Externalities in RD: a route to endogenous fluctuations

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Escola Superior de Comunicação Social - Instituto Politécnico de Lisboa

2007

Externalities in R&D:

a Route to Endogenous Fluctuations

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Abstract

Technological progress produces both positive and negative economy wide externalities. Although positive spillovers seem to prevail most of the times, there is evidence and logical arguments revealing that investment in R&D can exceed the corresponding socially optimal level. Taking on board the assumption that the two kinds of externalities are possible and that, therefore, one is able to define the pace of technical progress required to maximize social welfare, we develop a standard two-sector optimal growth model with externalities in the production of technology. The added assumption allows for introducing endogenous business cycles in the Walrasian growth setup. The undertaken stability analysis discusses the local properties of a difference equation two-dimensional system, identifying the occurrence of a flip bifurcation, and looks at global dynamics, through a numerical example, in order to better illustrate and describe the non linear nature of the system.

Keywords: Technology, Externalities, Endogenous business cycles, Two-sector growth models, Nonlinear dynamics and chaos.

JEL classification: C61, E32, O41

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Acknowledgements: Financial support from the Fundação Ciência e Tecnologia, Lisbon, is grateful acknowledged, under the contract No POCTI/ECO/48628/2002, partially funded by the European Regional Development Fund (ERDF).

1. Introduction

Typically, the decentralized economy invests less than what is socially optimal in R&D activities. The properties of technology and knowledge as public goods lead us directly to the conclusion that innovators are unable to capture all the consumer surplus of their output, and therefore private investment tends to remain below the level needed to guarantee the maximum degree of economy wide welfare. Such an observation has implied, since the first analytical work on growth and learning, as in Solow (1956) and Arrow (1962), to the technology based endogenous growth framework of Romer (1986, 1990), Grossman and Helpman (1991), Aghion and Howitt (1992) and Jones (1995), that the social stimulus to invest in R&D will always exist, independently of the pace of technological progress and of the way the society is able to absorb such progress.

Jones and Williams (2000) share this reasonable point of view that the decentralized economy generally under-invests in innovation (that is, positive externalities of technological progress are strong and, hence, the private return to R&D is lower than its social return). Nevertheless, for these authors it is also reasonable to ask whether negative externalities that trigger private investment above optimal levels exist. Two arguments can be put forward at this respect: first, distortions arise in terms of patent races; second, the intertemporal rent transfer, provoked by the creative destruction process, tends to be relevant as well.

The patent race issue is related to a congestion negative externality, in the sense that parallel R&D programs will take place; in this way, the research effort towards a given result will be simultaneously undertaken by different researchers and accordingly the average productivity of the innovation investment is lowered. The appropriation of monopoly rents is also a potential form of producing an over-incentive to generate knowledge: for each innovation, there is a clear rent distribution from earlier innovators to newly arrived researchers, as the new research outcome turns the previous results obsolete; thus, the rent of the new innovator corresponds to his effort, but also to the effort of previous producers, given the cumulative nature of technology and knowledge.

Both, the congestion negative externalities and the creative destruction are arguments in favour of the idea that the decentralized economy can over-invest in R&D, as the private return becomes eventually higher than the social return (although, as stated, the opposite is empirically the most plausible and frequent result).

Under the previous arguments, if the economy was ruled by a social planner than it would be possible to define, in each moment of time, a finite optimal level of technology in the economy. This level would correspond to the degree of knowledge for which positive spillovers and negative externalities would exactly offset each other. Below this level, the society is receptive to more investment in research and the various agents will stimulate such creation of technical knowledge: the government may attribute pecuniary rewards to innovators, consumers will reveal their preferences towards new goods and more technically sophisticated goods, and other firms with backward and forward linkages with the research sector will make this sector know how much it is relevant for the economic system as a whole. Above the socially optimal level, negative externalities introduce a penalty over technological progress: the government no longer contributes actively to technological progress, consumers will lose interest in new goods (since, e.g., they are yet getting acquainted with the previous waves of innovation), and related firms will also show less interest, since they cannot keep up with what the research sector is able to offer.

The previous reasoning constitutes the main idea in the way we introduce technology in a standard growth setup along the following sections. We assume that it is possible to define an optimal level of technology (and an optimal rate of technical progress). If the available level of technology is below this level, the research activity will be subject to a positive social stimulus that is derived from the positive external effect innovation has to offer. If the technology index reflects a relatively higher weight of the factors that induce investment in R&D above the social benchmark, a negative effect over the production of knowledge is introduced by the lack of social acceptance of a too high efficiency of the research sector, which is not accompanied either by other productive sectors or by consumer preferences.

Basically, the usual two-sector competitive growth model is developed, under the new imposed assumption (the sectors are the final goods production sector and a technology sector that incorporates the referred feature). As a result, nonlinear dynamics arise. The model will no longer be characterized by a saddle-path equilibrium that generally this framework precludes [see, for instance, the two-sector growth models in Barro and Sala-i-Martin (1995)], but periodic and a-periodic cycles can be observed for particular values of the parameters. In this way, we are able to put together an endogenous growth setup and an explanation for business cycles with endogenous foundations.

The proposed model contributes to the theory of endogenous business cycles (EBC), initially proposed by Medio (1979), Stutzer (1980), Benhabib and Day (1981), Day (1982) and Grandmont (1985), to cite some of the most relevant. The fundamental concern in this literature relates to the idea that cycles should be explained through endogenous economic mechanisms that can be expressed under nonlinear dynamic relations between variables, rather than being the result of some external event. Thus, one can think of EBC as an alternative interpretation of economic fluctuations, relatively to the popular and meaningful explanation provided by the Real Business Cycles theory (RBC) of Kydland and Prescott (1982), Long and Plosser (1983) and Christiano and Eichenbaum (1992). In RBC models growth and cycles are combined, under the usual utility maximization intertemporal framework, through external technology shocks or government expenditures disturbances. These shocks and disturbances have a direct impact over the labour market, generating a decision process relating the labour-leisure trade-off which leads to non constant labour participation through time, and therefore to a non linear evolution of per capita income.

The EBC theory has is revival with the work of Christiano and Harrison (1999), who have found that once we introduce externalities in the production of physical goods into a deterministic RBC model (that is, an intertemporal framework with labour-leisure decisions but without external disturbances), this is able to generate endogenous cycles. Strong increasing returns to scale provide the possibility of achieving a system of nonlinear difference equations where routes to nonlinear dynamics or chaos are evidenced – a series of flip bifurcations or a Neimark-Sacker bifurcation allow, for some parameter values, to transform fixed point results in cycles of various orders (frequently through a period-doubling process), including the possibility of finding completely irregular time series, with no identifiable order.

The result of Christiano and Harrison (1999) can be subject to criticism. In Coury and Wen (2005), it is argued that the level of externalities, and therefore the level of increasing returns to scale, needed to generate long run cycles is unrealistically high, and therefore although analytically appealing, the model will hardly be adequate to explain real phenomena. In this respect, EBC loses clearly to RBC. On the other hand, EBC gains in terms of encountering inside the economic system the roots of nonlinear behaviour; no external source triggers the cycles.

Other work concerning EBC and increasing returns includes Schmitt-Grohé (2000), Guo and Lansing (2002), Goenka and Poulsen (2004) and Weder (2004). An interesting new approach as been proposed by Cellarier (2006), who also searches for endogenous business cycles in the standard competitive growth setup, but focusing on expectations and learning. Agents are rational but they do not have the ability to make all lifetime decisions in a given initial moment. They will make decisions as time unfolds and, thus, perfect foresight gives place to a mechanism of adaptation and learning, that allow us to understand the behaviour of the agents as a boundedly rational behaviour. The constant gain learning framework of Cellarier (2006) brings to the EBC literature the important work on expectations in macroeconomics that has been developed in the previous years [see, e.g., Evans and Honkapohja (2001), Kurz (1994, 1997), Kurz, Jin and Motolese (2003), Brock and Hommes (1997, 1998), Hommes (2005*a*, 2005*b*)].

