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# Consumer Confidence, Endogenous Growth and Endogenous Cycles

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**Abstract:** Endogenous growth models are generally designed to address long term trends of growth. They explain how the economy converges to or diverges from a balanced growth path and they characterize aggregate behaviour given the optimization problem faced by a representative agent that maximizes consumption utility. In such frameworks, only potential output matters and all decisions, by firms and households, are taken assuming that any output gap does not interfere with the agents' behaviour. In this paper, we develop growth models (without and with optimization) that depart from the conventional framework in the sense that consumption decisions take into account output fluctuations. Households will raise their propensity to consume in periods of expansion and they will lower it in phases of recession. Such a framework allows to introduce nonlinear features into the model, making it feasible to obtain, for reasonable parameter values, endogenous fluctuations. These are triggered by a Neimark-Sacker bifurcation.

**Keywords:** Endogenous growth, Endogenous business cycles, Nonlinear dynamics, Neimark-Sacker bifurcation.

**JEL classification:** O41, E32, C61.

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## 1. Introduction

Standard growth models commonly overlook any possible reaction of households relatively to short run economic performance (i.e., to business cycles). Such growth models are long term paradigms, where a permanent coincidence between effective output and potential output is implicitly assumed. In this paper, we reinterpret the conventional *AK* endogenous growth model when this is modified to include consumers' response to previous periods' deviations of output relatively to its potential level. This response relates to a simple mechanism that involves confidence: when the output gap in the previous periods is systematically positive, demand side agents become increasingly confident, and they will consume an amount of resources that is tendentially higher than the consumption level derived from the benchmark growth models (the optimal consumption level of a Ramsey-like setup or the consumption level that arises from assuming a constant marginal propensity to save); if, alternatively, the observed output gap in the near past corresponds to negative value, then the contemporaneous level of consumption falls below the reference level.

The described mechanism intends to add realism to the simple growth model. It is well accepted by the economics profession that households in fact take into account short term macroeconomic fluctuations in order to plan their income allocation decisions. Links between consumer confidence and business cycles have been extensively reported in the empirical literature: for instance, McNabb and Taylor (2002) find evidence of causality between GDP movements and consumer confidence indexes, for several of the most important economies in Europe (UK, France, Italy and the Netherlands). A similar conclusion is highlighted by Goh (2003) for the economy of New Zealand; this author, in particular, states that consumer confidence reflects current economic conditions, which confirms the reasonability of our assumption: households are influenced by the perceived macro performance and will adopt a more or less enthusiastic attitude towards consumption accordingly.

Some authors remark that the consumer sentiment is often biased and does not reflect exactly the true amplitude of business cycles [it is the case of Souleles (2004), who studies consumer confidence for the American state of Michigan]; nevertheless, even when the extent of the relation between the cycle and consumers sentiment and attitude is questionable, it seems unreasonable to drop completely this relation as it happens in most of the contemporaneous growth analysis.

Note that our argument can be separated in two causal relations: first, cycles influence consumers' confidence; second, confidence automatically generates a reaction in terms of the relative level of consumption out of income. The second relation is even less subject to doubt than the first. Studies like Bram and Ludvigson (1998), Souleles (2004) and Dion (2006) clearly reveal that higher confidence is related to lower savings, given the logical argument that increases in expected future resources reduce the strength of the precautionary motive to save. However, some other authors, like Croushore (2004) have difficulty in finding a statistically significant relation between the measured degree of consumer optimism and effective levels of consumption. Even though the evidence on increasing confidence regarding short run aggregate performance cannot always confirm a direct correlation with rising consumption shares out of income, this is an intuitive relation that is reasonable to include in a theoretical framework that aims at combining the evidence on cyclical movements with an explanation of long run growth.

Back to our modelling setup, we should stress that the simple additional assumption that we introduce provokes relevant changes over the way one understands the growth process. This is no longer invariably materialized in a steady state positive constant growth rate that remains unchanged unless some exogenous disturbance occurs; instead, the response of the representative consumer to deviations from potential output might imply, for reasonable parameter values (e.g., technology level, savings rate or discount factor), everlasting fluctuations in the growth rate of the main economic aggregates. Essentially, one may infer from the analysis that business cycles are, under certain circumstances, self sustained, i.e., because deviations from the observable growth trend do exist, households will modify their behaviour, and these systematic changes on behaviour induce cycles to persist, originating a process that tends to repeat itself endlessly.

The analysis we develop may be associated with the literature on endogenous business cycles (EBC), a strand of thought that justifies economic fluctuations through assumptions that imply nonlinear modelling structures, which are able to generate long term cyclical behaviour that commonly arises after some type of bifurcation (that is provoked by a change in a parameter value). This literature goes back to the influential work by Stutzer (1980), Benhabib and Day (1981), Day (1982), Grandmont (1985), Boldrin and Montrucchio (1986) and Deneckere and Pelikan (1986), among others. These authors saw on basic nonlinear mathematical models (like the logistic map) a fruitful field to explore endogenous fluctuations associated with growth processes.

Responding to the real business cycles (RBC) theory, these authors found a way to conciliate into a same theoretical structure business cycles and growth. The main criticism relating to these first approaches to endogenous cycles had to do essentially with the somehow unreasonable hypotheses that were underlying the theoretical structures; it seemed that nonlinearities were not a consequence of economic assumptions, but the other way around: the need for nonlinearities forced some questionable assumptions.

Recently, various routes to endogenous cycles have been explored. The work by Brock and Hommes (1997) constitutes a fundamental reference because it has initiated a great deal of discussion on deterministic fluctuations. This work has inspired relevant contributions, mainly in what concerns financial analysis [Brock and Hommes (1998), Gaunersdorfer (2000), Lux and Marchesi (2000), Chiarella, Dieci and Gardini (2002), Chiarella and He (2003), Westerhoff (2004), De Grauwe and Grimaldi (2005), Hommes, Sonnemans, Tuinstra and van de Velden (2005) are relevant examples of this extensive literature]. We find as well relevant contributions with the same inspiration relating real analysis, as it is the case of the work by Goeree and Hommes (2000) and Onozaki, Sieg and Yokoo (2000, 2003).

The previous references relate to endogenous cycles generated by agent heterogeneity, but the most growth oriented reflections on endogenous fluctuations continue to address a scenario of representative agent. These contributions can be mainly separated in two groups. First, we find the overlapping generations analysis of economies with production technologies subject to increasing returns; this analysis comes in the tradition of Grandmont (1985) and has been developed by Cazavillan, Lloyd-Braga and Pintus (1998), Aloï, Dixon and Lloyd-Braga (2000), Cazavillan and Pintus (2004) and Lloyd-Braga, Nourry and Venditti (2006), among others. The second approach is also based on the presence of production externalities that generate increasing returns to scale, but this takes the optimization setup of the RBC models (i.e., an utility maximization control problem, with consumption and leisure as arguments of the utility function). In this respect, it is worth to mention the work by Christiano and Harrison (1999), Schmitt-Grohé (2000), Guo and Lansing (2002), Goenka and Poulsen (2004) and Coury and Wen (2005), among others.

There are other approaches to endogenous fluctuations in aggregate economic models that deserve to be mentioned. We refer only two additional contributions: the technical work by Nishimura, Sorger and Yano (1994), Nishimura and Yano (1994, 1995) and related papers, who study extreme conditions under which the competitive

growth scenario can generate long term non linear motion; and Cellarier (2006), who drops the optimal plan of conventional growth models and replaces it by a constant gain learning mechanism which is capable of producing endogenous fluctuations.

In the model developed along the following sections, which is likely to generate cyclical behaviour, fluctuations are triggered by a Neimark-Sacker bifurcation or Hopf bifurcation in discrete time. This is a type of bifurcation that fits well the economic data on business cycles, in the sense that the bifurcation induces a quasi-periodic movement (something between period cycles and chaotic motion) where several periods of expansion are followed by several periods of slower growth, which is similar to what real data time series reveal [see Dosi, Fagiolo and Roventini (2006) for a review of the main stylized facts concerning business cycles]. In fact, the importance of the link between the Hopf bifurcation (mostly in continuous time) and the inquiry about the nature of business cycles has been highlighted in the literature, as it happens with Semmler (1994), Asada and Semmler (1995) and Manfredi and Fanti (2004).

The remainder of the paper is organized as follows. Section 2 presents the general properties of the type of dynamic system we intend to approach; some definitions concerning nonlinearities, in the specific environment we consider, are set forth. Section 3 develops the endogenous growth model with consumer confidence on a scenario where no optimization by a representative consumer is assumed. Section 4 repeats the analysis of section 3 for a model with consumption utility maximization. In both sections, the local properties of the model are explored and global dynamics are discussed through a numerical / graphical analysis. Finally, section 5 concludes.

## 2. Useful definitions

The dynamic systems to consider in the following sections correspond to pairs  $(X, h)$ , with  $h$  a map defined in the state space  $X \subseteq \mathbb{R}$  (we assume that  $X$  is a non-empty and compact subset of  $\mathbb{R}$ ). The map defines the law of motion of a variable  $k_t \in X$ , with  $t=0, 1, 2, \dots$ , and the first  $k_i$  given ( $i=0, 1, 2, \dots, n$ ), with  $n$  some positive integer. This law of motion assumes the generic form  $k_{t+1} = h(k_t, k_{t-1}, \dots, k_{t-n})$ .

Let  $h^{(1)}(k_n, k_{n-1}, \dots, k_0)$  be the first iteration of  $h$ , and let  $h^{(t)}(k_n, k_{n-1}, \dots, k_0) = h \circ h \circ \dots \circ h^{(1)}(k_n, k_{n-1}, \dots, k_0)$  correspond to the iteration  $t$  of the map.

The class of models we propose takes the endogenous variable  $k_t$  as a variable that grows at a constant rate in the long run. Thus, we define a steady state or balanced growth path of the system as,

**Definition 1:** Consider that  $k_t$  grows at a constant rate in the long term, that is,  $\lim_{t \rightarrow +\infty} \left[ \frac{h^{(t+1)}(k_n, k_{n-1}, \dots, k_0)}{h^{(t)}(k_n, k_{n-1}, \dots, k_0)} - 1 \right] = \gamma$ , with  $\gamma \in \mathbb{R}$ . Let  $\hat{k}_t \equiv \frac{k_t}{(1+\gamma)^t}$ . A balanced growth path or steady state corresponds to the set  $E = \{\bar{k} \mid \bar{k} = h(\bar{k}, \bar{k}, \dots, \bar{k})\}$ , i.e., corresponds to a set of one or more positive constant values that are obtained through the dynamic system under the condition  $\bar{k} \equiv \hat{k}_{t+1} = \hat{k}_t = \hat{k}_{t-1} = \dots = \hat{k}_{t-n}$ .

