Calibration of the subdiffusive arithmetic Brownian motion with tempered stable waiting-times

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Abstract

In the classical analysis many models used to real data description are based on the standard Brownian diffusion-type processes. However, some real data exhibit characteristic periods of constant values. In such cases the popular systems seem not to be applicable. Therefore we propose an alternative approach, based on the combination of the popular arithmetic Brownian motion and tempered stable subordinator. The probability density function of the proposed model can be described by a Fokker-Planck type equation and therefore it has many similar properties as the popular arithmetic Brownian motion. In this paper we propose the estimation procedure for the considered tempered stable subdiffusive arithmetic Brownian motion and calibrate the analyzed process to the real financial data.
Key words: Subdiffusion Tempered stable distribution Calibration

1 Introduction

Anomalous behavior characterized through constant time periods (called also trapping events) is observed in variety of physical systems, including charge carrier transport in amorphous semiconductors [29, 28, 25], transport in micelles [24], intracellular transport [4], motion of mRNA molecules inside E. coli cells [9], for a review including discussion of different applications see [5]. This specific behavior is also typical for some financial data especially corresponding to interest rates and stock prices, for which the constant time periods occur when the liquidity of the analyzed assets is low [12]. Description and modeling of such systems require the appropriate mathematical tools that correspond to fundamental physical laws and capture the most significant properties of the phenomena. Such tools as generalized Langevin equations, fractional Fokker-Planck-type equations (FFPEs) or fractional Brownian motion are well-known and usually applicable to anomalous diffusions, [5, 31, 21, 16].

In the domain of subdiffusion the typical approach is based on continuous time random walk (CTRW), [29, 22], and subordinated Lévy processes as a limit in distribution of CTRW, [20, 17]. The key issue in the framework of CTRW as well as in subordination technique is the waiting-times distribution corresponding to periods of constant values in which a test particle is immobilized. Let us note, that a family of nonnegative infinitely divisible (ID) distributions is enough rich to capture waiting-times distributions appearing in real physical systems, [18]. In the class of the ID distributions of the special importance are one-sided Lévy stable, Pareto, gamma, Mittag-Leffler, and tempered stable distributions.

Especially tempered stable distributions are the most appropriate in modeling of waiting-times in intermediate case between sub and normal diffusion, [32, 8]. Moreover, it is worth noticing that the tempered stable distributions have many interesting properties (i.e. finite moments), but simultaneously they remain close to the purely α-stable distribution, [27]. The tempered stable distributions have found many practical applications for instance in finance [15, 14], biology [11], physics to description of anomalous diffusion and relaxation phenomena [32, 8], turbulence [7] and in plasma physics [13], see also [30, 6]. Some physical systems that also demonstrate subdiffusive behavior at short time, and normal (Gaussian) at long times are analyzed in [3, 26].

In this paper we consider the model based on the combination of the classical arithmetic Brownian motion (ABM) and tempered stable subordinator. Let us mention that the extended model based on subordinated ABM with general ID subordinator was recently applied to option pricing, [19]. The considered subdiffusive ABM with tempered stable waiting-times capture the aforementioned property, i.e. it demonstrates the subdiffusive behavior for small time scale and Gaussian for large times. Moreover, it is based on the classical ABM therefore it is not complicated from the theoretical point of view. For these reasons, the presented practical methods of data analysis, especially parameters estimation, can be easily applied to the real data. In this paper we overview the main properties of the considered ABM with tempered stable waiting-times. As a main result we present in details the estimation procedure for the considered process and additionally describe the simple method of distinction between strictly $\alpha-$stable and tempered stable distribution of the subordinator. In order to demonstrate theoretical results we calibrate the subdiffusive ABM with tempered stable waiting-times to the real financial data. The similar considerations for Brownian diffusion with purely α-stable subordinator are presented in [12, 23].