Other approach is followed by Gomes (2006), who considers that firms do not predict optimally future demand. As a result, their investment decisions will be biased. The difference between optimal investment decisions (the ones underlying the benchmark growth model) and the effectively undertaken ones, gives rise to a distortion in the capital accumulation process that leads to endogenous cycles (analytically, a logistic equation regarding demand expectations is added to the conventional Solow capital accumulation constraint).

The debate between EBC and RBC can be synthesized in the words of Diebolt (2006), who states that "*There are two contrasting viewpoints concerning the explanation of observed fluctuations in economics. According to the first view the main source of fluctuations is to be found in exogenous, random shocks to fundamentals. According to the second view a significant part of observed fluctuations is caused by non-linear economic laws. Even in the absence of any external shocks, non-linear market laws can generate endogenous business fluctuations. (…) By the late 1970s and early 1980s, the debate concerning the main source of business cycle fluctuations seemed to have been settled in favour of the exogenous shock hypothesis. An important critique on this hypothesis has been that it does not provide an economic explanation of observed fluctuations, but rather attributes these fluctuations to external, non-economic forces. Due to the discovery of deterministic chaos however, a renewed interest in endogenous economic dynamics emerged*." (pages 86 and 87).

The model to develop in the following sections is strongly motivated by the idea that cycles can be explained in the basis of 'nonlinear economic market laws'. It furnishes a new candidate source of fluctuations: the co-existence of positive and negative externalities affecting the production of technology. The intuition behind the proposed mechanism is as follows: if the generation of knowledge is below the social optimal level there is a force that pushes technology indexes upward; if negative externalities dominate, the production of technology is forced to slowdown. These two contradicting effects push and pull in different directions generating the endogenous cycles.

The remainder of the paper is organized as follows. Section 2 formalizes the analytical structure under evaluation, giving special attention to the novel features of the technology sector. Sections 3 and 4 study the stability properties of the model, both locally and globally. Section 5 presents some final remarks.

2. The Growth Setup and Technology Dynamics

Consider a two-sector economy. The first sector produces final goods, while the second translates the way in which R&D activities are developed. Relatively to the final goods sector, we assume that aggregate output is generated through a production function with labour augmenting technological progress, $Y_t=F(uK_t,vL_tA_t)$, where Y_t respects to aggregate output, and the inputs K_t, L_t and A_t are, respectively, the amount of physical capital, the aggregate labour input and the technology level. Labour is considered to evolve at a constant non negative rate $n\geq 0$, over time. Variables $u<1$ and $v<1$ are positive shares of capital and labour, respectively, used in the final goods production process (and, therefore, 1-u and 1-v are the shares of capital and labour in the R&D sector). Technology is non rival and, thus, the available level of this input can be integrally used in both activity sectors.

Production function F has standard neoclassical features, as described in assumption 1.

Assumption 1. Production function $F:\mathbf{R}_+^2 \rightarrow \mathbf{R}_+$ is twice continuously differentiable and exhibits positive and diminishing marginal returns with respect to each input. Furthermore, it yields constant returns to scale (it is homogeneous of degree 1) and the following conditions (Inada conditions) are satisfied: $\lim_{K\to 0} F_K = \lim_{L\to 0} F_L = \lim_{A\to 0} F_A = \infty$ and $\lim_{K\to\infty} F_K = \lim_{L\to\infty} F_L = \lim_{A\to\infty} F_A = 0$.

According to the aggregate production function properties in assumption 1, one may write the production function in intensive form, i.e., $y_t=f(uk_t,vA_t)$, with $y_t\equiv Y_t/L_t$ and $k_t\equiv K_t/L_t$. The dynamics of the final goods sector are given by the conventional definition of capital accumulation, that is, $K_{t+1}-K_t=I_t-\delta K_t$, with I_t aggregate investment and $\delta>0$ the depreciation rate of physical capital. Investment is defined as the difference between aggregate income and aggregate consumption, which we present as C_t.

In intensive form, the capital accumulation constraint comes

$$k_{t+1} = \frac{1}{1+n} \cdot \left[f(uk_t, vA_t) - c_t + (1-\delta) \cdot k_t \right], \ k_0 \text{ given.} \tag{1}$$

with $c_t \equiv C_t/L_t$. Equation (1) is the usual constraint of the representative household intertemporal problem (the Ramsey problem), relatively to which it is well known that a saddle-path equilibrium is obtainable [see Barro and Sala-i-Martin (1995), Romer (2001) or Heer and Maussner (2005)]. Saddle-path stability means that a one-dimensional stable trajectory exists in a two-dimensional space, and therefore to guarantee stability one has to consider that the level of consumption is chosen in an initial moment in order to locate exactly over the stable trajectory, otherwise the system will be unstable and the convergence process towards the unique steady state point will not hold. Since we are interested in studying the dynamics and the stability properties of the pair technology – capital stock, it is fruitful to assume from the beginning that most likely the convergence process will take place. Hence, assumption 2 is taken.

Assumption 2. Consumption grows in time at exactly the same rate as the stock of capital; a constant value $\psi \equiv c_t/k_t$ is considered. This assumption is equivalent to say that a constant marginal propensity to consume holds, and thus we analyze a Solow (1956) – type growth model rather than the Ramsey (1928) – Cass (1965) – Koopmans (1965) intertemporal optimization framework.

We now define the dynamics of the R&D sector. An accumulation process similar to the one characterized for physical capital can be established. First assume a production function for technology, $Z_t = H\left[(1-u) \cdot K_t, (1-v) \cdot L_t A_t\right]$. Function H has the same inputs as F, and their properties are also identical, as stated in assumption 3.

Assumption 3. The production function for technological goods, $H : \mathbf{R}_+^2 \to \mathbf{R}_+$, is twice continuously differentiable and exhibits positive and diminishing returns with respect to each input. The Inada conditions must be satisfied and constant returns to scale hold.

In intensive form, $z_t = h[(1-u)\cdot k_t, (1-v)\cdot A_t]$, with $z_t \equiv Z_t/L_t$.

The process of technology accumulation is given by $L_{t+1}A_{t+1} - L_t A_t = Z_t - \rho L_t A_t$, with $\rho>0$ a depreciation / obsolescence rate of technical resources. Note that the difference equation concerning technological progress can also be presented in intensive form: $A_{t+1} = \frac{1}{1+n}\cdot[h((1-u)\cdot k_t, (1-v)\cdot A_t) + (1-\rho)\cdot A_t]$, A_0 given.

In the presented formulation, the R&D sector is in fact a sector of human capital accumulation, where it is easy to separate the labour force component (that we assumed as growing exogenously) and the technological component (that we intend to study with the considered dynamic rule).

So far, the displayed model is a simple two-sector growth model similar to the ones proposed by Lucas (1988), Romer (1990) or Jones (1995). The new feature arises with assumption 4.

Assumption 4. The production of technology is subject to externalities. Given a benchmark socially optimal level of technology, B_t, positive externalities over the production of knowledge arise for $A_t<B_t$, while negative external effects will prevail if $A_t>B_t$.

