{r}\right), \left((0)_{i\in S}, \left(c_i - c_i^E + \frac{1}{r}\right)_{i\in\mathcal{N}\setminus S}\right)\right)$$

for all $r \in \mathbb{N}$ such that

$$\frac{1}{r} < \min\left\{\min_{i\in\mathcal{N}\setminus S}\{c_i\}, c_n - E\right\}.$$

Now, by considering the limit when r goes to infinity in the previous equation and taking into account that φ is continuous, we obtain

$$\varphi(E, c) = \varphi\left(E, c^E\right) + \varphi\left(0, c - c^E\right) = \varphi\left(E, c^E\right).$$

Case 2: $E \leq c_{n-1} < c_n$

By *μ-Additivity*,

$$\varphi(E, c) = \varphi\left(E - \frac{1}{r}, \left((0)_{i\in S}, \left(c_i^E - \frac{1}{r}\right)_{i\in\mathcal{N}\setminus S\cup\{n\}}, c_n^E - \frac{1}{2r}\right)\right) + \\ +\varphi\left(\left(\frac{1}{r}\right), \left((0)_{i\in S}, \left(c_i - c_i^E + \frac{1}{r}\right)_{i\in\mathcal{N}\setminus S\cup\{n\}}, c_n - c_n^E + \frac{1}{2r}\right)\right)$$

for all $r \in \mathbb{N}$ such that

$$\frac{1}{r} < \min\left\{\min_{i\in\mathcal{N}\setminus S}\{c_i\}, \max_{i\in\mathcal{N}:E<c_i<c_n}\{c_i - E\}, 2(c_n - c_{n-1})\right\}.$$

Now, by considering the limit when r goes to infinity in the previous equation and taking into account that φ is continuous, we obtain

$$\varphi(E,c)=\varphi\left(E,c^E\right)+\varphi\left(0,c-c^E\right)=\varphi\left(E,c^E\right).$$

$\square$

Proof of Theorem 1

Firstly, it is straightforward to verify that the *Minimal Overlap rule* satisfies *Anonymity*, *Continuity* and *μ-Additivity.*

Now, let φ be a rule satisfying these axioms. Given a problem $(E,c)\in\mathcal{B}$, let us consider the following two cases:

Case 1: $E\leq c_n$.

By Proposition 1 we have that

$$\varphi(E,c)=\varphi\left(E,c^E\right)$$

Let us denote $P^1=c_1^E$; for $1<i\leq n$, $P^i=c_i^E-c_{i-1}^E$. And for each $i\in\mathcal{N}$, let denote $c^{P^i}=\left((0)_{j<i},(P^i)_{j\geq i}\right)$.

Now, let us consider the following two subcases:

Subcase 1.a: $c_n^E=c_{n-1}^E$.

μ-Additivity implies that

$$\varphi\left(E,c^E\right)=\sum_{i\in\mathcal{N}}\varphi\left(P^i,c^{P^i}\right).$$

Therefore, by *Anonymity* and Proposition 1,

$$\varphi_j\left(P^i,c^{P^i}\right)=\begin{cases}0 & \text{if } j<i\\ \frac{P^i}{n-i+1} & \text{if } j\geq i\end{cases},$$

i.e.

$$\varphi_j\left(P^i, c^{P^i}\right) = \begin{cases} 0 & \text{if } \ j < i \\ \frac{c_i^E - c_{i-1}^E}{n-i+1} & \text{if } j \geq i \end{cases}$$

with $c_0 = 0$.[10] And, thus, for each agent h

$$\varphi_h\left(E, c^E\right) = \sum_{i\in\mathcal{N}} \varphi_h\left(P^i, c^{P^i}\right)$$

Since, for any $j > h$ we have that $c_h^{Pj} = 0$,

$$\begin{aligned}\varphi_h\left(E, c^E\right) &= \sum_{i=1}^{h} \varphi_h\left(P^i, c^{P^i}\right) = \sum_{i=1}^{h} \frac{c_i^E - c_{i-1}^E}{n-i+1} = \\ &= \sum_{i=1}^{h} \frac{\min\{c_i, E\} - \min\{c_{i-1}, E\}}{n-i+1} = \varphi_h^{MO}(E, c).\end{aligned}$$

Subcase 1.b: $c_n^E \neq c_{n-1}^E$.