The previous one-dimensional system can be transformed into a  $(n+1)$ -equations system and only one time lag in  $h(\cdot)$ . To obtain this system, consider variables  $\tilde{k}_t \equiv \hat{k}_t - \bar{k}$ ,  $\tilde{z}_{1,t} \equiv \tilde{k}_{t-1}$ ,  $\tilde{z}_{2,t} \equiv \tilde{z}_{1,t-1}$ , ...,  $\tilde{z}_{n,t} \equiv \tilde{z}_{n-1,t-1}$ . A new system, that includes  $n+1$  difference equations, arises:  $\tilde{k}_{t+1} = h(\tilde{k}_t + \bar{k}, \tilde{z}_{1,t} + \bar{k}, \tilde{z}_{2,t} + \bar{k}, \dots, \tilde{z}_{n,t} + \bar{k}) - \bar{k}$ ,  $\tilde{z}_{1,t+1} = \tilde{k}_t$  and  $\tilde{z}_{i,t+1} = \tilde{z}_{i-1,t}$ ,  $i=2, \dots, n$ . Note that the steady state of this system is the origin,  $(\bar{k}, \bar{z}_1, \bar{z}_2, \dots, \bar{z}_n) = (0, 0, 0, \dots, 0)$ .

We redefine the initial problem as system  $(X \times X \times \dots \times X, \mathbf{h})$ , that is,  $(X^{n+1}, \mathbf{h})$ , with the law of motion given by  $\mathbf{z}_{t+1} = \mathbf{h}(\tilde{k}_t, \tilde{z}_{1,t}, \tilde{z}_{2,t}, \dots, \tilde{z}_{n,t})$ , and where  $\mathbf{z}_{t+1} = [\tilde{k}_{t+1} \quad \tilde{z}_{1,t+1} \quad \tilde{z}_{2,t+1} \quad \dots \quad \tilde{z}_{n,t+1}]'$  and  $\mathbf{h}(\tilde{k}_t, \tilde{z}_{1,t}, \tilde{z}_{2,t}, \dots, \tilde{z}_{n,t}) = [h(\tilde{k}_t + \bar{k}, \tilde{z}_{1,t} + \bar{k}, \tilde{z}_{2,t} + \bar{k}, \dots, \tilde{z}_{n,t} + \bar{k}) - \bar{k} \quad \tilde{k}_t \quad \tilde{z}_{1,t} \quad \dots \quad \tilde{z}_{n-1,t}]'$ . For the new presentation of the system,  $\mathbf{h}^{(1)}(\tilde{k}_n, \tilde{z}_{1,n}, \tilde{z}_{2,n}, \dots, \tilde{z}_{n,n})$  represents its first iteration and, as before,  $\mathbf{h}^{(t)}(\tilde{k}_n, \tilde{z}_{1,n}, \tilde{z}_{2,n}, \dots, \tilde{z}_{n,n}) = \mathbf{h} \circ \mathbf{h} \circ \dots \circ \mathbf{h}^{(1)}(\tilde{k}_n, \tilde{z}_{1,n}, \tilde{z}_{2,n}, \dots, \tilde{z}_{n,n})$  relates to the iteration number  $t$ .

The trajectory of the endogenous variable and the orbit of the system can be defined as follows,

**Definition 2:**

The sequence  $\tau(\tilde{k}_n, \tilde{z}_{1,n}, \tilde{z}_{2,n}, \dots, \tilde{z}_{n,n}) = (\mathbf{h}^{(t)}(\tilde{k}_n, \tilde{z}_{1,n}, \tilde{z}_{2,n}, \dots, \tilde{z}_{n,n}))_{t=1}^{+\infty}$  represents the trajectory of  $k_t$ , as described by the evolution of  $\tilde{k}_t, \tilde{z}_{1,t}, \tilde{z}_{2,t}, \dots, \tilde{z}_{n,t}$  in the space of

motion (i.e., in time), starting from a given point  $(\tilde{k}_n, \tilde{z}_{1,n}, \tilde{z}_{2,n}, \dots, \tilde{z}_{n,n})$ . The orbit of the system can be formally presented as the set of points  $\omega(\tilde{k}_n, \tilde{z}_{1,n}, \tilde{z}_{2,n}, \dots, \tilde{z}_{n,n}) = \{(\tilde{k}, \tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_n) \mid (\tilde{k}, \tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_n) = \mathbf{h}^{(t)}(\tilde{k}_n, \tilde{z}_{1,n}, \tilde{z}_{2,n}, \dots, \tilde{z}_{n,n}), \text{ for some } t \geq 1\}$ . The orbit corresponds to the evolution of the system in the state space (i.e., the space of variables), starting from  $(\tilde{k}_n, \tilde{z}_{1,n}, \tilde{z}_{2,n}, \dots, \tilde{z}_{n,n})$ .

The map  $\mathbf{h}(\tilde{k}_t, \tilde{z}_{1,t}, \tilde{z}_{2,t}, \dots, \tilde{z}_{n,t})$  might be a nonlinear map (the assumptions underlying the growth models to discuss afterwards produce such kind of map); this means that its underlying dynamics are tendencialy morphologically rich, i.e., it can give place to cyclical or complex trajectories for the assumed endogenous variable. Hence, one should emphasize that dynamic results are not necessarily limited to fixed point stability or instability; cycles of different periodicities, a-periodicity and chaotic motion may as well reflect the behaviour over time of variable  $k_t$ , depending on the specification of  $h(\cdot)$ .

Note that we are essentially referring to types of long run outcomes (i.e., once the transient phase is overcome), and thus the notion of stability (or possibility of convergence to the long term outcome) becomes central in our argument. With respect to this point, we take the conventional concept of asymptotic stability.

**Definition 3:** Let  $W$  be an invariant compact subset of  $X^{n+1}$ . Asymptotic stability of the map  $\mathbf{h}$  towards set  $W$  requires:

a) for every neighbourhood  $U$  of  $W$ , there exists a point  $(\tilde{k}_n^U, \tilde{z}_{1,n}^U, \tilde{z}_{2,n}^U, \dots, \tilde{z}_{n,n}^U)$  such that any orbit starting at  $(\tilde{k}_n^U, \tilde{z}_{1,n}^U, \tilde{z}_{2,n}^U, \dots, \tilde{z}_{n,n}^U)$  is entirely contained in  $U$ , that is,  $\omega(\tilde{k}_n^U, \tilde{z}_{1,n}^U, \tilde{z}_{2,n}^U, \dots, \tilde{z}_{n,n}^U) \subset U$ ;

b) the set  $B(W) = \{(\tilde{k}_t, \tilde{z}_{1,t}, \tilde{z}_{2,t}, \dots, \tilde{z}_{n,t}) \in X^{n+1} \mid \lim_{t \rightarrow +\infty} d[\omega(\tilde{k}_t, \tilde{z}_{1,t}, \tilde{z}_{2,t}, \dots, \tilde{z}_{n,t}), W] = 0\}$

is a neighbourhood of  $W$ . In this set,  $d$  is some distance measure between the position of the endogenous variable, given by its orbit, and set  $W$ .

Definition 3 deserves some comments. First, set  $W$  is known as an attractor or attracting set, as long as it is a topologically transitive set. As mentioned, we may have several types of attractors, that range from a fixed point to periodic or a-periodic points. If chaotic motion is identifiable, the set to which the system converges into in the long



term takes generally the designation of a strange attractor. Second, the stability property may apply solely to a subset of  $X^{n+1}$ , which was presented as set  $B(W)$ . This set is the basin of attraction, that is, the set of all initial points corresponding to orbits that converge to the attracting set. According to the definition, the orbits originating in points inside the basin will coincide with the attractor in the long term (i.e., the distance between the orbits and the attractor tends to zero). Of course, by the definition,  $W \subseteq B(W)$ .

Let us now characterize the several types of attracting sets. Definition 4 relates to the two simplest categories.

**Definition 4:** For the system of difference equations  $\mathbf{z}_{t+1} = \mathbf{h}(\tilde{k}_t, \tilde{z}_{1,t}, \tilde{z}_{2,t}, \dots, \tilde{z}_{n,t})$ , a point  $(\tilde{k}^*, \tilde{z}_i^*)$ ,  $i=1, \dots, n$ , is a periodic point of (minimal) period  $p \geq 1$  if  $\mathbf{h}^{(p)}(\tilde{k}^*, \tilde{z}_i^*) = (\tilde{k}^*, \tilde{z}_i^*)$ . A fixed point is a period 1 periodic point.

Considering the definition, the orbit

$\omega(\tilde{k}^*, \tilde{z}_i^*) = \{(\tilde{k}^*, \tilde{z}_i^*) \mid (\tilde{k}^*, \tilde{z}_i^*) = \mathbf{h}^{(p)}(\tilde{k}^*, \tilde{z}_i^*), \mathbf{h}^{(p-1)}(\tilde{k}^*, \tilde{z}_i^*), \dots, \mathbf{h}^{(1)}(\tilde{k}^*, \tilde{z}_i^*)\}$  is a sequence of  $p$  distinct points that are visited repeatedly by the system in a given order.

Periodic orbits correspond to a first level of complexity that the dynamics of nonlinear models can contain; a higher degree of complexity can be defined as a-periodic motion, which corresponds to orbits relatively to which no periodicity is identifiable but where the dynamics are relatively simple to be considered as chaos (we will deal with chaos below).

A-periodic or quasi-periodic orbits are most of the cases the result of a Neimark-Sacker bifurcation, which contrarily to other types of bifurcations (e.g., flip) do not involve a process of period doubling cycles; with the referred kind of bifurcation, generally we have a process where a fixed point stable equilibrium gives place to a-periodic cycles that can eventually degenerate in chaotic motion before the system ends up in an unstable equilibrium. Thus, while a flip bifurcation implies increasing (doubling) the number of cycles as some parameter value is changed, a Neimark-Sacker bifurcation implies a similar process but where doubling periodicity is replaced by quasi-periodicity of increasing order, that, as stated, in the limit can lead to chaos (as any other bifurcation). We will discuss further this specific type of bifurcation later, in the end of this section.

There is no easy definition of quasi-periodicity, so we describe it by default as the intermediate case between identifiable periodicity (of any order) and chaotic motion.

To get to chaos, one needs some definitions; one of these is the notion of scrambled set.