The paper is scheduled as follows. In Section 2, we recall the construction of the subordinated ABM with tempered stable waiting-times. The estimation procedure for considered process is presented in details in Section 3. The practical applications of theoretical results are presented in Section 4 to the real financial data, i.e. United States Government Bonds (Inflation-Indexed 3.875%, Yield) from the period 09.04.1999-03.09.2008. Last Section contains conclusions.

2 The arithmetic Brownian motion with tempered stable waiting-times

Let us consider the arithmetic Brownian motion with tempered stable waiting-times, i.e. process $\{Y(t)\}$ defined as follows [18]:

$$Y(t) = X(S_{\alpha,\lambda,c}(t)), \tag{2.1}$$

where $\{X(\tau)\}$ is ABM with parameters μ and σ, represented by the following stochastic differential equation:

$$dX(\tau) = \mu d\tau + \sigma dB(\tau). \tag{2.2}$$

The inverse subordinator $\{S_{\alpha,\lambda,c}(t)\}$, called inverse tempered stable subordinator, is defined as follows, [18, 8]:

$$S_{\alpha,\lambda,c}(t) = \inf\{\tau > 0 : T_{\alpha,\lambda,c}(t) > \tau\}, \tag{2.3}$$

where $\{T_{\alpha,\lambda,c}(t)\}$ is a Lévy process with tempered stable increments and Laplace transform given by, [32]:

$$E\left(e^{-uT_{\alpha,\lambda,c}(t)}\right) = e^{-t\Psi(u)} = e^{-tc((\lambda+u)^{\alpha}-\lambda^{\alpha})}, \tag{2.4}$$

where $\lambda > 0$, $0 < \alpha < 1$, $c > 0$. When $\lambda = 0$, then the Lévy process $\{T_{\alpha,0,c}(t)\}$ becomes simply $\alpha-$stable with the scale parameter $c^{1/\alpha}$. We assume the processes $\{X(\tau)\}$ and $\{T_{\alpha,\lambda,c}(t)\}$ are independent. The probability density function of $\{T_{\alpha,\lambda,c}(t)\}$ can be expressed in the following form:

$$\tilde{p}_{\alpha,\lambda,c}(x,t) = e^{-\lambda x + c\lambda^{\alpha} t} p_{\alpha,\sigma,1,0}(x,t), \tag{2.5}$$

where $\sigma = \left(c * cos\frac{\pi\alpha}{2}\right)^{1/\alpha}$ and $p_{\alpha,\sigma,\beta,\mu}(x,t)$ is a probability density function of the α-stable Lévy motion with the index of stability α, scale parameter σ, skewness β and shift μ, [1].

On Fig. 1 we present an exemplary sample path of the inverse tempered stable subordinator $S_{\alpha,\lambda,c}(t)$, the arithmetic Brownian motion $X(\tau)$ and the tempered subdiffusion process $Y(t) = X(S_{\alpha,\lambda,c}(t))$. Let us recall, that the constant periods of trajectories of subdiffusion process $\{Y(t)\}$ correspond to the waiting-times that are distributed according to the tempered stable law.