To model the previous assumption, one considers that B_t grows at a constant positive rate γ: $B_{t+1}=(1+\gamma)\cdot B_t$. The externality will be associated to function $H(\cdot)$ and translated in function $\xi(A_t,B_t)$, in such a way that we replace, in the accumulation of technology, function $H(\cdot)$ by $\tilde{H}(\cdot) = H(\cdot)\cdot\xi(A_t,B_t)$. When $A_t=B_t$, the externality function should yield a value equal to 1, so that $\tilde{H}(\cdot) = H(\cdot)$; for $A_t<B_t$ we should expect $\xi(\cdot)>1$ and for $A_t>B_t$, $\xi(\cdot)<1$. The functional form $\xi(A_t,B_t) = e^{\theta\cdot[(B_t-A_t)/B_t]}$, $\theta>0$, serves the required purposes.

Figure 1 draws the relation between the technology level and the externality component for $\gamma=0$.

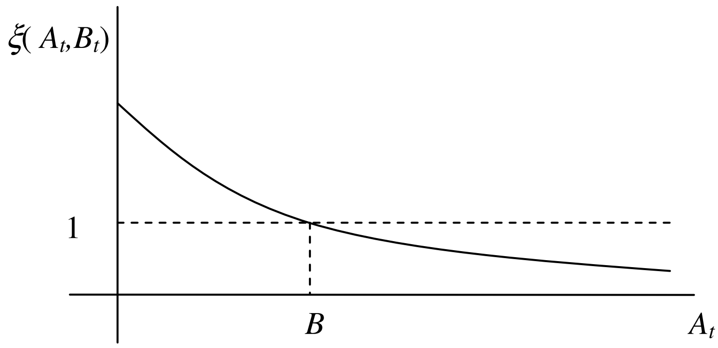

Figure 1 – Technology externalities function.

Assumption 4 and figure 1 translate the concerns referred in the introduction. There is an optimal level of technology, B. If the observable level of technology is below the socially optimal benchmark, positive external effects prevail; if the technology level is above B, then the economy will show that it is not prepared for such a high level of R&D, and negative external effects will rule.

Adding the external effect to the technology dynamic equation, one gets

$$A_{t+1} = \frac{1}{1+n} \cdot \left[h((1-u) \cdot k_t, (1-v) \cdot A_t) \cdot e^{\theta \cdot [(B_t - A_t)/B_t]} + (1-\rho) \cdot A_t \right], A_0, B_0 \text{ given.} \tag{2}$$

3. Steady State and Local Analysis

The system we are interested in studying is (1)-(2). The first result concerning this system is presented in proposition 1.

Proposition 1. Variables k_t and A_t grow in the steady state at rate γ.

Proof: The steady state is defined as the point in which variables k_t, A_t and B_t grow at constant rates (null or positive). From equation (1), given the constant returns to scale property, we find that $f\left(u, v \cdot \frac{\overline{A}}{\overline{k}}\right) = (1+n) \cdot \left(\frac{\overline{k_{t+1}}}{k_t}\right) + \psi - 1 + \delta$. Accordingly, we guarantee a constant long run growth rate for per capita capital if and only if $\overline{A}/\overline{k}$ is a constant value. Likewise, for equation (2),

$h\left((1-u)\cdot\frac{\bar{k}}{\bar{A}},1-v\right)\cdot e^{\theta\cdot(1-\bar{A}/\bar{B})}=(1+n)\cdot\left(\frac{\bar{A}_{t+1}}{A_t}\right)-1+\rho$, and therefore the ratio $\bar{A}/\bar{B}$ must assume a constant value as well. Variable B_t evolves at a constant rate for all periods of time, including the steady state, and hence $\bar{A}$ should grow too at rate γ. The constancy of the technology – capital ratio in the steady state implies that also the per capita stock of capital grows at rate γ in the long run ■

The dynamic analysis of the problem requires defining variables that do not grow in the steady state. Two ratios are defined: the ratio between the level of technology generated by the decentralized economy and the optimal social level of accumulated techniques, $G_t \equiv A_t/B_t$; for the case of absence of external effects, $G_t=1$. And the capital stock by unit of accumulated skills, $\omega_t \equiv k_t/A_t$.

To transform system (1)-(2) in a system of endogenous variables G_t and ω_t we regard that $\frac{G_{t+1}}{G_t}=\frac{A_{t+1}}{A_t}\times\frac{B_t}{B_{t+1}}$ and $\frac{\omega_{t+1}}{\omega_t}=\frac{k_{t+1}}{k_t}\times\frac{A_t}{A_{t+1}}$. The following stystem is accomplished,

$$\begin{cases} G_{t+1}=\dfrac{1}{(1+n)\cdot(1+\gamma)}\cdot\left[h((1-u)\cdot\omega_t,1-v)\cdot e^{\theta\cdot(1-G_t)}+1-\rho\right]\cdot G_t \\ \omega_{t+1}=\left[\dfrac{f(u,v/\omega_t)-\psi+1-\delta}{h((1-u)\cdot\omega_t,1-v)\cdot e^{\theta\cdot(1-G_t)}+1-\rho}\right]\cdot\omega_t \end{cases} \tag{3}$$

The study of the dynamics of system (3) should be made in three phases: (i) Analyze steady state properties; (ii) Study the dynamics in the vicinity of the steady state; (iii) Through numerical simulation, investigate global dynamic properties. The first two points are the concern of the remainder of this section. Global dynamics are addressed in section 4.

Proposition 2. The steady state of system (3) exists and it is unique.

Proof: Taking conditions $G_{t+1}=G_t$ and $\omega_{t+1}=\omega_t$ over (3), we get $f(u,v/\bar{\omega})=(1+n)\cdot(1+\gamma)+\psi-1+\delta$ and $\bar{G}=1+\frac{1}{\theta}\cdot\ln\left[\frac{h((1-u)\cdot\bar{\omega},1-v)}{(1+n)\cdot(1+\gamma)-1+\rho}\right]$.

Since f is continuous, positive and concave in $\mathbf{R}_{+}$, there is one and only one value $\overline{\varpi}$ that satisfies the first condition. For a unique $\overline{\varpi}$, the second condition clearly states that a unique $\overline{G}$ exists, because h is also a continuous, positive and concave function in $\mathbf{R}_{+}$. The values that parameters may possess are such that positive $\overline{\varpi}$ and $\overline{G}$ values are always guaranteed ■

The steady state properties become clearer under specific functional forms for f and h. The most common functions that obey to neoclassical properties are Cobb-Douglas production functions. Take parameters μ, $\alpha \in$ (0,1), such that

$$f(uk_t, vA_t) = a \cdot (uk_t)^{\alpha} \cdot (vA_t)^{1-\alpha}, \quad a > 0. \tag{4}$$

$$h[(1-u)\cdot k_t, (1-v)\cdot A_t] = g \cdot [(1-u)\cdot k_t]^{1-\mu} \cdot [(1-v)\cdot A_t]^{\mu}, \quad g > 0. \tag{5}$$

For these specific production functions, it is straightforward to find equilibrium values:

$$\overline{\varpi} = v \cdot \left[\frac{au^{\alpha}}{(1+n)\cdot(1+\gamma)+\psi-1+\delta}\right]^{1/(1-\alpha)}$$

and

$$\overline{G} = 1 + \frac{1}{\theta}\cdot \ln\left[\frac{g\cdot((1-u)\cdot\overline{\varpi})^{1-\mu}\cdot(1-v)^{\mu}}{(1+n)\cdot(1+\gamma)-1+\rho}\right].$$

From the steady state results, it is possible to highlight in a straightforward way that the higher are the values of v, a and u, then the larger is the amount of accumulated capital in the steady state per unit of technology; the opposite occurs for n, γ, ψ and δ. Relatively to the technology ratio, we should stress that the higher is the value of parameter θ, the lower is the relative value of $\overline{A}$ in terms of $\overline{B}$. Other parameters have also unambiguous effects over $\overline{G}$: positive changes in g, u and v contribute to a higher $\overline{G}$, and higher n, γ, ψ, δ and ρ lead to a lower $\overline{G}$. This implies that a high level of A_t is attainable in the long run before negative externalities set in if the rates of growth of population, growth of the technological frontier, depreciation and obsolescence are low. The same is true for a low consumption – capital ratio and a low participation of rival inputs in the production of technology.