**Definition 5** (this definition relies on Mitra, Nishimura and Sorger (2005)) A subset  $S$  of  $X^{n+1}$  is a scrambled set for the dynamic system  $(X^{n+1}, \mathbf{h})$  if the following conditions are satisfied:

i) For any  $(\tilde{k}_n', \tilde{z}_{1,n}', \tilde{z}_{2,n}', \dots, \tilde{z}_{n,n}')$ ,  $(\tilde{k}_n'', \tilde{z}_{1,n}'', \tilde{z}_{2,n}'', \dots, \tilde{z}_{n,n}'') \in S$  the following condition is verified:

$$\liminf_{t \rightarrow \infty} \left| \mathbf{h}^{(t)}(\tilde{k}_n', \tilde{z}_{1,n}', \tilde{z}_{2,n}', \dots, \tilde{z}_{n,n}') - \mathbf{h}^{(t)}(\tilde{k}_n'', \tilde{z}_{1,n}'', \tilde{z}_{2,n}'', \dots, \tilde{z}_{n,n}'') \right| = 0;$$

ii) For any  $(\tilde{k}_n', \tilde{z}_{1,n}', \tilde{z}_{2,n}', \dots, \tilde{z}_{n,n}') \in S$  and either  $(\tilde{k}_n', \tilde{z}_{1,n}', \tilde{z}_{2,n}', \dots, \tilde{z}_{n,n}') \neq (\tilde{k}_n'', \tilde{z}_{1,n}'', \tilde{z}_{2,n}'', \dots, \tilde{z}_{n,n}'') \in S$  or  $(\tilde{k}_n'', \tilde{z}_{1,n}'', \tilde{z}_{2,n}'', \dots, \tilde{z}_{n,n}'') \in P$  (with  $P$  the set of all periodic points of the dynamic system  $(X^{n+1}, \mathbf{h})$ ) it holds that:

$$\limsup_{t \rightarrow \infty} \left| \mathbf{h}^{(t)}(\tilde{k}_n', \tilde{z}_{1,n}', \tilde{z}_{2,n}', \dots, \tilde{z}_{n,n}') - \mathbf{h}^{(t)}(\tilde{k}_n'', \tilde{z}_{1,n}'', \tilde{z}_{2,n}'', \dots, \tilde{z}_{n,n}'') \right| > 0.$$

According to the definition,  $S$  is a scrambled set in the sense that any orbit starting in this set does not asymptotically converge to any periodic orbit. This means that inside a scrambled set there is sensitive dependence with respect to initial conditions (SDIC), which corresponds to stating that nearby orbits tend to diverge exponentially. This is a well accepted notion of chaos; a model with SDIC can be associated with the presence of chaotic motion, although one should be careful since this is not a complete and rigorous notion of chaos. The definition, as assembled by Sarkovskii (1964) and Li and York (1975), requires the scrambled set to be uncountable (i.e., to be infinite and without a one-to-one correspondence with the set of natural numbers); since this definition has in consideration the topological / geometrical properties of sets, it is known as the definition of topological chaos [as opposed to ergodic chaos, which deals with the statistic properties of ensembles of deterministic orbits; concerning ergodicity and chaos see Huang and Day (2001) and Huang (2005)].

**Definition 6:** The dynamic system  $(X^{n+1}, \mathbf{h})$  exhibits topological chaos if its orbits are defined in an uncountable scrambled set and some of these orbits (at least one) correspond to periodic points of period that is not a power of 2.

A practical way to distinguish between periodic cycles, quasi-periodicity and chaos consists in computing Lyapunov characteristic exponents (LCEs); these respect to a measure of exponential divergence of nearby orbits, and are defined, in the case of our system, by  $LCEs = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \ln \left\| \prod_{i=0}^{n-1} Dh(\tilde{k}_i, \tilde{z}_{1,i}, \tilde{z}_{2,i}, \dots, \tilde{z}_{n,i}) \right\|$ , where  $Dh(\tilde{k}_i, \tilde{z}_{1,i}, \tilde{z}_{2,i}, \dots, \tilde{z}_{n,i})$  is a  $(n+1) \times (n+1)$  matrix with the elements given by the derivative of each one of the equations of the system relatively to each one of the variables of the system.

In the case of a  $n+1$  dimensional system,  $n+1$  LCEs are determinable and their signs allow to classify types of orbits. In particular, it is known that:

*i)* for an asymptotically stable fixed point, all LCEs are negative. This reflects the fact that for stable fixed points the distance between orbits originating at different initial conditions tend to decrease in time, as these orbits converge to the same long run equilibrium value;

*ii)* for the cases of periodicity and quasi-periodicity, at least one of the LCEs is equal to zero, while the others remain negative. In this case orbits do not approach or diverge relatively to a same long run locus;

*iii)* finally, a positive LCE signals that nearby orbits exponentially diverge and, thus, the presence of at least one positive LCE relates to the lack of predictability in the system, which is often a good argument to support the presence of chaos.

With the computation of Lyapunov exponents it becomes relatively easy to distinguish between chaos (which implies unpredictability) and quasi-periodicity, where the dynamics are predictable, although no order is identifiable for the underlying cycles. Another measure that allows for such a distinction is metric entropy (a measure of the degree of unpredictability of a deterministic system): only chaotic systems display positive entropy, because these are the ones with associated unpredictable dynamics. Quasi-periodic systems, as we have defined them to be a-periodic but not chaotic, will display zero entropy as any other system with predictable dynamics.

Measures of complexity and chaos are not our main concern here. Alongside with the analytical treatment of the growth models in next sections, we will look at LCEs as a way to clarify the qualitative nature of the steady state of the system. Additional insights about the general nature of nonlinear systems and measures of chaos can be found in the

literature. See, for instance, Alligood, Sauer and Yorke (1997), Lorenz (1997), Elaydi (2000) or Medio and Lines (2001).

An insightful approach to nonlinear dynamics in models as the ones we propose imply the need for an attentive analysis of both local and global dynamics. Global dynamics promote the accurate understanding of how orbits evolve towards the long term attracting set, but they are dependent on fully specifying the array of parameter values, because each set of parameter values may give place to a unique attractor. Local analysis allows for a more general investigation of the properties of the system but it can only go so far as to distinguish between regions of stability (stable node or stable focus), saddle-path stability and instability (unstable node or unstable focus); these regions are separated by bifurcation lines, and the movement from one region to another is made through a varying parameter that gains the designation of bifurcation parameter. Thus, local analysis hides the possible presence of periodic, quasi-periodic or / and chaotic motion that eventually characterizes the model's dynamics. For instance, in the growth models of the following sections, a bifurcation separates locally a region of stability from a region of instability; global analysis allows to realize that once the stable region is abandoned, quasi-periodicity arises for a given interval of some parameter value, before instability begins to prevail.

To close this section, and given that the growth models to be developed present non conventional dynamics (i.e., dynamics besides fixed-point stability or periodic stability) as a result of a Neimark-Sacker bifurcation, we concentrate in the fundamental properties underlying this type of bifurcation.

Recover system  $(X^{n+1}, \mathbf{h})$  and assume the following family of maps:  $\mathbf{h}(\tilde{k}_t, \tilde{z}_{1,t}, \tilde{z}_{2,t}, \dots, \tilde{z}_{n,t}; \zeta)$ , with  $\zeta \in \mathbb{R}$  a parameter. Let also  $\lambda_1(\bar{\zeta}), \lambda_2(\bar{\zeta}), \dots, \lambda_{n+1}(\bar{\zeta})$  be the eigenvalues of the Jacobian matrix of the system in the vicinity of  $(\bar{k}, \bar{z}_1, \bar{z}_2, \dots, \bar{z}_n) = (0, 0, 0, \dots, 0)$ , for a given value of  $\zeta$  ( $\zeta = \bar{\zeta}$ ).

**Definition 7:** A Neimark-Sacker bifurcation, or Hopf bifurcation for maps, occurs when the conditions below are simultaneously satisfied:

i) Any two eigenvalues of the Jacobian matrix of the system  $(X^{n+1}, \mathbf{h})$  for a parameter value  $\zeta = \bar{\zeta}$ ,  $\lambda_i(\bar{\zeta})$  and  $\lambda_j(\bar{\zeta})$ , are complex conjugate eigenvalues, with  $\lambda_i(\bar{\zeta}) \cdot \lambda_j(\bar{\zeta}) = 1$ ;

ii) The derivative  $\frac{d(\lambda_i(\bar{\zeta}) \cdot \lambda_j(\bar{\zeta}))}{d\bar{\zeta}}$  is not a null value;

iii)  $[\lambda_i(\bar{\zeta})]^m \neq 1$  and  $[\lambda_j(\bar{\zeta})]^m \neq 1$  for  $m=1, 2, 3$  and  $4$  [according to Medio and Lines (2001), this property is needed in order to get eigenvalues that are not low roots of unity].

Therefore, the central condition for a Neimark-Sacker bifurcation to occur is that at least a pair of eigenvalues from the Jacobian matrix of the linearized dynamic system in the steady state vicinity has to be a pair of complex conjugate eigenvalues, with the corresponding modulus equal to one (this is condition *i* of definition 7); the other conditions complete the required properties for the referred bifurcation to take place.

### 3. The model without optimization

Consider a typical closed economy where government intervention is absent. Let  $y_t$ ,  $k_t$ ,  $i_t$  and  $c_t$  be the levels of per capita output, physical capital, investment and consumption, respectively, and assume that the growth rate of population / labour is zero. Capital accumulation is defined as investment less capital depreciation,  $k_{t+1} - k_t = i_t - \delta k_t$ , with  $\delta > 0$  the depreciation rate and  $k_0$  given. We take as well the level of investment as corresponding to households' savings, i.e.,  $i_t = y_t - c_t$ . Under the standard Solow capital accumulation equation, consumption respects to a fixed amount of output or income, that is, consumers adopt the simplest possible rule: marginal propensity to consume does not change over time, unless some exogenous disturbance occurs.

Over this basic growth structure, we introduce the change suggested by the assumption in the following paragraph.

**Main assumption:** Consumers react to recent deviations of income relatively to the potential income level. If last periods' output gap (difference between effective and potential output / income) is positive, then aggregate consumption will be more than proportional relatively to today's income; if last periods' output gap is negative, then contemporaneous aggregate consumption will be less than proportional relatively to the present amount of aggregate income.

The previous assumption indicates that the framework we are proposing reflects the trivial capital accumulation process of growth models only if the output gap remains at a null value; in other words, the Solow accumulation equation is a particular analysis of growth when considering that there is a permanent coincidence between effective and potential levels of output and, even if there is not such coincidence, consumers do not respond to observed gaps.

To formalize the above assumption, let  $x_t$  be the output gap measured in logs ( $x_t = \ln y_t - \ln y_t^*$ , with  $y_t^*$  the potential level of per capita output; this is a variable that is supposed to follow a growth trend corresponding to the balanced growth path). Consider the case where previous periods' output gap is zero as the benchmark case, so that  $s \in (0,1)$  defines the marginal propensity to save (savings rate) when  $x_{t-1} = x_{t-2} = \dots = x_{t-n} = 0$ , with  $n$  the number of past periods that households estimate as relevant to base present consumption decisions. Per capita consumption is given by  $c_t = (1-s) \cdot y_t \cdot g(x_{t-1}, x_{t-2}, \dots, x_{t-n})$ . According to our main assumption,  $g(0,0,\dots,0)=1$ ,  $g(x_{t-1}, x_{t-2}, \dots, x_{t-n}) > 1$  if the weighted average of  $x_{t-1}, x_{t-2}, \dots, x_{t-n}$  is positive, and  $g(x_{t-1}, x_{t-2}, \dots, x_{t-n}) < 1$  if the weighted average of  $x_{t-1}, x_{t-2}, \dots, x_{t-n}$  is negative; below we will address the way these averages are computed. From an economic point of view, we are stating that the level of consumption is a fixed percentage of output  $(1-s)$  if no output gap is observable in the previous periods; when the output gap is, on average, positive then households will react by rising consumption above the benchmark level in  $t$ ; finally, in the circumstance where the output gap is predominantly negative in previous periods, the reaction of consumers will be lowering consumption levels in  $t$  further and further below the benchmark level as the output gap widens.