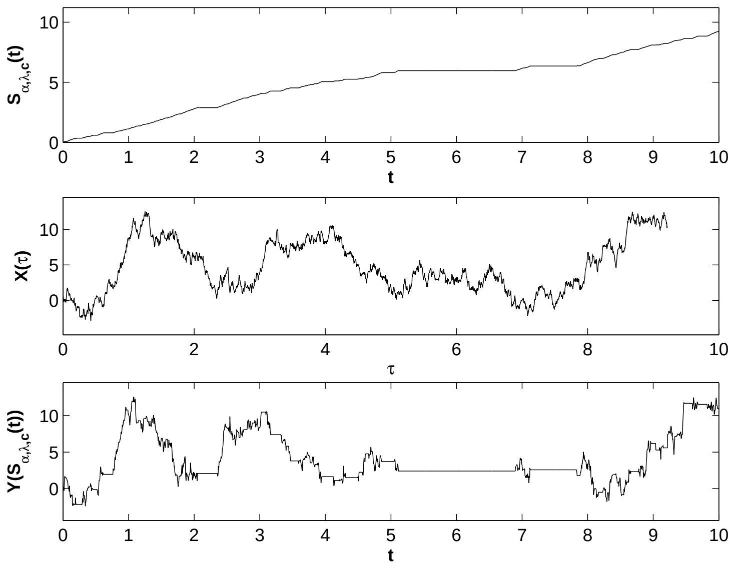

Figure 1: An exemplary sample path of the inverse tempered stable subordinator $S_{\alpha,\lambda,c}(t)$ (top panel), the classical arithmetic Brownian motion $X(\tau)$ (middle panel) and the subdiffusion process $Y(t) = X(S_{\alpha,\lambda,c}(t))$ (bottom panel). The parameters of the subdiffusive process are: $\alpha = 0.85$, $\lambda = 0.05$, $c = 0.05$, $\mu = 1$ and $\sigma = 5$.

Main properties and the simulation procedure for the process $\{Y(t)\}$ one can find in [18, 8, 19]. We only mention here that the probability density function of the process $\{Y(t)\}$ satisfies the following generalized fractional Fokker-Planck equation:

$$\frac{\delta w(x,t)}{\delta t} = \left[-\mu\frac{\delta}{\delta x} + \frac{\sigma^2}{2}\frac{\delta^2}{\delta x^2}\right]\Phi w(x,t), \tag{2.6}$$

where the operator Φ is defined as:

$$\Phi f(t) = \frac{d}{dt}\int_0^t M(t-y)f(y)dy.$$

According to [18], the memory kernel $M(t)$ is defined via its Laplace transform

$$\int_0^\infty e^{-ut}M(t)dt = \frac{1}{\Psi(u)} = \frac{1}{c((\lambda+u)^\alpha - \lambda^\alpha)}. \tag{2.7}$$

Let us point out that in case $\lambda \to 0$ the operator Φ is proportional to the fractional Riemann-Liouville derivative, therefore (2.6) tends to fractional Fokker-Planck equation. Using formula (2.7) we can obtain the form of the memory kernel $M(t)$:

$$M(t) = \frac{e^{-\lambda t} t^{\alpha-1}}{c} E_{\alpha,\alpha}((\lambda t)^{\alpha}), \tag{2.8}$$

where

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

is a generalized Mittag-Leffler function, [10].

3 Estimation procedure

The estimation procedure for parameters of the ABM with tempered stable waiting-times is based on the fact that lengths of constant time periods observed in a real data set are realizations of independent identically distributed (i.i.d.) random variables that are distributed according to the tempered stable law and the process that arises after removing the trapping events is the classical ABM. For detailed explanation of this algorithm in case of purely α-stable subordinator see [12, 23]. To estimate the parameters we apply the following scheme:

(a) From the data set $Y_1, Y_2, ..., Y_n$ determine the length of the constant time periods $T_1, ..., T_k$ that constitute i.i.d. random variables from tempered stable distribution. For simplicity we assume the parameter $c = 1$. Therefore the Laplace transform of the random variable T_i is given by

$$E\left(e^{-uT_i}\right) = e^{\lambda^{\alpha} - (\lambda+u)^{\alpha}},\ i = 1, 2, ..., k.$$

(b) Estimate the α and λ parameters from the sample $T_1, ..., T_k$ obtained in point (a). This can be done by using the method of moments that in this case proceeds as follows.