Let us look now to the dynamics in the steady state vicinity.

Proposition 3. On a general evaluation of stability, one concludes that fold, transcritical or pitchfork bifurcations cannot occur, while flip and Neimark-Sacker bifurcations are possible. Furthermore, given that condition 1-*Tr*(*J*)+*Det*(*J*)>0 always holds [with *J* the Jacobian matrix of system (3)], one verifies that the eigenvalues of the Jacobian matrix are both higher than 1 or, alternatively, they are both lower than one.

<u>Proof</u>: In the steady state vicinity, the linearization of (3) allows for displaying the system in the following matrix form,

$$\begin{bmatrix} G_{t+1}-\overline{G} \\ \omega_{t+1}-\overline{\omega} \end{bmatrix} = \begin{bmatrix} 1-\theta\cdot\dfrac{(1+n)\cdot(1+\gamma)-1+\rho}{(1+n)\cdot(1+\gamma)}\cdot\overline{G} & \dfrac{h_\omega\cdot e^{\theta\cdot(1-\overline{G})}}{(1+n)\cdot(1+\gamma)}\cdot\overline{G} \\ \theta\cdot\dfrac{(1+n)\cdot(1+\gamma)-1+\rho}{(1+n)\cdot(1+\gamma)}\cdot\overline{\omega} & 1+\dfrac{f_\omega-h_\omega\cdot e^{\theta\cdot(1-\overline{G})}}{(1+n)\cdot(1+\gamma)}\cdot\overline{\omega} \end{bmatrix}\cdot\begin{bmatrix} G_t-\overline{G} \\ \omega_t-\overline{\omega} \end{bmatrix}$$

with $f_\omega < 0$ and $h_\omega > 0$.

The trace and determinant of the Jacobian matrix, *J*, are respectively,

$$Tr(J) = 2-\theta\cdot\frac{(1+n)\cdot(1+\gamma)-1+\rho}{(1+n)\cdot(1+\gamma)}\cdot\overline{G}+\frac{f_\omega-h_\omega\cdot e^{\theta\cdot(1-\overline{G})}}{(1+n)\cdot(1+\gamma)}\cdot\overline{\omega}$$

$$Det(J) = Tr(J)-1-\chi(J),\ \text{ with } \chi(J)=\theta\cdot\frac{(1+n)\cdot(1+\gamma)-1+\rho}{\left[(1+n)\cdot(1+\gamma)\right]^2}\cdot f_\omega\cdot\overline{G}\cdot\overline{\omega}<0.$$

Conditions for stability are: 1-*Det*(*J*)>0, 1-*Tr*(*J*)+*Det*(*J*)>0 and 1+*Tr*(*J*)+*Det*(*J*)>0. The second condition always holds, because 1-*Tr*(*J*)+*Det*(*J*)=-χ(*J*); this implies that the eigenvalues of *J* are both above the upper bound of the limit circle or they are both below 1. This result means that saddle-path can only prevail for an eigenvalue inside the unit circle and the other eigenvalue below -1.

We observe that 1-*Det*(*J*)=2-*Tr*(*J*)+χ(*J*), which can be a positive or a negative value [a Neimark-Sacker bifurcation occurs when *Tr*(*J*)-χ(*J*)=2; the particular case that is presented in the end of this section and the global analysis of the following section show that it is unlikely to find this type of bifurcation for reasonable parameter values]. Relatively to the last condition, note that 1+*Tr*(*J*)+*Det*(*J*)=2·*Tr*(*J*)-χ(*J*), which can be a positive or negative expression. A flip bifurcation occurs when 2·*Tr*(*J*)=χ(*J*); recall that

$\chi(J)<0$ but $Tr(J)$ can also be a negative value. The flip bifurcation occurs when one of the eigenvalues is equal to -1, and thus it separates a zone of stability from the unstable outcome region.

Synthesizing, computing trace and determinant of the Jacobian matrix one realizes that instability, saddle-path stability or a stable node outcome are all eventual results. Bifurcations are possible for one or both eigenvalues equal to -1. The eigenvalues can be, eventually, complex values, and if the modulus of the eigenvalues is equal to 1, a Neimark-Sacker bifurcation occurs ■

The result in proposition 3, although a generic one, gives little information about the constraints over parameters that have to be imposed to guarantee a given stability result. To explore further local stability properties, we recover the Cobb-Douglas particular case and impose the constraint $\mu=1$, that is, we exclude capital as a technology sector input; this means also that $u=1$. We also assume $n=0$ and $\gamma=0$. In this case, proposition 4 can be stated.

Proposition 4. For Cobb-Douglas production functions and absence of physical capital as an input in the R&D sector, a flip bifurcation occurs for the following combination of parameters: $\theta=\frac{2}{\rho}-\ln\left(\frac{g\cdot(1-v)}{\rho}\right)$. If $\theta<\frac{2}{\rho}-\ln\left(\frac{g\cdot(1-v)}{\rho}\right)$ a stable node characterizes the model's dynamics in the steady state vicinity, while if $\theta>\frac{2}{\rho}-\ln\left(\frac{g\cdot(1-v)}{\rho}\right)$ then saddle-path stability prevails. These results hold for $n=0$ and $\gamma=0$.

Proof: The linearized system in the proof of proposition 3 becomes now

$$\begin{bmatrix} G_{t+1}-\overline{G} \\ \omega_{t+1}-\overline{\omega} \end{bmatrix}=\begin{bmatrix} 1-\theta\rho\cdot\left[1+\frac{1}{\theta}\cdot\ln\left(\frac{g\cdot(1-v)}{\rho}\right)\right] & 0 \\ \theta\rho v\cdot\left(\frac{a}{\psi+\delta}\right)^{1/(1-\alpha)} & 1-(1-\alpha)\cdot(\psi+\delta) \end{bmatrix}\cdot\begin{bmatrix} G_t-\overline{G} \\ \omega_t-\overline{\omega} \end{bmatrix}$$

In this particular case, the eigenvalues of J can be directly displayed; they are simply the elements in the main diagonal, i.e., $\lambda_1 = 1-\theta\rho\cdot\left[1+\frac{1}{\theta}\cdot\ln\left(\frac{g\cdot(1-v)}{\rho}\right)\right]$ and $\lambda_2 = 1-(1-\alpha)\cdot(\psi+\delta)$. Clearly, $\lambda_1<1$ and $\lambda_2<1$. For both eigenvalues a bifurcation can theoretically be found at the lower bound of the unit circle, for given combinations of parameter values. However, for the second eigenvalue the bifurcation is very unlikely, because it would impose a level of consumption several times higher than the stock of accumulated capital. Thus, we take the second eigenvalue as remaining inside the unit circle.

Given the first eigenvalue's expression, a flip bifurcation occurs for $\lambda_1=-1$, condition from which the relation between parameters in the proposition is withdrawn; similarly, $\lambda_1<-1$ means instability (in the case, saddle-path stability, given the other eigenvalue possible value) and $\lambda_1>-1$ is the condition that allows for writing the expression in the proposition that translates the case of stability ■

Let us illustrate the previous result with a small example. Take the following reasonable values for parameters: $g=1$ and $v=0.25$. With these, one may draw the areas of stability and instability, as well as a bifurcation line, in the space of parameters (ρ,θ). Figure 2 identifies such areas. The bifurcation line is the representation of function $\theta=\frac{2}{\rho}-\ln\left(\frac{1}{4\rho}\right)$.