Therefore, we are basically defining a mechanism of response of households relatively to business cycles; in a favourable phase of the business cycle, agents will be more optimistic and will react by applying a larger share of their disposable income to consumption; likewise, if the precedent time periods indicates a recession phase, the reaction is to reduce the consumption share out of income. This optimism-pessimism response mechanism to economic fluctuations introduces an important link between today's process of capital accumulation and consumption, and past economic performance; furthermore, it attributes to households a role that is commonly absent in this kind of economic analysis: they are no longer well informed agents fully aware of the economy's growth trend and insensitive to any other factor; rather, they incorporate in their decisions about income allocation an adjustment term that reflects how the

economy moves in the short run. Because this new framework is able to produce endogenous cycles under some circumstances, we might say that cycles are self-fulfilling: they exist as the result of the agents' reaction to their existence.

We define function  $g(x_{t-1}, x_{t-2}, \dots, x_{t-n})$  as a continuous, positive and increasing function that obeys to condition  $g(0,0,\dots,0)=1$ . The analytical tractability of the model requires a specific functional form; the following function contemplates the referred properties:  $g(x_{t-1}, x_{t-2}, \dots, x_{t-n}) = \exp(a_1 x_{t-1} + a_2 x_{t-2} + \dots + a_n x_{t-n})$ ,  $a_1 > a_2 > \dots > a_n > 0$ .

Parameters  $a_i, i=1,2,\dots,n$  are ordered in a descending way to reflect the logical idea that consumers give more importance to recent output gaps than to far in the past deviations from the potential output, when taking decisions about consumption. Thus, these constant values may be interpreted as the weights the consumer attributes to past economic performance. To simplify the analysis, we consider a rate of discount for the mentioned weights; assuming a constant rate  $\mu > 0$ , one defines a new parameter  $a$  that obeys to the condition  $a \equiv a_1 = (1 + \mu) \cdot a_2 = (1 + \mu)^2 \cdot a_3 = \dots = (1 + \mu)^{n-1} \cdot a_n$ ; as a consequence, we should rewrite function  $g$  as

$g(x_{t-1}, x_{t-2}, \dots, x_{t-n}) = \exp\left[ a \cdot \left( x_{t-1} + \frac{1}{1 + \mu} \cdot x_{t-2} + \dots + \frac{1}{(1 + \mu)^{n-1}} \cdot x_{t-n} \right) \right]$ . Recalling the

definition of output gap, an equivalent presentation of this function comes,

$$g(x_{t-1}, x_{t-2}, \dots, x_{t-n}) = \prod_{i=1}^n \left[ \left( \frac{y_{t-i}}{y_{t-i}^*} \right)^{a \cdot [1/(1+\mu)]^{i-1}} \right].$$

To close our model, one needs to define a production function  $y_t = f(k_t)$ . This setup corresponds to an endogenous growth framework, and therefore the production function must exhibit constant returns, that is,  $f'(k_t) = \frac{f(k_t)}{k_t} = A > 0$ , with this parameter reflecting the technological level concerning the production of final goods.

The above characterization can be synthesized through a simple one difference equation system, which is

$$k_{t+1} = f(k_t) - (1 - s) \cdot f(k_t) \cdot \prod_{i=1}^n \left[ \left( \frac{f(k_{t-i})}{y_{t-i}^*} \right)^{a \cdot [1/(1+\mu)]^{i-1}} \right] + (1 - \delta) \cdot k_t \quad (1)$$

We are assuming an endogenous growth setup and, thus, the economy is supposed to grow at a positive and constant rate in the steady state. Defining this growth rate by letter  $\gamma$ , the potential output corresponds to  $y_t^* = \hat{y}^* \cdot (1 + \gamma)^t$ , with  $\hat{y}^*$  a positive constant. Notice that we are just saying that potential output follows, in every time moment, the balanced or steady state growth path.

Recover variable  $\hat{k}_t \equiv \frac{k_t}{(1 + \gamma)^t}$ , already presented in the previous section. This

new capital variable is not constant for all  $t$ , but it should be constant in the steady state. Replacing variables  $y_t^*$  and  $k_t$  in equation (1) by the respective detrended values, it is straightforward to encounter the following equation (note that the production function is homogeneous of degree one),

$$\hat{k}_{t+1} = \frac{1}{1 + \gamma} \cdot \left[ f(\hat{k}_t) - (1 - s) \cdot (1 / \hat{y}^*)^{a \cdot \sum_{i=1}^n [1/(1+\mu)]^{i-1}} \cdot f(\hat{k}_t) \cdot \prod_{i=1}^n [f(\hat{k}_{t-i})]^{a \cdot [1/(1+\mu)]^{i-1}} \right] + (1 - \delta) \cdot \hat{k}_t \quad (2)$$

Equation (2) takes us to the steady state result,

**Proposition 1:** *The capital accumulation equation with consumption levels adjusted to last periods' economic performance has a unique steady state, where the detrended per capita value corresponds to the following expression:*

$$\bar{k} = \frac{\hat{y}^*}{A} \cdot \left( \frac{A - \gamma - \delta}{(1 - s) \cdot A} \right)^{1 / a \cdot \sum_{i=1}^n [1/(1+\mu)]^{i-1}}.$$

**Proof:** see appendix.

Note that condition  $A > \gamma + \delta$  is essential for a steady state result with economic meaning.

To analyze the dynamics of equation (2) we follow the procedure suggested in section 2. Taking variables  $\tilde{k}_t \equiv \hat{k}_t - \bar{k}$ ,  $\tilde{z}_{1,t} \equiv \tilde{k}_{t-1}$ ,  $\tilde{z}_{2,t} \equiv \tilde{z}_{1,t-1}$ , ...,  $\tilde{z}_{n,t} \equiv \tilde{z}_{n-1,t-1}$ , we replace these in the referred equation to obtain



$$\tilde{k}_{t+1} = \frac{1}{1+\gamma} \cdot \left[ (1+A-\delta) \cdot \tilde{k}_t + (A-\gamma-\delta) \cdot \bar{k} - (1-s) \cdot A \cdot (A/\hat{y}^*)^{a \sum_{i=1}^n [1/(1+\mu)]^{i-1}} \cdot (\tilde{k}_t + \bar{k}) \cdot \prod_{i=1}^n [(\tilde{z}_{i,t} + \bar{k})^{a[1/(1+\mu)]^{i-1}}] \right] \quad (3)$$

The dynamics of the model is addressable by studying a  $(n+1)$ -equation,  $(n+1)$ -endogenous variables system. The  $n+1$  equations are (3),  $\tilde{z}_{1,t+1} = \tilde{k}_t$  and  $\tilde{z}_{i,t+1} = \tilde{z}_{i-1,t}$ ,  $i=2, \dots, n$ , while the endogenous variables are obviously  $\tilde{k}_t$  and  $\tilde{z}_{i,t}$ ,  $i=1, \dots, n$ .

### 3.1 Local dynamics

Local dynamics are straightforward to interpret, although high dimensions turn computation of stability conditions and bifurcation points a cumbersome task. Hence, we shortly address the general properties of the linearized system in the vicinity of the steady state and we proceed to the investigation of local stability conditions for three particular cases:  $n=1$ ,  $n=2$  and  $n=3$ .

Given the unique steady state point, the linearization of the system in the neighbourhood of such point allows to present it in matricial form,

$$\begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{z}_{1,t+1} \\ \tilde{z}_{2,t+1} \\ \vdots \\ \tilde{z}_{n,t+1} \end{bmatrix} = \begin{bmatrix} 1 & -a \cdot \frac{A-\gamma-\delta}{1+\gamma} & -\frac{a}{1+\mu} \cdot \frac{A-\gamma-\delta}{1+\gamma} & \dots & -\frac{a}{(1+\mu)^{n-1}} \cdot \frac{A-\gamma-\delta}{1+\gamma} \\ \hline & & \mathbf{I} & & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \tilde{k}_t \\ \tilde{z}_{1,t} \\ \tilde{z}_{2,t} \\ \vdots \\ \tilde{z}_{n,t} \end{bmatrix} \quad (4)$$

In system (4),  $\mathbf{I}$  is an identity matrix of order  $n$ , and  $\mathbf{0}$  is a column vector with  $n$  elements. The peculiar shape of the Jacobian matrix of system (4) allows for a direct computation of its trace, determinant and sums of principal minors of any order. The calculus leads to:

$$Tr(J)=1;$$

$$\sum M_{\times 2}(J) = a \cdot \frac{A-\gamma-\delta}{1+\gamma};$$

$$\Sigma M_{\times 3}(J) = -\frac{a}{1+\mu} \cdot \frac{A-\gamma-\delta}{1+\gamma};$$

$$\Sigma M_{\times 4}(J) = \frac{a}{(1+\mu)^2} \cdot \frac{A-\gamma-\delta}{1+\gamma};$$

...

$$\Sigma M_{\times n}(J) = \left(-\frac{1}{1+\mu}\right)^{n-2} \cdot a \cdot \frac{A-\gamma-\delta}{1+\gamma};$$

$$\text{Det}(J) = \left(-\frac{1}{1+\mu}\right)^{n-1} \cdot a \cdot \frac{A-\gamma-\delta}{1+\gamma},$$

where  $\Sigma M_{\times i}(J), i = 2, \dots, n$  represents the sum of the principal minors of order  $i$ .

A generic result concerning local stability can be stated as follows.

**Proposition 2:** *The system is locally stable if all the roots of the characteristic polynomial*

$$P_J(\lambda) = (-1)^{n+1} \cdot [-(1+\mu)]^{n-1} \cdot \frac{1+\gamma}{a \cdot (A-\gamma-\delta)} \cdot \lambda^{n+1} + (-1)^n \cdot [-(1+\mu)]^{n-1} \cdot \frac{1+\gamma}{a \cdot (A-\gamma-\delta)} \cdot \lambda^n \\ + \sum_{j=1}^{n-1} (-1)^{n-j} \cdot [-(1+\mu)]^{n-j} \cdot \lambda^{n-j} + 1$$

*lie inside the unit circle.*

**Proof:** see appendix.

The expression in proposition 2 does not allow for explicitly discussing the specific economic conditions concerning different local stability results. Thus, we study the most straightforward cases.