Let us consider the cumulant generating function K that is defined as:

$$K(u) = log\left(Ee^{uT_i}\right) = \lambda^{\alpha} - (\lambda - u)^{\alpha}.$$

The cumulants $c_m = E(T_i - c_1)^m$ (for each $i = 1, 2, ..., k$ and $m = 1, 2, 3$) we can obtain computing $m-$th derivative of K function in point $u = 0$:

$$c_1 = E(T_i) = \alpha\lambda^{\alpha-1},$$

$$c_2 = Var(T_i) = -\alpha(\alpha - 1)\lambda^{\alpha-2},$$

$$c_3 = E(T_i - c_1)^3 = \alpha(\alpha-1)(\alpha-2)\lambda^{\alpha-3}.$$

Therefore we obtain

$$\alpha = 1 + \frac{c_2^2}{c_2^2 - c_1c_3},\ \lambda = \frac{(1-\alpha)c_1}{c_2} = \frac{c_1c_2}{c_1c_3 - c_2^2}.$$

Using method of moments (i.e. replacement of the theoretical central moments c_m by the empirical ones) we obtain the formulas for estimators:

$$\hat{\alpha} = 1 + \frac{\hat{c}_2^2}{\hat{c}_2^2 - \hat{c}_1\hat{c}_3},\ \hat{\lambda} = \frac{\hat{c}_1\hat{c}_2}{\hat{c}_1\hat{c}_3 - \hat{c}_2^2},$$

where $\hat{c}_m$ is an empirical central $m-$th moment ($m = 1, 2, 3$) calculated on the basis of the vector $(T_1, T_2, ..., T_k)$, i.e.

$$\hat{c}_1 = \frac{1}{k}\sum_{i=1}^{k} T_i,\ \hat{c}_m = \frac{1}{k}\sum_{i=1}^{k}(T_i - \hat{c}_1)^m \text{ for } m = 2, 3.$$

(c) After removing the constant time periods we obtain the realization of the classical ABM $\{X(\tau)\}$. The parameters μ and σ of the ABM we estimate by using discrete version of equation (2.2), i.e.

$$X(\tau) - X(\tau-1) = \mu + \sigma Z(\tau),\ \tau = 1, 2, ...$$

where $\{Z(\tau)\}$ is a sequence of i.i.d. random variables with standard normal distribution. Therefore the estimator $\hat{\mu}$ is equal to the mean of the differenced (with order 1) series $\{X(\tau)\}$ while the estimator $\hat{\sigma}$ is equal to the empirical standard deviation of the differenced (with order 1) series $\{X(\tau)\}$.

4 Applications

In this Section we consider the real data set of United States Government Bonds (Inflation-Indexed 3.875%, Yield) expressed in USD from the period 09.04.1999-03.09.2008 (2350 observations). Let us notice, the data demonstrate characteristic trap-behavior typical for the subdiffusive processes, see Fig. 2.

In the first step of our analysis we divide considered data into two sets: the first one represents lengths of the observed trapping events (DATA1), while the second describes the data after removing the constant time periods (DATA2). According to the theoretical results, DATA1 constitute the length of constant time periods of the inverse subordinator, that is a significant component of the data, since lengths of traps have large values and number of trapping events is 436, see Fig. 3. For

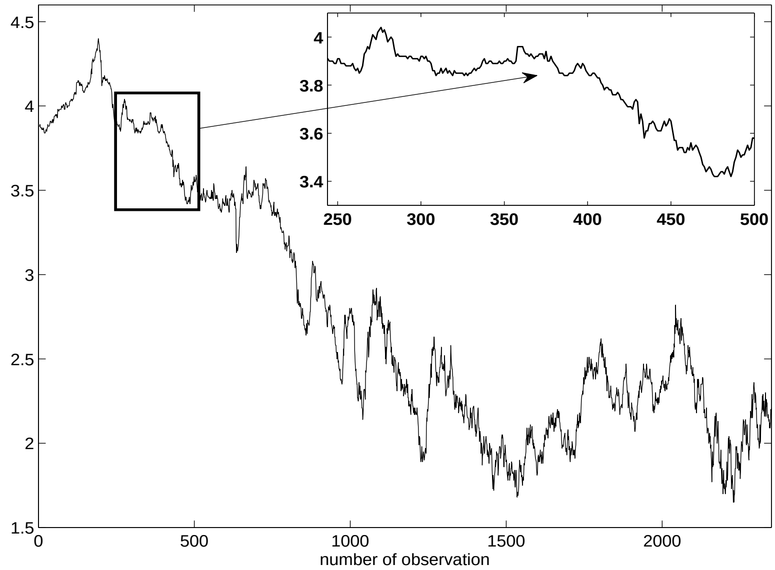

Figure 2: The examined real data set of United States Government Bonds (Inflation-Indexed 3.875%, Yield) expressed in USD. The considered period is 09.04.1999-03.09.2008. The data demonstrate characteristic trap-behavior typical for the subdiffusive processes.