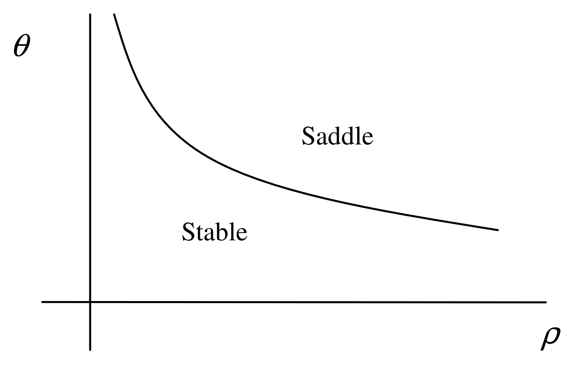

Figure 2 – Stability areas in the space of parameters (ρ,θ).

Figure 2 indicates that stability holds for a low level of either of the assumed parameters – the externality parameter and / or the rate of obsolescence of technology.

Only through severe constraints over parameter values we were able to extract explicit local stability results. Even in this circumstance, however, the results do not fully address the true properties of the dynamic system. This is because endogenous fluctuations are present, and the local analysis is unable to capture them. A global analysis is undertaken in the next section, considering various numerical examples and providing a graphical characterization of the dynamics.

4. Global Dynamics

The motivation for our discussion of positive and negative externalities in R&D resides, as explained in the introduction, in the possibility of arising endogenous fluctuations that, however, cannot be observed under a study of steady state vicinity properties. Therefore, we now engage in a discussion of global dynamic properties.

Let us begin by the simple example addressed in the final part of the previous section (take μ=1, u=1, n=0, γ=0, g=1 and v=0.25, as before; assume also a=1, α=0.25, ψ=0.5 and δ=0.05). For the selected parameter values we may begin by drawing a figure similar to figure 2, that analyzes stability in the space of parameters (ρ,θ); this new figure is presented in order to understand that local and global dynamics share the same stable node result [the area of stability (two eigenvalues inside the unit circle) is the same in both figures], but what locally is an area of saddle-path stability corresponds in global terms to an area of cycles with various periodicities and as we depart from the line of bifurcation, complete a-periodicity emerges. Figure 3 is drawn after withdrawing the first 1,000 transient observations, and considering any reasonable pair of initial values G_0, ω_0 (the basin of attraction for the system is a large area around the steady state, so that any reasonable initial values are feasible).[1]

[1] To draw figure 3, and all the following figures, we have used iDMC (interactive Dynamical Model Calculator). This is a free software program available at www.dss.uniud.it/nonlinear, and copyright of Marji Lines and Alfredo Medio.

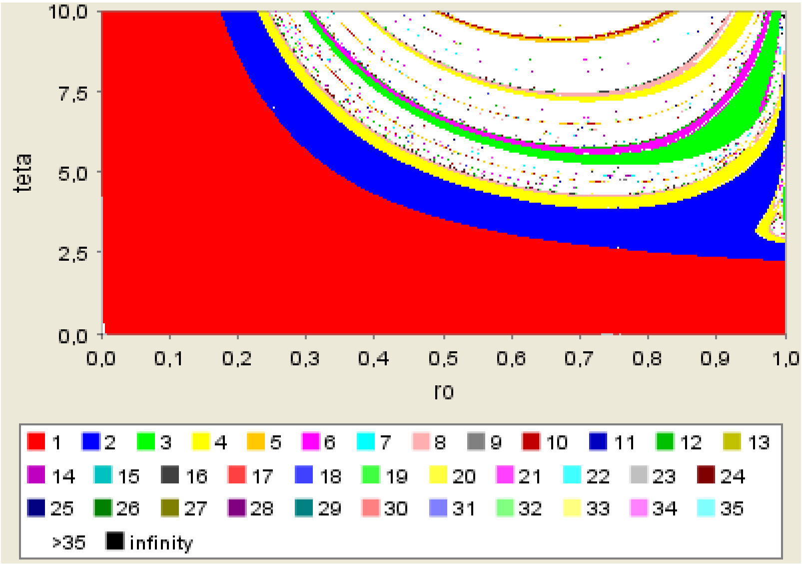

Figure 3 – Stability area and cycles in the space of parameters (ρ,θ), under a global dynamics point of view.

Bifurcation diagrams could be displayed for several of the assumed parameters. To illustrate the type of bifurcation that occurs, we take ρ=0.5 and let the externality parameter, θ, vary. We present bifurcation diagrams for both endogenous variables, and in both cases one observes that a kind of period doubling flip bifurcation gives rise to a zone of a-periodicity. Figures 4 and 5 are presented for 1,000 iterations and after excluding the first 1,000 observations.

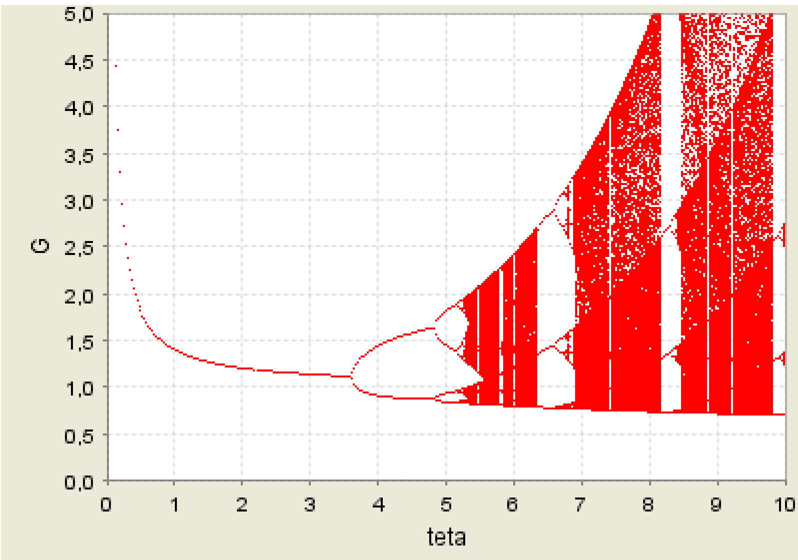

Figure 4 – Bifurcation diagram for variable G_t (with $0<\theta<10$).

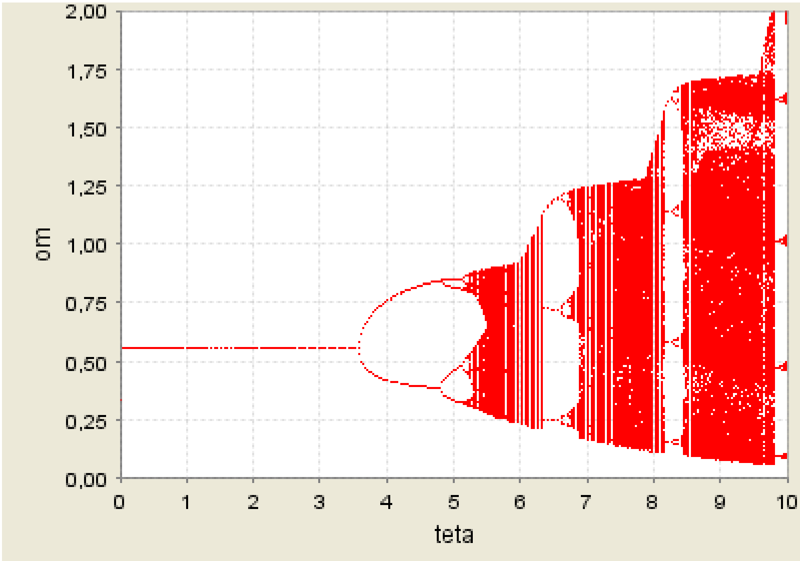

Figure 5 – Bifurcation diagram for variable ω_t (with $0<\theta<10$).