**Case 1:**  $n=1$ . The simplest case is the one in which the representative consumer considers solely the previous period output gap to base consumption decisions. In this case, the linearized system contains a 2×2 Jacobian matrix,

$$\begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{z}_{1,t+1} \end{bmatrix} = \begin{bmatrix} 1 & -a \cdot \frac{A-\gamma-\delta}{1+\gamma} \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \tilde{k}_t \\ \tilde{z}_{1,t} \end{bmatrix} \quad (5)$$

The trace and the determinant of the Jacobian matrix of system (5) are, respectively,  $Tr(J)=1$  and  $Det(J) = \frac{A-\gamma-\delta}{1+\gamma} > 0$ . Depending on the values of parameters  $a$ ,  $A$ ,  $\gamma$  and  $\delta$ , different local dynamic results are obtainable. Figure 1 graphically represents the dynamic possibilities.

\*\*\* Figure 1 here \*\*\*

In figure 1, three bifurcation lines are represented. The space corresponding to the area inside the triangle formed by these three lines is the region of stability. Unstable outcomes are found outside the bifurcation lines. As one regards, only one kind of bifurcation is admissible in this framework, the one for which  $Det(J)=1$ , that is, a Neimark-Sacker bifurcation. The interesting point is that by introducing a slight change in a basic capital accumulation equation, regarding the amount of consumption, one generates a type of bifurcation that cannot be found in one dimensional systems. As we will observe through global dynamic analysis, invariant cycles will arise for certain values of parameters.

Proposition 3 synthesizes the result on local dynamics.

**Proposition 3:** *The capital accumulation equation with consumption levels adjusted to last period's economic performance implies the following local dynamics:*

a)  $A = \frac{1}{a} + \left(\frac{1+a}{a}\right) \cdot \gamma + \delta$  is the condition that defines the point where the

Neimark-Sacker bifurcation occurs;

b) the system is locally unstable for  $A > \frac{1}{a} + \left(\frac{1+a}{a}\right) \cdot \gamma + \delta$ ;

c) the system is locally stable for  $A < \frac{1}{a} + \left(\frac{1+a}{a}\right) \cdot \gamma + \delta$ .

**Proof:** see appendix.

**Case 2:**  $n=2$ . As we increase the number of periods that the representative household takes into consideration to form contemporaneous consumption decisions, the computation of a stability condition becomes harder from a calculus point of view,

but qualitatively we find that no significant changes arise: a Neimark-Sacker bifurcation continues to separate a region of stability for a relatively low technology level and a region of instability for a relatively high technology index. The analysis of this case is synthesized through proposition 4.

**Proposition 4:** *The model of capital accumulation without optimization and with consumption decisions based on the economic performance of the last two periods is locally characterized by the following conditions:*

a) A Neimark-Sacker bifurcation exists under

$$A = \frac{1+\gamma}{a} \cdot (1+\mu) \cdot \left[ \sqrt{\left(\frac{2+\mu}{2}\right)^2 + 1} - \frac{2+\mu}{2} \right] + \gamma + \delta;$$

b) Local stability holds for  $A < \frac{1+\gamma}{a} \cdot (1+\mu) \cdot \left[ \sqrt{\left(\frac{2+\mu}{2}\right)^2 + 1} - \frac{2+\mu}{2} \right] + \gamma + \delta;$

c) Local instability holds for  $A > \frac{1+\gamma}{a} \cdot (1+\mu) \cdot \left[ \sqrt{\left(\frac{2+\mu}{2}\right)^2 + 1} - \frac{2+\mu}{2} \right] + \gamma + \delta.$

**Proof:** see appendix.

Recalling that, in our model and for  $n=2$ ,  $Det(J) = -\frac{\sum M_{\times 2}(J)}{1+\mu}$ , we can clarify

the dynamics underlying this specific case by drawing a diagram that relates the sum of the order two minors of  $J$  with the determinant of the matrix. Figure 2 reveals a line segment near to the origin where stability holds, while, after the bifurcation point, instability rules.

\*\*\* Figure 2 here \*\*\*

**Case 3:**  $n=3$ . When consumption decisions are based on the previous three periods' aggregate economic outcomes, the Jacobian matrix of the linearized system in the steady state vicinity will display a trace equal to 1, a positive sum of principal minors of order 1, a negative sum of principal minors of order 2, and a positive determinant. These signs lead to the unquestionable observation that two of the eigenvalues of the Jacobian matrix are negative and that the other two are positive

values. This information is vital to highlight a stability result concerning the three-period case.

**Proposition 5:** *In the growth model without optimization, taking consumption decisions by evaluating the previous three periods implies the following stability result.*

a) *Local stability:*

$$\frac{\mu \cdot (2 + \mu)}{(1 + \mu)^6} \cdot \left( a \cdot \frac{A - \gamma - \delta}{1 + \gamma} \right)^3 + \left( \frac{1 - \mu - \mu^2}{(1 + \mu)^4} \right) \cdot \left( a \cdot \frac{A - \gamma - \delta}{1 + \gamma} \right)^2 + \frac{4 + 3\mu + \mu^2}{(1 + \mu)^2} \cdot \left( a \cdot \frac{A - \gamma - \delta}{1 + \gamma} \right) < 1$$

$$\text{and } a \cdot \frac{A - \gamma - \delta}{1 + \gamma} < (1 + \mu)^2;$$

b) *Neimark-Sacker bifurcation:*

$$\frac{\mu \cdot (2 + \mu)}{(1 + \mu)^6} \cdot \left( a \cdot \frac{A - \gamma - \delta}{1 + \gamma} \right)^3 + \left( \frac{1 - \mu - \mu^2}{(1 + \mu)^4} \right) \cdot \left( a \cdot \frac{A - \gamma - \delta}{1 + \gamma} \right)^2 + \frac{4 + 3\mu + \mu^2}{(1 + \mu)^2} \cdot \left( a \cdot \frac{A - \gamma - \delta}{1 + \gamma} \right) = 1$$

$$\text{or } a \cdot \frac{A - \gamma - \delta}{1 + \gamma} = (1 + \mu)^2;$$

c) *Local instability:*

$$\frac{\mu \cdot (2 + \mu)}{(1 + \mu)^6} \cdot \left( a \cdot \frac{A - \gamma - \delta}{1 + \gamma} \right)^3 + \left( \frac{1 - \mu - \mu^2}{(1 + \mu)^4} \right) \cdot \left( a \cdot \frac{A - \gamma - \delta}{1 + \gamma} \right)^2 + \frac{4 + 3\mu + \mu^2}{(1 + \mu)^2} \cdot \left( a \cdot \frac{A - \gamma - \delta}{1 + \gamma} \right) > 1$$

$$\text{or } a \cdot \frac{A - \gamma - \delta}{1 + \gamma} > (1 + \mu)^2.$$

**Proof:** see appendix.

To extend the analysis of local dynamics beyond  $n=3$  is not a worthwhile task, since computation leads to heavy expressions that are progressively less informative. Nevertheless, we have found a pattern: stability holds for certain combinations of parameter values; this stability can only be broken when the product of two eigenvalues is equal to one (Neimark-Sacker bifurcation), and after the bifurcation instability will prevail (next subsection will reveal, through a numerical example, that after the bifurcation and before instability sets in it is possible that an area of quasi-periodic cycles exist).

### 3.2 Global dynamics

Depending on the kind of nonlinearities involved, a bifurcation such as the one just characterized can separate regions of fixed point stability from regions of instability or it can produce a region where cycles of different periodicities emerge following the bifurcation and before instability begins to prevail. In the present model, the Neimark-Sacker bifurcation gives effectively place to endogenous fluctuations for certain combinations of parameter values. The following graphical analysis concentrates on the simplest case  $n=1$ , but since we are not limited by relevant computation problems in the numerical study of the global properties of the system, we end the section with an example that assumes  $n=4$ . We will observe that a same kind of dynamics is revealed for the two discussed cases (similar dynamics can be found for any other value of  $n$ ).

Take the following set of benchmark values, which represent reasonable economic conditions (in particular, we consider a long run growth rate of 4%):  $[s \ \delta \ \gamma \ \hat{y}^* \ a]=[0.25 \ 0.05 \ 0.04 \ 1 \ 10]$  and let  $n=1$ . We elect  $A$  as the bifurcation parameter. In figure 3, we display the bifurcation diagram; it clearly indicates that the system undergoes a bifurcation that generates a region of cyclical behaviour for a limited interval of the parameter's value.<sup>1</sup>

\*\*\* Figure 3 here \*\*\*

With figure 4, we take a closer look to the multiple bifurcations that this dynamic process generates, as the technology parameter is varied.

\*\*\* Figure 4 here \*\*\*

In section 2, one has discussed the possibility of the cycles arising from the bifurcation process to correspond either to quasi-periodic cycles or to chaotic motion. In particular, quasi-periodicity does not imply a divergence of nearby orbits or SDIC, and therefore one of the two Lyapunov exponents (in this case, we have an order 2 Jacobian matrix and, thus, 2 LCEs exist) can take a null value, but it will not be positive. With figure 5, that represents LCEs for different values of the index  $A$ , we clarify this point. In particular, we observe that the upper LCE is, for most of the values of the parameter, very close to zero, indicating the presence of quasi-periodicity; the jumps in the

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<sup>1</sup> All the figures concerning global dynamics presented in this paper are drawn using IDMC software (interactive Dynamical Model Calculator). This is a free software program available at [www.dss.uniud.it/nonlinear](http://www.dss.uniud.it/nonlinear), and copyright of Marji Lines and Alfredo Medio.

trajectory of both LCEs just signal small regions of instability that one can confirm to exist by looking at the bifurcation diagram in figure 3.

\*\*\* Figure 5 here \*\*\*

Proceeding with the visual analysis, figure 6 represents an attracting set for a value of the technology parameter with which quasi-periodicity holds, and in figure 7 a chaotic attractor is revealed, for another value of the considered parameter.

\*\*\* Figures 6 and 7 here \*\*\*

The graphical analysis becomes complete with a basin of attraction (figure 8), that we present for  $A=0.33$ , and a time series of the endogenous variable of the model (figure 9), for  $A=0.25$ . Recall that this endogenous variable,  $\tilde{k}_t$ , has undertaken a double transformation: it was detrended and scaled to correspond to zero in the steady state.

\*\*\* Figures 8 and 9 here \*\*\*

With figure 9, we reveal our main result; a simple change in the conventional capital accumulation equation, that involves taking into account considerations about the business cycle when consumption decisions are made, results in endogenous business cycles. It is, as expressed in the introduction, the behaviour of households and their confidence regarding the short run economic scenario that generates and sustains economic fluctuations over time.

Note, as well, that the steady state value of the per capita physical capital variable, when the growth of trend is withdrawn, is, for the considered set of parameters,

$\bar{k} = \frac{1}{A} \cdot \left( \frac{A-0.09}{0.75A} \right)^{0.1}$ . Consider one of the assumed technology values, e.g.  $A=0.25$ ;

with this value,  $\bar{k} = 5.238$ . Thus, the true capital variable is given by

$k_t = (\tilde{k}_t + 5.238) \cdot (1.04)^t$ , and its growth rate is  $\gamma(k_t) = \left( \frac{\tilde{k}_t + 5.238}{\tilde{k}_{t-1} + 5.238} \right) - 1$ .