Let $q(j)$ denote the cardinality of the set $\{i \leq j \colon P^i \neq 0\}$. By μ-*Additivity* we have

$$\begin{aligned}\varphi\left(E, c^E\right) = \varphi\left(P^1 + \frac{1}{r}, \left(\left(c^{P^1}\right)_{i<n}, c^{P^1} + \frac{1}{r}\right)\right) + \\ + \sum_{i<n:P^i\neq 0} \varphi\left(P^i - \frac{1}{r\left(q\left(n\right)-1\right)}, \left(\max\left\{0, c^{P^i} - \frac{1}{r\left(q\left(n\right)-1\right)}\right\}\right)_{j\in\mathcal{N}}\right) + \\ + \varphi\left(P^n - \frac{1}{r\left(q\left(n\right)-1\right)}, \left(0, \left(\min\left\{c^{p^i}, \frac{1}{r}\right\}\right)_{1<i<n}, c_n^{P^n} - \frac{1}{r\left(q\left(n\right)-1\right)}\right)\right),\end{aligned}$$

where $r \in \mathbb{N}$ is such that

$$\frac{1}{r} < \min\left\{\left(1 - \frac{1}{q(n)}\right)P^n, \min_{i:P^i\neq 0}\left\{P^i\right\}\right\}.$$

[10] Throughout this proof, and for notational convenience, we will consider $c_0 = c_0^E = 0$.

Now, by considering the limit when r goes to infinity in the previous equation, by *Continuity* we obtain

$$\varphi\left(E, c^{E}\right)=\sum_{i \in \mathcal{N}} \varphi\left(P^{i}, c^{P^{i}}\right).$$

By using the reasoning of the Subcase 1.a above, we obtain that for each agent h

$$\varphi_{h}\left(E, c^{E}\right)=\varphi_{h}^{MO}(E, c).$$

Case 2: $E > c_n$.

In such a case, there is a unique t, $0 \leq t < c_n$, such that

$$\sum_{i \in \mathcal{N}} \max\{0, c_i - t\} = E - t.$$

Let k be the unique agent such that $c_k - t > 0$, and $c_{k-1} - t \leq 0$. Note that this implies that, for each agent j, with $j \leq k$, we have that c_j is *under-transferred*. Then, for each $r \in \mathbb{N}$ such that

$$\frac{1}{r} < \min\{E - t, (n-k-1)(c_k - t)\}$$

by μ-*Additivity* we have that,

$$\begin{aligned} \varphi(E, c) = \quad & \varphi\left(t+\tfrac{1}{r}, \left((c_i)_{i<k}, \left(t+\tfrac{1}{r(n-k+1)}\right)_{i \geq k}\right)\right) \\ & + \varphi\left(E-t-\tfrac{1}{r}, \left((0)_{i<k}, \left(c_i - t - \tfrac{1}{r(n-k+1)}\right)_{i \geq k}\right)\right) \end{aligned} \tag{A.1}$$

Let us consider the limit, when r goes to infinity, in the equation $(A.1)$ above. By *Continuity*, we have

$$\varphi(E, c) = \varphi\left(t, (\min\{c_i, t\})_{i \in \mathcal{N}}\right) + \varphi\left(E-t, (\max\{0, c_i - t\})_{i \in \mathcal{N}}\right).$$

Observe that the problem $\left(t, (\min\{c_i, t\})_{i\in\mathcal{N}}\right)$ was analyzed in Case 1 above. Therefore, for each agent h, we have that

$$\varphi_h\left(t, (\min\{c_i, t\})_{i\in\mathcal{N}}\right) = \sum_{i=1}^{h} \frac{\min\{c_i, t\} - \min\{c_{i-1}, t\}}{n-i+1}. \tag{A.2}$$

Moreover, note that for agent h we have that

$$\max\{0, c_h - t\} = \begin{cases} 0 & \text{if } h < k \\ c_h - t & \text{if } h \geq k \end{cases}.$$

Since by construction

$$\sum_{i=1}^{n} \max\{0, c_i - t\} = E - t,$$

we can conclude that for each agent h,

$$\varphi_h\left(E - t, (\max\{0, c_i - t\})_{i\in\mathcal{N}}\right) = \max\{0, c_h - t\}. \tag{A.3}$$

Therefore, by combining equations $(A.1)$, $(A.2)$, and $(A.3)$, we have that, for each agent h,

$$\varphi_h(E, c) = \sum_{i=1}^{h} \frac{\min\{c_i, t\} - \min\{c_{i-1}, t\}}{n-i+1} + \max\{0, c_h - t\} = \varphi_h^{MO}(E, c).$$

$\square$