Figures 4 and 5 confirm the nonlinear nature of the model's dynamics for a specific value of the technology obsolescence parameter. Our main conclusion is that endogenous fluctuations effectively arise when the externality over R&D activities is considered. The constant values that characterize the steady state of the effective technology – potential technology ratio and of the capital - technology ratio, tend to give place, for strong externality effects, to fluctuations that indicate the presence of long term business cycles generated endogenously by the dynamics of the model.

To emphasize the previous results, one presents, in figures 6 to 8, the time paths of the endogenous variables and an attractor that describes the long run relation between the two variables. These figures are drawn for the set of parameters indicated above and assuming θ=7.5. The first 1,000 transients are excluded and the attractor in figure 8 is drawn with 100,000 observations.

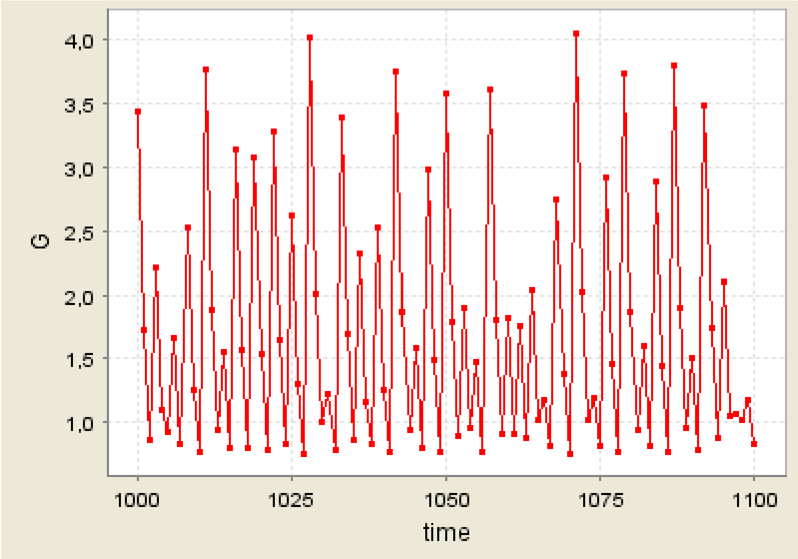

Figure 6 –Long run time path for variable G_t .

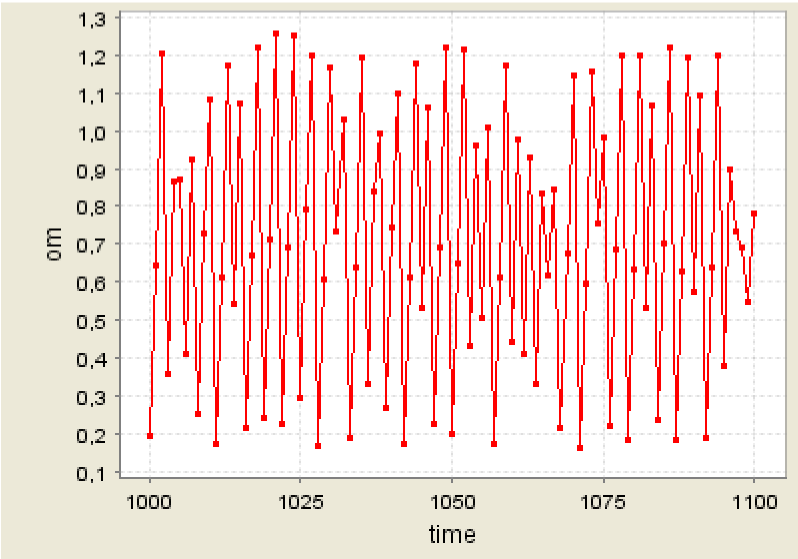

Figure 7 –Long run time path for variable ω_t .

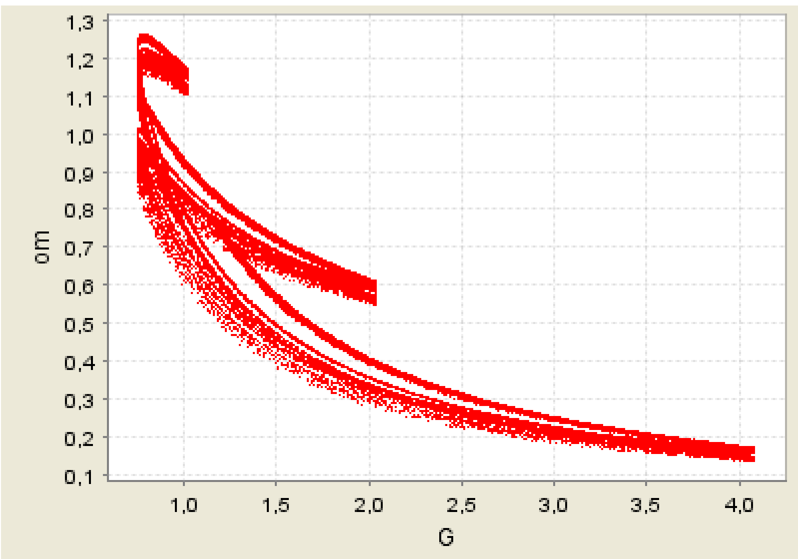

Figure 8 – Attractor (long run relation between G_t and ω_t) .

The presence of chaotic motion can be confirmed through the computation of Lyapunov characteristic exponents (LCEs). These are a measure of local average asymptotic exponential divergence of nearby orbits and the presence of at least one positive LCE implies sensitive dependence on initial conditions (SDIC). SDIC, in turn, can be interpreted as the lack of predictability of a dynamic system, which is an essential feature of chaotic behaviour.

Figure 9 takes, once again, $0<\theta<10$, and assumes the several benchmark values considered before, revealing that chaotic motion is indeed present for most of the values of the externality parameter above 5.

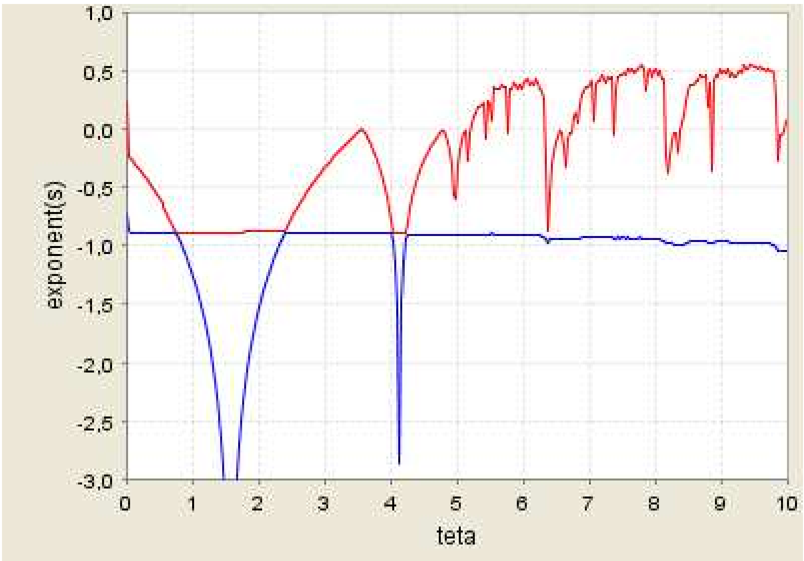

Figure 9 – Lyapunov characteristic exponents ($0<\theta<10$).

The previous graphical analysis reveals that in our simplest case, where population and the benchmark level of technology do not grow and where no capital is used in the R&D sector, endogenous fluctuations are found. This particular result can be generalized for many other combinations of parameter values. We present just one more case to emphasize that endogenous fluctuations are a common outcome of the proposed theoretical framework.