As in standard endogenous growth models, the long term average growth rate  $\langle \gamma(\bar{k}) \rangle$  is

constant, but unlike the standard endogenous growth model deterministic fluctuations on the growth rate are observable.

For  $n > 1$ , a similar type of analysis is possible to undertake. To save in space, we just consider  $n=4$  and present a bifurcation diagram like the one in figure 3 (figure 10). The same array of parameters is assumed, along with  $\mu=0.05$ . In this case, it is clear the occurrence of the bifurcation, of the same type as in the above example, and therefore the same kind of time trajectory as in figure 9 is obtainable for variable  $\tilde{k}_t$ , when  $A$  is above 0.105.

\*\*\* Figure 10 here \*\*\*

#### 4. The model with optimization

The model with an exogenous savings rate can easily be sophisticated in order to incorporate consumption utility maximization. Let us now define the per capita consumption variable in the following way:  $c_t = E_{t-1}c_t + c_t^c$ .

Term  $E_{t-1}c_t$  is the level of consumption when the output gap is permanently zero, that is, if the expected level of output coincides with the potential level, then individuals will consume an amount  $E_{t-1}c_t$  of final goods. To this amount of consumption we call expected consumption. In this problem, the representative agent chooses to maximize expected utility, that is, the objective function is  $V_1 = \sum_{t=1}^{+\infty} U(E_{t-1}c_t) \cdot \beta^t$ . Parameter  $\beta < 1$  is the discount factor; note that we consider an infinite horizon.

The other component of consumption respects to a reaction to fluctuations. As in the non optimization case, we consider that this fraction of consumption is measured as a percentage of per capita output. Letting  $\sigma$  be a positive parameter, we define  $c_t^c \equiv \sigma \cdot y_t \cdot [g(x_{t-1}, x_{t-2}, \dots, x_{t-n}) - 1]$ . Function  $g$  is the same as in the previous section. This expression indicates that if all  $x_{t-i}$ ,  $i=1, 2, \dots, n$  are equal to zero then  $c_t^c = 0$ , that is, the problem becomes the standard Ramsey growth model (in this case with a constant returns production function); it is straightforward to verify that if the weighted average of  $x_{t-1}, x_{t-2}, \dots, x_{t-n}$  is positive then  $c_t^c > 0$  and, also, if the weighted average of  $x_{t-1}, x_{t-2}, \dots, x_{t-n}$  is negative then  $c_t^c < 0$ .



Hence, the logic underlying the theoretical structure is exactly the same as in the case without optimization: households respond to previous periods' economic performance by consuming more (expansions) or less (recessions) than they would if one considers the benchmark model that is designed only for the case where no difference between effective and potential output is taken into account.

The utility maximization model with consumers reaction to deviations of output from its trend corresponds to the maximization of  $V_1$ , subject to the resource constraint  $k_{t+1} = y_t - c_t + (1 - \delta) \cdot k_t$ , with  $k_0$  given, the variable per capita consumption as defined above and  $E_{t-1}c_t$  a control variable.

Since the derivation of optimality conditions is standard, we neglect the details that give place to the motion of expected per capita consumption in optimal conditions, and just present this difference equation that arises from such computation:  $E_t c_{t+1} = \beta \cdot (1 + A - \delta) \cdot E_{t-1} c_t$ ; because the production function is linear, the rule that defines the time path of consumption reflects the existence of a constant growth rate of this aggregate over time. Thus, contrarily to the non optimization case, now we have an explicit growth rate  $\gamma \equiv \beta \cdot (1 + A - \delta) - 1$ . Note that this is the growth rate of expected consumption, but it is also the steady state growth rate of capital and output, as one easily observes through the examination of the long term condition underlying the resource constraint.

Hence, we may define  $\hat{k}_t \equiv \frac{k_t}{[\beta \cdot (1 + A - \delta)]^t}$  as a non growing variable in the steady state. We also consider the following constant values:  $\hat{y}^* \equiv \frac{y^*}{[\beta \cdot (1 + A - \delta)]^t}$  and  $\hat{c} \equiv \frac{E_{t-1}c_t}{[\beta \cdot (1 + A - \delta)]^t}$ . The constant ratio  $\frac{\hat{c}}{\hat{k}}$  will be designated by letter  $\psi$ , so that we present the detrended expected consumption value as  $\hat{c} = \psi \cdot \bar{k}$ .

Rewriting the capital accumulation equation taking in consideration the above variables and values, we get

$$\hat{k}_{t+1} = \frac{1}{\beta \cdot (1 + A - \delta)} \cdot \left[ (1 + \sigma) \cdot f(\hat{k}_t) - \psi \cdot \bar{k} - \sigma \cdot (1/\hat{y}^*)^{a \cdot \sum_{i=1}^n [1/(1+\mu)]^{i-1}} \cdot f(\hat{k}_t) \cdot \prod_{i=1}^n [f(\hat{k}_{t-i})^{a \cdot [1/(1+\mu)]^{i-1}}] + (1 - \delta) \cdot \hat{k}_t \right] \quad (6)$$

Because we have transformed the consumption variable into a detrended constant, we have, as in the previous non optimization case, a one-dimensional system of the form  $\hat{k}_{t+1} = h(\hat{k}_t, \hat{k}_{t-1}, \dots, \hat{k}_{t-n})$ . Therefore, we must use a same kind of procedure to study the dynamics associated with this model. First, we state the balanced growth path result.

**Proposition 6:** *The utility maximization problem with consumption reaction to short run economic conditions has a unique steady state, which is found by imposing the condition  $\bar{k} \equiv \hat{k}_{t+1} = \hat{k}_t = \hat{k}_{t-1} = \dots = \hat{k}_{t-n}$  to (6). The balanced growth level of*

*accumulated capital is given by  $\bar{k} = \frac{\hat{y}^*}{A} \cdot \left[ \frac{(1-\beta) \cdot (1+A-\delta) + \sigma A - \psi}{\sigma \cdot A} \right]^{1/\alpha \sum_{i=1}^n [1/(1+\mu)]^{i-1}}$ .*

**Proof:** see appendix.

We must guarantee a positive long run capital stock and, thus, condition  $\psi < (1-\beta) \cdot (1+A-\delta) + \sigma A$  must hold.

As in the non optimization case, take  $\tilde{k}_t \equiv \hat{k}_t - \bar{k}$ ,  $\tilde{z}_{1,t} \equiv \tilde{k}_{t-1}$ ,  $\tilde{z}_{2,t} \equiv \tilde{z}_{1,t-1}$ , ...,  $\tilde{z}_{n,t} \equiv \tilde{z}_{n-1,t-1}$ , to present another version of the capital accumulation equation,

$$\begin{aligned} \tilde{k}_{t+1} = & \left[ \frac{1}{\beta} + \frac{\sigma A}{\beta \cdot (1+A-\delta)} \right] \cdot \tilde{k}_t + \left[ \frac{1-\beta}{\beta} + \frac{\sigma A - \psi}{\beta \cdot (1+A-\delta)} \right] \cdot \bar{k} \\ & - \frac{\sigma \cdot A \cdot (A/\hat{y}^*)^{\alpha \sum_{i=1}^n [1/(1+\mu)]^{i-1}}}{\beta \cdot (1+A-\delta)} \cdot (\tilde{k}_t + \bar{k}) \cdot \left( \frac{\tilde{z}_t + \bar{k}}{\hat{y}^*} \right)^\alpha \cdot \prod_{i=1}^n \left[ (\tilde{z}_{i,t} + \bar{k})^{\alpha [1/(1+\mu)]^{i-1}} \right] \end{aligned} \quad (7)$$

The system subject to analysis is now constituted by (7),  $\tilde{z}_{1,t+1} = \tilde{k}_t$  and  $\tilde{z}_{i,t+1} = \tilde{z}_{i-1,t}$ ,  $i=2, \dots, n$ , with  $(\tilde{k}, \tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_n) = (0, 0, 0, \dots, 0)$ .

#### 4.1 Local dynamics

Local analysis requires the computation of the Jacobian matrix elements. The calculus of partial derivatives and corresponding evaluation in the steady state vicinity leads to the following linearized system:



**Proposition 7:** *The linearized system, of the growth model with expected utility maximization and consumption decisions based on past economic performance, is locally stable if all the roots of the characteristic polynomial*

$$P_j(\lambda) = (-1)^{n+1} \cdot [-(1+\mu)]^{n-1} \cdot \frac{\beta \cdot (1+A-\delta)}{a \cdot [(1-\beta) \cdot (1+A-\delta) + \sigma A - \psi]} \cdot \lambda^{n+1} \\ + (-1)^n \cdot [-(1+\mu)]^{n-1} \cdot \frac{\beta \cdot (1+A-\delta)}{a \cdot [(1-\beta) \cdot (1+A-\delta) + \sigma A - \psi]} \cdot \left[1 + \frac{\psi}{\beta \cdot (1+A-\delta)}\right] \cdot \lambda^n \\ + \sum_{j=1}^{n-1} (-1)^{n-j} \cdot [-(1+\mu)]^{n-j} \cdot \lambda^{n-j} + 1$$

*lie inside the unit circle.*

**Proof:** see appendix.

To save space and to guarantee that we are working with tractable expressions, we now concentrate exclusively in the simplest case  $n=1$ .

For  $n=1$ , the linearized system is just

$$\begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{z}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 + \frac{\psi}{\beta \cdot (1+A-\delta)} & -a \cdot \frac{(1-\beta) \cdot (1+A-\delta) + \sigma A - \psi}{\beta \cdot (1+A-\delta)} \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \tilde{k}_t \\ \tilde{z}_t \end{bmatrix} \quad (8)$$

The trace and the determinant of the Jacobian matrix are, respectively,

$$Tr(J) = 1 + \frac{\psi}{\beta \cdot (1+A-\delta)} > 1 \quad \text{and} \quad Det(J) = a \cdot \frac{(1-\beta) \cdot (1+A-\delta) + \sigma A - \psi}{\beta \cdot (1+A-\delta)} > 0. \quad \text{To}$$

obtain a relation between trace and determinant regard that  $\frac{\psi}{\beta \cdot (1+A-\delta)} = Tr(J) - 1$

and that  $\frac{\psi}{\beta \cdot (1+A-\delta)} = \frac{1-\beta}{\beta} + \frac{\sigma A}{\beta \cdot (1+A-\delta)} - \frac{1}{a} \cdot Det(J)$ . Therefore,

$$Det(J) = \frac{a}{\beta} \cdot \left(1 + \frac{\sigma A}{1+A-\delta}\right) - a \cdot Tr(J). \quad \text{This relation can be displayed graphically in a}$$

same referential as the one used in figure 1. Comparing figures 1 and 11, we notice that the introduction of optimization changes somehow the local properties of the model.