detailed description of the algorithm, see [23]. Moreover, tail behavior of DATA2 (without constant time periods) is close to the Gaussian case because the estimated parameter of stability, so called tail index, calculated by using the McCulloch, regression and moments methods, [2], is close to 2. For these reasons, for the considered financial data we propose to use the subdiffusibe ABM.

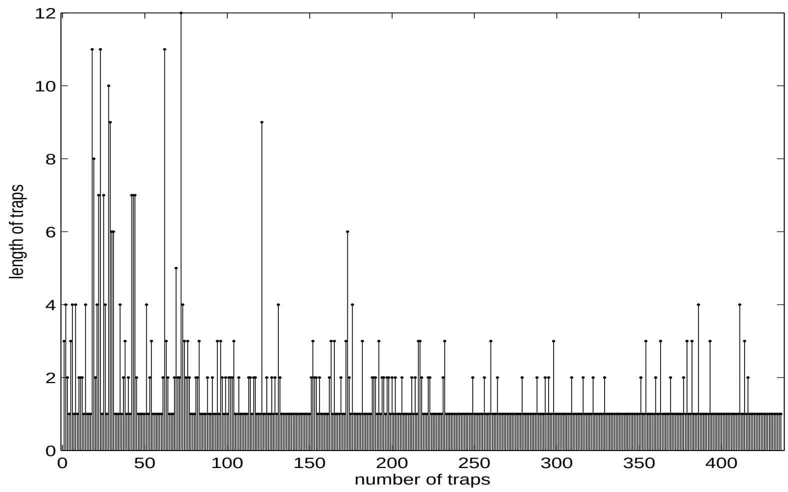

Figure 3: The lengths of constant time periods (DATA1). Constant periods are defined here as a number of days in which the changes of the bond prices are smaller than 0.01.

Firstly, let us examine the hypothesis, that the purely $\alpha-$stable distribution (i.e. tempered stable with $\lambda = 0$) better describes waiting-times behavior than the tempered stable one. To this end, from DATA1 we estimate, by using the McCulloch, regression and moments methods, the α parameter under the assumption that lengths of traps constitute i.i.d. random variables from strictly $\alpha-$stable distribution. All the estimation methods return $\hat{\alpha} > 1$, that contradicts the main assumption about the purely α-stable distribution of the subordinator (in this case the α parameter should be between 0 and 1). Therefore, as an alternative, we propose the tempered stable distribution. By using the estimation procedure presented in the previous Section we obtain the following values of α and λ estimators:

$$\hat{\alpha} = 0.6786, \ \hat{\lambda} = 0.2203.$$

According to our assumption, DATA2 (that arises after removing the trapping events) represents the classical ABM. Therefore, the estimators $\hat{\mu}$ and $\hat{\sigma}$ we calculate as the empirical mean and the standard deviation, respectively, of the differenced (with order 1) series DATA2. We obtain the following values:

$$\hat{\mu} = -0.001, \ \hat{\sigma} = 0.0443.$$

5 Conclusions

In this paper we have examined the subdiffusive ABM with tempered stable waiting-times that is the most appropriate in intermediate case between sub- and normal diffusion. As a main result we have presented the estimation scheme for parameters of the considered process. To distinguish between models with purely α-stable subordinator and tempered stable one, we have proposed a simple method based on analysis of the estimated (from data corresponding to constant time periods) index of stability. In order to present the motivation of the paper and the theoretical results we have calibrated the subordinated ABM with tempered stable subordinator to the real financial data.

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