Consider now μ=0.25, u=0.75, n=0.02 and γ=0.05. The other parameter values remain as in the first example. Figures 10 to 15 refer to the same graphical analysis as before: stability in the parameters space, bifurcation diagrams, long run time trajectories and an attractor are drawn under the same assumptions (namely, that 1,000 transient observations are taken into account).

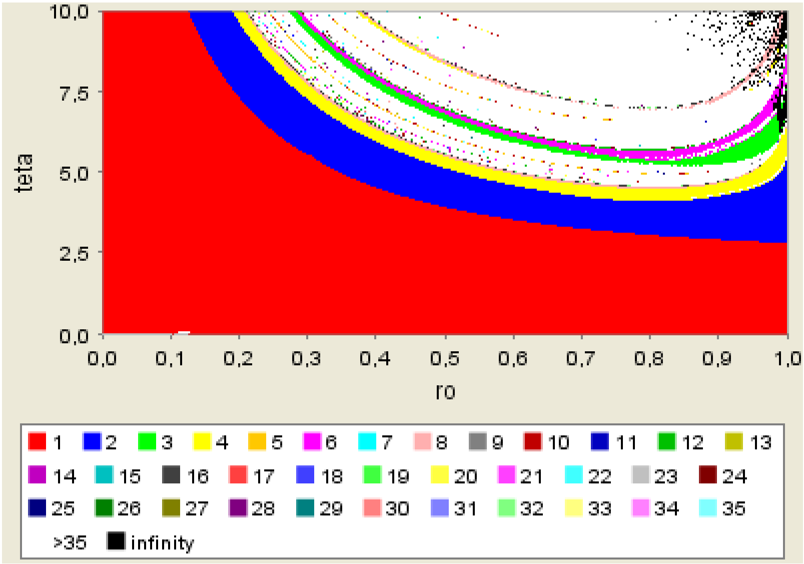

Figure 10 – Stability area and cycles in the space of parameters (ρ,θ), under a global dynamics point of view (example 2).

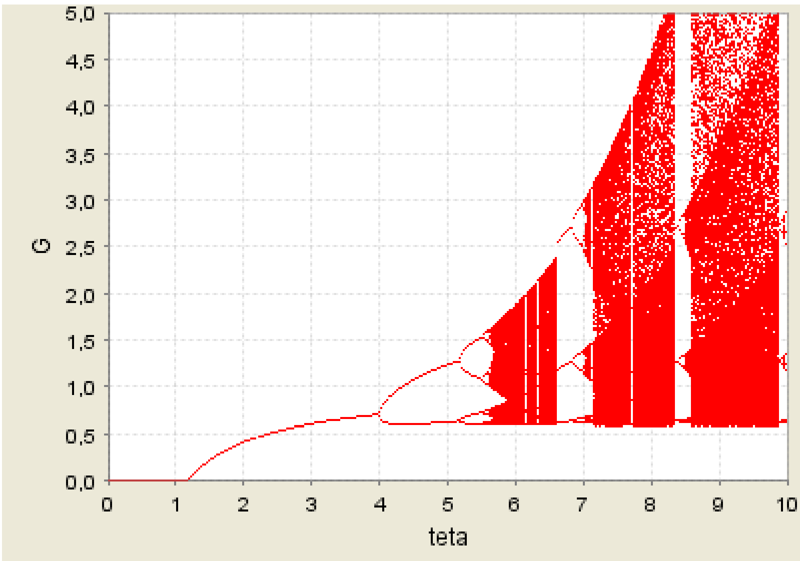

Figure 11 – Bifurcation diagram for variable G_t, with $0<\theta<10$ (example 2).

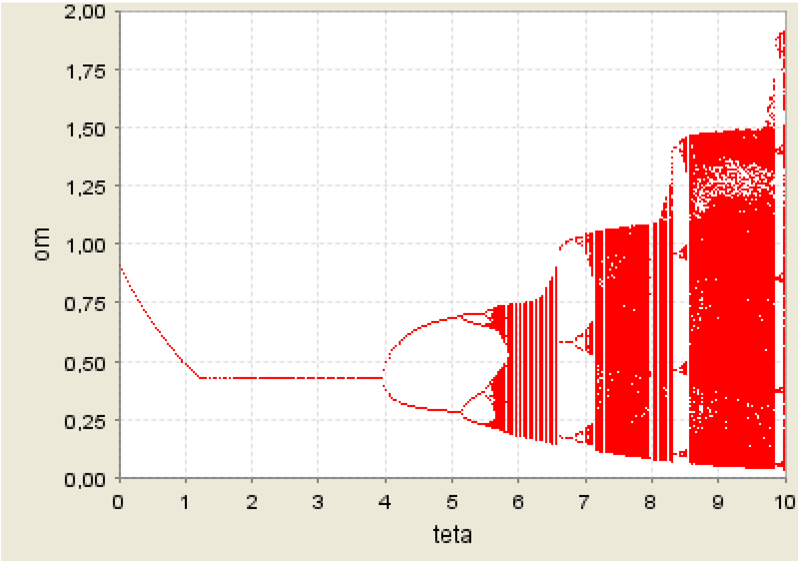

Figure 12 – Bifurcation diagram for variable ω_t, with $0<\theta<10$ (example 2).

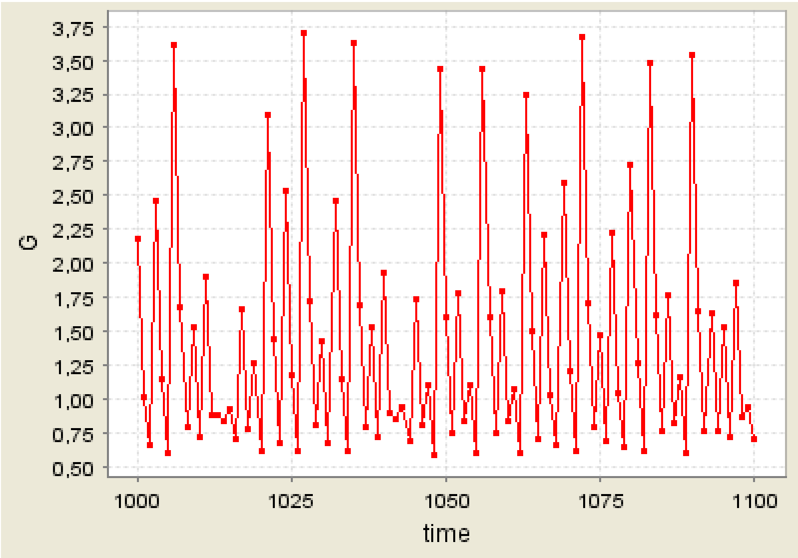

Figure 13 –Long run time path for variable G_t (example 2).

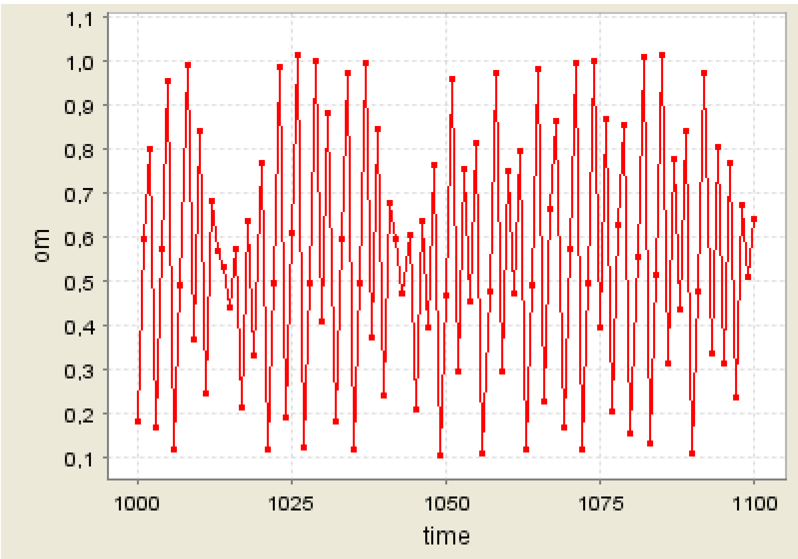

Figure 14 –Long run time path for variable ω_t (example 2).