\*\*\* Figure 11 here \*\*\*

Figure 11 allows to realize that optimization enlarges the dynamic possibilities of the model. Although a flip bifurcation continues excluded from the possible outcomes, another bifurcation (fold) besides the Neimark-Sacker one is admissible. Furthermore, if the slope of the dynamic line is low (a low  $a$ ), the Neimark-Sacker bifurcation will not take place, independently of other parameters' values. Proposition 8 synthesizes the stability results.

**Proposition 8:** Consider  $\theta \equiv (1 - \beta) \cdot (1 + A - \delta) + \sigma A$ . In the optimization growth model with consumers reaction to the previous period business cycle position, stability is guaranteed under the following condition:  $\theta - \frac{1 + A - \delta}{a} < \psi < \frac{a}{1 + a} \cdot \theta$ .

**Proof:** see appendix.

From proposition 8, one can withdraw several corollaries,

**Corollary 1:** Because  $\psi < \theta$  is a necessary condition for an economically meaningful steady state, and stability requires  $\psi < \frac{a}{1 + a} \cdot \theta$ , if the system is stable then the steady state capital value is necessarily economically meaningful (i.e., positive).

**Corollary 2:** A Neimark-Sacker bifurcation occurs if when  $\text{Tr}(J)=1$  then  $\text{Det}(J) \geq 1$ . Thus, a Neimark-Sacker bifurcation requires  $\theta \geq \frac{\beta \cdot (1 + A - \delta)}{a}$ .

**Corollary 3:** Besides stability (stable node or stable focus), as identified through proposition 8, other qualitative local results are possible:

i) Instability (unstable focus):  $\psi < \theta - \frac{1 + A - \delta}{a}$ ;

ii) Saddle-path stability:  $\psi > \frac{a}{1 + a} \cdot \theta$ .

A numerical example helps to get some further insights on the local dynamics; we consider the same values of the parameters that are common to the non optimization and to the optimization frameworks, i.e.,  $\delta=0.05$ ,  $\hat{y}^*=1$ ,  $a=10$ . To these, we add  $\sigma=0.1$

(recall that this is a relevant parameter, because it reflects the degree of importance agents give to previous economic conditions when taking their consumption decisions),  $\beta=0.86$  (a value that corresponds to a discount rate equal to 16.28%) and  $\psi=0.04$  (in the steady state, the representative agent consumes a value corresponding to 4% of the available stock of capital). As in the non optimization case, we let  $A$  be the varying parameter.

Given the assumed array of parameters, the trace and the determinant of the Jacobian matrix come:  $Tr(J) = \frac{0.857 + 0.86A}{0.817 + 0.86A}$  and  $Det(J) = \frac{1.33 + 2.4A}{0.817 + 0.86A}$ . Condition  $\psi < \theta$ , needed for a positive balanced growth stock of capital, is always satisfied, independently of the value of the positive technology index.

Note that if  $A=0$ , then  $Tr(J)=1.049$  and  $Det(J)=1.628$ ; observe as well that  $\lim_{A \rightarrow +\infty} Tr(J) = 1$  and that  $\lim_{A \rightarrow +\infty} Det(J) = 2.791$ . These values imply that no bifurcation occurs for a positive technology value; the system will locate in the region where a pair of complex conjugate eigenvalues associated to the Jacobian matrix exist, independently of  $A$ . However, this does not mean that cycles and chaos cannot occur; even though economically negative values of  $A$  have no meaning, we should understand that mathematically it is possible to find the point of transition between stable and unstable outcomes (i.e., the bifurcation point; in this case, the determinant of the Jacobian matrix is equal to zero when  $A=-0.333$ ). Figure 12 illustrates local dynamics under the discussed example.

\*\*\* Figure 12 here \*\*\*

Not all positive values of  $A$  are meaningful in our analysis. Recall that  $A$  is an argument in the expression of the steady state growth rate, and thus we must guarantee, through this parameter, a reasonable long run growth rate. Recalling that  $\gamma \equiv \beta \cdot (1 + A - \delta) - 1$ , we will work, on the global analysis that follows, with  $0.2 \leq A \leq 0.3$ ; this allows for a reasonable growth rate  $-1.1\% \leq \gamma \leq 7.5\%$ . Take in consideration that  $A=0.2$  implies  $[Tr(J), Det(J)] = [1.040, 1.830]$  and that  $A=0.3$  implies  $[Tr(J), Det(J)] = [1.037, 1.907]$

## 4.2 Global dynamics

To discuss global dynamics, we assume the set of parameter values presented above. The analysis is basically the same one has undertaken for the non optimization case. We study the case  $n=1$ .

The first step consists in presenting a bifurcation diagram. Figure 13 reveals this diagram, for values of  $A$  capable of producing a reasonable long term growth rate. In this case, we encounter essentially an a-periodicity result. Recall that we are in the region ‘after’ the Neimark-Sacker bifurcation that we can locate in the curve of figure 12.

\*\*\* Figure 13 here \*\*\*

Figure 14 represents an attracting set for a value of  $A$  for which a-periodicity is present (in this case, we choose a value of  $A$  that corresponds to a 4% steady state growth rate); because cycles come from the same type of bifurcation as in the non-optimization case, one sees some similarities with the attractor in figure 6. The similarities can be regarded also through the time series of the endogenous variable  $\tilde{k}_t$ ; there is no significant difference between the kind of endogenous business cycles in figure 9 and the fluctuations one observes in figure 15.

\*\*\* Figures 14 and 15 here \*\*\*

As in section 3, we complete the analysis with the basin of attraction that indicates the admissible set of initial values in order for the long run attracting set to be reached (figure 16). Note the similarities between this basin and the one in figure 8. This resemblance is an additional element that compels us to state that the optimization of consumption utility thus not change significantly the nature of cycles that arise when the representative agent takes in consideration previous economic performance in the moment of evaluation of consumption decisions (even if the weight of such evaluation is relatively low; recall that the value assumed for parameter  $\sigma$  is close to zero).

\*\*\* Figure 16 here \*\*\*

Remind that the time series in figure 15 corresponds to a transformed capital variable, and that the original capital variable is  $k_t = (\tilde{k}_t + \bar{k}) \cdot (1 + \gamma)^t$ , with  $\bar{k} = 4.6123$  and  $\gamma = 0.04$ .

### 4.3 Welfare considerations

One important doubt that the analysis may raise is whether the inclusion of consumer reaction to short run economic fluctuations is welfare enhancing or not. To clarify this point we should look at time series averages. Our graphics give some clues and essentially point to the inefficiency that the consumers' behaviour may produce.

If one considers the time series generated in our examples, for the assumed parameter values, for  $n=1$  and  $A=0.25$  (model without optimization) or  $A=0.2593$  (model with optimization), the cases depicted in figures 9 and 15, respectively, we find the following: for any reasonable number of observations (e.g., more than 50 iterations), the average of each one of the time series is found to be  $\langle \tilde{k}_t \rangle \approx -0.6$  and  $\langle \tilde{k}_t \rangle \approx -0.2$ , respectively.

In both cases, the average of the time series is below zero, which is the steady state result in the absence of fluctuations. Thus, one may infer that the cycles are the result of an inefficient response of individuals to the economic conditions they perceive in each moment. The proposed model generates inefficient endogenous business cycles over the benchmark growth models, which do not take in consideration any ability of consumers to respond to the economic behaviour they witness.

## 5. Final remarks

In this paper, we have asked a simple question: what if agents react to last periods' economic performance when making consumption decisions today? After all, economists unanimously consider that economic performance is directly attached to consumer confidence. If households are optimistic about the ability of the economy to grow above normal standards, they will be more willing to consume, and therefore the marginal propensity to consume out of income rises. If consumers are pessimistic, the opposite effect should be observed. Thus, we contrast the standard growth model under which agents believe that output converges in the long term to its potential level and act according to such belief, with a framework where agents are essentially myopic in the



sense they react to observable economic results rather than to a far in time benchmark that they do not have any guarantee to hold.

The proposed setup leads to endogenous business cycles that are triggered by households' behaviour; if agents did not respond to cycles, these would just fade out. Taking this argument seriously means that countercyclical economic policy may be adequate to attenuate the persistence of economic fluctuations; if agents do not perceive that the economy departs from the potential output level, then they will adjust savings rates obeying only to requisites of optimality under a frictionless scenario. Another major conclusion is that cycles are inefficient, in the sense that the economy will, on average, be worse off with than without them (our numerical examples point in this direction). Cycles penalize long term welfare, and therefore private agents should be educated not to give a strong response to short run fluctuations with regard to their decisions on the allocation of income between consumption and savings.

## Appendix

**Proof of proposition 1:** Section 2 has defined the steady state as the long run locus that satisfies condition  $\bar{k} \equiv \hat{k}_{t+1} = \hat{k}_t = \hat{k}_{t-1} = \dots = \hat{k}_{t-n}$ . Applying this condition to

equation (2), one gets  $\bar{k} = \frac{1}{1+\gamma} \cdot \left[ f(\bar{k}) - (1-s) \cdot f(\bar{k}) \cdot (f(\bar{k})/\hat{y}^*)^{\alpha \sum_{i=1}^n [1/(1+\mu)]^{i-1}} + (1-\delta) \cdot \bar{k} \right]$ ; in our

AK framework,  $f(k_t) = Ak_t$ , and thus  $f(\bar{k}) = A\bar{k}$ . To get to the expression in the proposition we just have to solve the above expression in order to  $\bar{k}$ , for the explicit functional form of the production function  $\square$

**Proof of proposition 2:** The proposition states a trivial definition of stability; thus, this prove only involves clarifying that the presented characteristic polynomial  $P_j(\lambda)$  is in fact the polynomial under which condition  $P_j(\lambda) = 0$  allows for the determination of the  $n+1$  eigenvalues of matrix  $J(\lambda_j, j=1, \dots, n+1)$ .

A characteristic polynomial is generically defined as  $Q_j(\lambda) = (-1)^{n+1} \cdot \lambda^{n+1} + (-1)^n \cdot Tr(J) \cdot \lambda^n + (-1)^{n-1} \cdot \sum M_{\times 2}(J) \cdot \lambda^{n-1} + \dots + (-1) \cdot \sum M_{\times n}(J) \cdot \lambda + Det(J)$ .