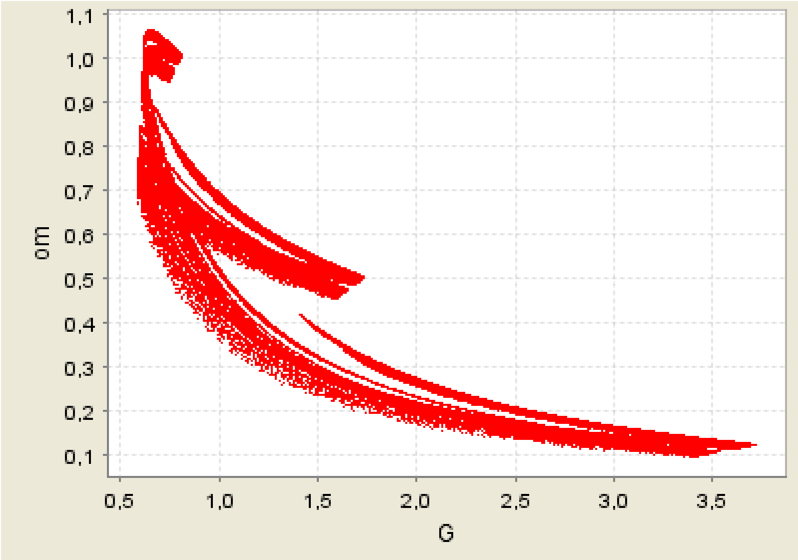

Figure 15 – Attracting set (example 2).

Comparing the two sets of figures, relating to each of the examples, one encounters no significant differences, and thus it seems reasonable to conclude that, qualitatively, the results found for the case of no physical capital in the production of technology and absence of population growth and socially optimal technological progress can be considered identical to the ones found for the scenario where physical capital is an input of the R&D sector and where the two rates n and γ grow positively through time.

We leave one final note regarding global dynamics. We have defined G_t and ω_t as constant long run values; as observed, these are not necessarily constant after the fixed point giving place to a series of bifurcations inducing endogenous fluctuations. Thus, in reality, the original variables A_t and k_t will not grow at the constant rate γ, for the combinations of parameters implying endogenous cycles. In this case, the steady state will be given by A_t=B_tG_t and k_t=$A_t\omega_t$, where B_{t+1}=$(1+\gamma)\cdot B_t$ and G_t and ω_t are subject to fluctuations in the conditions described above. To illustrate that A_t and k_t grow at a rate around γ (the model is an endogenous growth setup), but do not grow exactly at rate γ (the model is an endogenous fluctuations setup), we present figures 16 and 17. Note that the time series of A_t is much more volatile then the time series of k_t, what is not a surprising result given that it is variable A_t that is directly influenced by the externality effect.

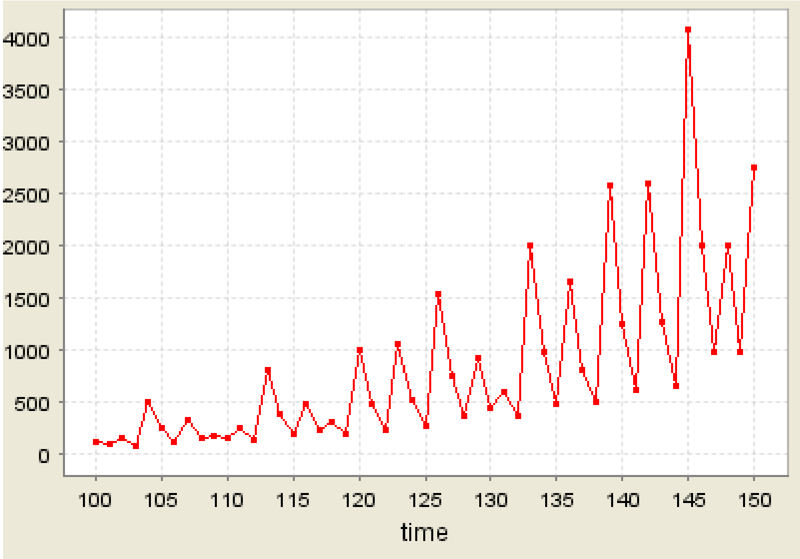

Figure 16 –Long run time trajectory for variable A_t (the parameter values are the ones in the second example; the trajectory is drawn for the 50 observations after the first 100).

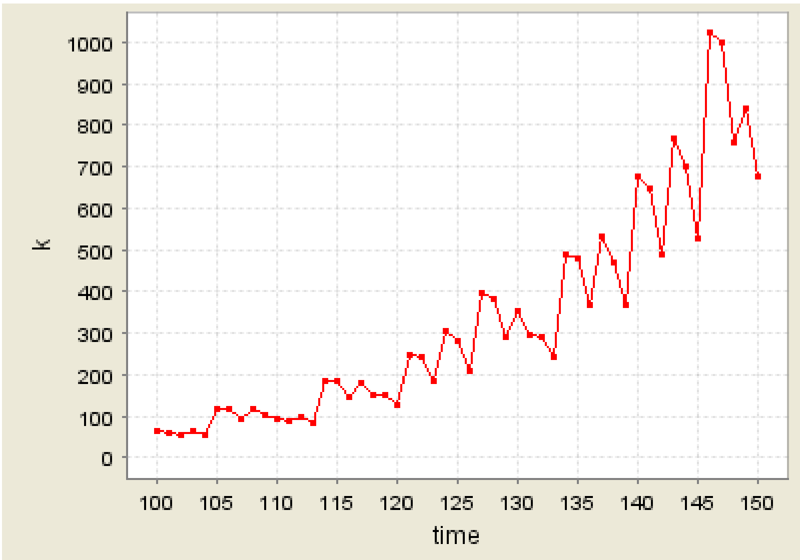

Figure 17 –Long run time trajectory for variable k_t (the parameter values are the ones in the second example; the trajectory is drawn for the 50 observations after the first 100).

5. Conclusions

Business cycles imply some kind of push and pull mechanism that is hard to attach to a competitive market clearing framework. The standard optimal growth model with decreasing marginal returns and constant returns to scale is unable to reveal the existence of business cycles (at least for reasonable parameter values), and, therefore, to capture these, one has to search for market inefficiencies that change the notion of perfect allocation of resources. The literature as pointed to some candidate sources of perturbation over the optimal growth paradigm, namely technological shocks that generate exogenous fluctuations and final goods production positive externalities that are able to reveal endogenous fluctuations.

We have identified and explored one additional potential source of endogenous business cycles. Externalities affecting the pace of technological progress are assumed, and these are either positive or negative externalities, depending on the confrontation between the effective level of technology and the amount of R&D the society is able to accept in each time moment. If one realizes that technology levels can evolve to a point in which the society and the economic system are not prepared to deal with them, then a negative externality arises, which can be thought as symmetric to the positive externality that is associated to a research sector where investment is typically below optimal social levels.

The decentralized economy has in this way the power to create cycles, and their magnitude and extent are associated essentially to the values of several parameters on the technical progress equation, namely the externality parameter and the rate of technological obsolescence. The cycles arise through a flip bifurcation. A period doubling process leads from a stability outcome, for low levels of the cited parameters, to chaotic motion that arises for relatively high levels of technology obsolescence and strong externality effects.

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