Dividing all terms by the determinant, we continue to have a polynomial that allows for finding exactly the same  $n+1$  solutions; thus, we define  $P_j(\lambda) \equiv \frac{Q_j(\lambda)}{Det(J)}$ , and present

$$P_J(\lambda) = \frac{(-1)^{n+1}}{Det(J)} \cdot \lambda^{n+1} + \frac{(-1)^n \cdot Tr(J)}{Det(J)} \cdot \lambda^n + \frac{(-1)^{n-1} \cdot \sum M_{\times 2}(J)}{Det(J)} \cdot \lambda^{n-1} + \dots + \frac{(-1) \cdot \sum M_{\times n}(J)}{Det(J)} \cdot \lambda + 1.$$

Replacing the trace, the sums of the principal minors and the determinant that we have computed for our linearized system in the above polynomial, we find the expression in the proposition  $\square$

**Proof of proposition 3:** Given the values of the trace and determinant of  $J$ , one verifies that  $1+Tr(J)+Det(J)>0$  and  $1-Tr(J)+Det(J)<0$ , independently of parameter values. Relatively to the condition  $1-Det(J)=0$ , that defines a Neimark-Sacker bifurcation, this is equivalent to the expression in item *a*) of the proposition. Local stability and local instability conditions are obtained by solving, respectively,  $Det(J)<1$  and  $Det(J)>1$   $\square$

**Proof of proposition 4:** For  $n=2$ , the trace of the Jacobian matrix is equal to 1 (and, thus, the three eigenvalues associated with the Jacobian matrix cannot be all negative values); the determinant is negative (what implies that an odd number of negative eigenvalues exists). The two previous conditions imply with certainty that  $\lambda_1 < 0, \lambda_2 > 0, \lambda_3 > 0$  are the eigenvalues of matrix  $J$ .

With these eigenvalues signs, condition  $(1 + \lambda_1) \cdot (1 + \lambda_2) \cdot (1 + \lambda_3) > 0$  holds only if  $\lambda_1 > -1 \wedge \lambda_2 > -1 \wedge \lambda_3 > -1$ ; condition  $(1 - \lambda_1) \cdot (1 - \lambda_2) \cdot (1 - \lambda_3) > 0$  will be a true condition if, alternatively,  $\lambda_1 < 1 \wedge \lambda_2 < 1 \wedge \lambda_3 < 1$  or  $\lambda_1 < 1 \wedge \lambda_2 > 1 \wedge \lambda_3 > 1$ . Furthermore,  $(1 - \lambda_1 \cdot \lambda_2) \cdot (1 - \lambda_1 \cdot \lambda_3) \cdot (1 - \lambda_2 \cdot \lambda_3) > 0$  is a condition that requires  $\lambda_1 \cdot \lambda_2 < 1 \wedge \lambda_1 \cdot \lambda_3 < 1 \wedge \lambda_2 \cdot \lambda_3 < 1$ .

Considering simultaneously the three above conditions implies that all the eigenvalues will lie inside the unit circle, that is, we guarantee stability. The three conditions may be rewritten as

- (a)  $1 + Tr(J) + \sum M_{\times 2}(J) + Det(J) > 0$ ;
- (b)  $1 - Tr(J) + \sum M_{\times 2}(J) - Det(J) > 0$ ;
- (c)  $1 - \sum M_{\times 2}(J) + Det(J) \cdot [Tr(J) + Det(J)] > 0$ .

Conditions (a) and (b) correspond to relations that hold for any admissible combination of parameter values, while condition (c) requires the inequality

$\frac{1}{(1+\mu)^2} \cdot \left( a \cdot \frac{A-\gamma-\delta}{1+\gamma} \right)^2 + \left( \frac{2+\mu}{1+\mu} \right) \cdot \left( a \cdot \frac{A-\gamma-\delta}{1+\gamma} \right) - 1 < 0$  to hold. Solving this inequality, one gets:

$$-(1+\mu) \cdot \left[ \sqrt{\left( \frac{2+\mu}{2} \right)^2 + 1 + \frac{2+\mu}{2}} \right] < a \cdot \frac{A-\gamma-\delta}{1+\gamma} < (1+\mu) \cdot \left[ \sqrt{\left( \frac{2+\mu}{2} \right)^2 + 1 - \frac{2+\mu}{2}} \right];$$

the first inequality of this solution is an universal condition, given the boundaries on parameters values; the second relation can be rearranged to give place to the stability inequality in proposition 4.

A Neimark-Sacker bifurcation will occur when the following equality is observed:  $1 - \sum M_{\times 2}(J) + Det(J) \cdot [Tr(J) + Det(J)] = 0$ . In such circumstance, we will have a pair of complex conjugate eigenvalues equal to unity:  $\lambda_2 \cdot \lambda_3 = 1$ . This condition is equivalent to the expression presented in proposition 4 respecting the bifurcation point. Finally, if the system is not stable and we are not over a bifurcation point, then the system will be locally unstable, as referred through point c) of proposition 4  $\square$

**Proof of proposition 5:** In the case  $n=3$ , one has realized that two of the eigenvalues of matrix  $J$  are positive while the other two are negative. This observation allows to compile a set of inequalities that if jointly satisfied imply that local stability holds. These conditions are the following:

- i)  $(1 + \lambda_1) \cdot (1 + \lambda_2) \cdot (1 + \lambda_3) \cdot (1 + \lambda_4) > 0$ ;
- ii)  $(1 - \lambda_1) \cdot (1 - \lambda_2) \cdot (1 - \lambda_3) \cdot (1 - \lambda_4) > 0$ ;
- iii)  $(1 - \lambda_1 \cdot \lambda_2) \cdot (1 - \lambda_1 \cdot \lambda_3) \cdot (1 - \lambda_1 \cdot \lambda_4) \cdot (1 - \lambda_2 \cdot \lambda_3) \cdot (1 - \lambda_2 \cdot \lambda_4) \cdot (1 - \lambda_3 \cdot \lambda_4) > 0$ ;
- iv)  $1 - \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4 > 0$ .

Note that condition i) is satisfied under  $\lambda_1 > -1 \wedge \lambda_2 > -1$  or  $\lambda_1 < -1 \wedge \lambda_2 < -1$ . Likewise, condition ii) requires  $\lambda_3 > 1 \wedge \lambda_4 > 1$  or  $\lambda_3 < 1 \wedge \lambda_4 < 1$ . Condition iii) is true for  $\lambda_1 \cdot \lambda_2 > 1 \wedge \lambda_3 \cdot \lambda_4 > 1$  or  $\lambda_1 \cdot \lambda_2 < 1 \wedge \lambda_3 \cdot \lambda_4 < 1$ . Finally, inequality iv) says that  $\lambda_1 \cdot \lambda_2 > 1 \wedge \lambda_3 \cdot \lambda_4 > 1$  is not a feasible condition, and thus  $\lambda_1 \cdot \lambda_2 < 1 \wedge \lambda_3 \cdot \lambda_4 < 1$  holds; this last expression guarantees that all eigenvalues are inside the unit circle, given the first two conditions (i and ii).

The stability conditions may be transformed, after some calculus (that is particularly tedious for the third condition), into the following:

- i)  $1 + Tr(J) + \sum M_{\times 2}(J) + Det(J) > 0$ ;

$$\text{ii) } 1 - \text{Tr}(J) + \sum M_{x_2}(J) - \text{Det}(J) > 0;$$

iii)

$$1 - \sum M_{x_2}(J) - [\sum M_{x_3}(J)]^2 + [2 \sum M_{x_2}(J) - (\text{Tr}(J))^2] \cdot \text{Det}(J) \\ + [\text{Tr}(J) \cdot \sum M_{x_3}(J) - \text{Det}(J)] \cdot [1 + \text{Det}(J)] - \sum M_{x_2}(J) \cdot (\text{Det}(J))^2 + (\text{Det}(J))^3 > 0;$$

$$\text{iv) } 1 - \text{Det}(J) > 0.$$

The first two conditions hold under the system's assumptions; condition iv) is true

$$\text{for } a \cdot \frac{A - \gamma - \delta}{1 + \gamma} < (1 + \mu)^2 \quad \text{and} \quad \text{condition iii) is compatible with}$$

$$\frac{\mu \cdot (2 + \mu)}{(1 + \mu)^6} \cdot \left( a \cdot \frac{A - \gamma - \delta}{1 + \gamma} \right)^3 + \left( \frac{1 - \mu - \mu^2}{(1 + \mu)^4} \right) \cdot \left( a \cdot \frac{A - \gamma - \delta}{1 + \gamma} \right)^2 + \frac{4 + 3\mu + \mu^2}{(1 + \mu)^2} \cdot \left( a \cdot \frac{A - \gamma - \delta}{1 + \gamma} \right) < 1.$$

These are the conditions for stability; instability prevails if one of them is not fulfilled.

A Neimark-Sacker bifurcation requires once again the product of eigenvalues to be equal to one, that is,  $\lambda_1 \cdot \lambda_2 = 1$  or  $\lambda_3 \cdot \lambda_4 = 1$  (or both)  $\square$

**Proof of proposition 6:** It is straightforward to verify that the balanced growth path is unique and that it corresponds to the capital value in the proposition. For such, just solve the following equation in order to  $\bar{k}$ ,

$$\bar{k} = \frac{1}{\beta \cdot (1 + A - \delta)} \cdot \left[ (1 + \sigma) \cdot f(\bar{k}) - \psi \cdot \bar{k} - \sigma \cdot (1/\hat{y}^*) \cdot f(\bar{k}) \cdot \prod_{i=1}^n [f(\bar{k})^{a \cdot [1/(1+\mu)]^{i-1}}] + (1 - \delta) \cdot \bar{k} \right] \Leftrightarrow \\ \bar{k} = \frac{1}{\beta \cdot (1 + A - \delta)} \cdot \left[ (1 + \sigma) \cdot A\bar{k} - \psi \cdot \bar{k} - \sigma \cdot (A\bar{k}/\hat{y}^*) \cdot \prod_{i=1}^n [f(\bar{k})^{a \cdot [1/(1+\mu)]^{i-1}}] \cdot A\bar{k} + (1 - \delta) \cdot \bar{k} \right] \square$$

**Proof of proposition 7:** By solving equation  $P_J(\lambda) = 0$ , we obtain the  $n+1$  eigenvalues of the system, if  $P_J(\lambda)$  is the characteristic polynomial of the system or any equivalent relation. The notion of stability requires those eigenvalues to be above -1 and below 1. Thus, since the expression in the proposition corresponds to

$$P_J(\lambda) = \frac{(-1)^{n+1}}{\text{Det}(J)} \cdot \lambda^{n+1} + \frac{(-1)^n \cdot \text{Tr}(J)}{\text{Det}(J)} \cdot \lambda^n + \frac{(-1)^{n-1} \cdot \sum M_{x_2}(J)}{\text{Det}(J)} \cdot \lambda^{n-1} + \dots + \frac{(-1) \cdot \sum M_{x_n}(J)}{\text{Det}(J)} \cdot \lambda + 1, \quad \text{we}$$

can assert that, as in proposition 2, the eigenvalues of  $J$  can be found through the referred expression  $\square$

**Proof of proposition 8:** Just note that the stability condition  $1+Tr(J)+Det(J)>0$  is always guaranteed; the left inequality in the proposition  $\left(\theta - \frac{1+A-\delta}{a} < \psi\right)$  holds under  $1-Det(J)>0$  and the right inequality in the proposition  $\left(\psi < \frac{a}{1+a} \cdot \theta\right)$  is satisfied for  $1-Tr(J)+Det(J)>0$ , given the values of the trace and the determinant of the matrix in linearized system (8)□

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## Figures

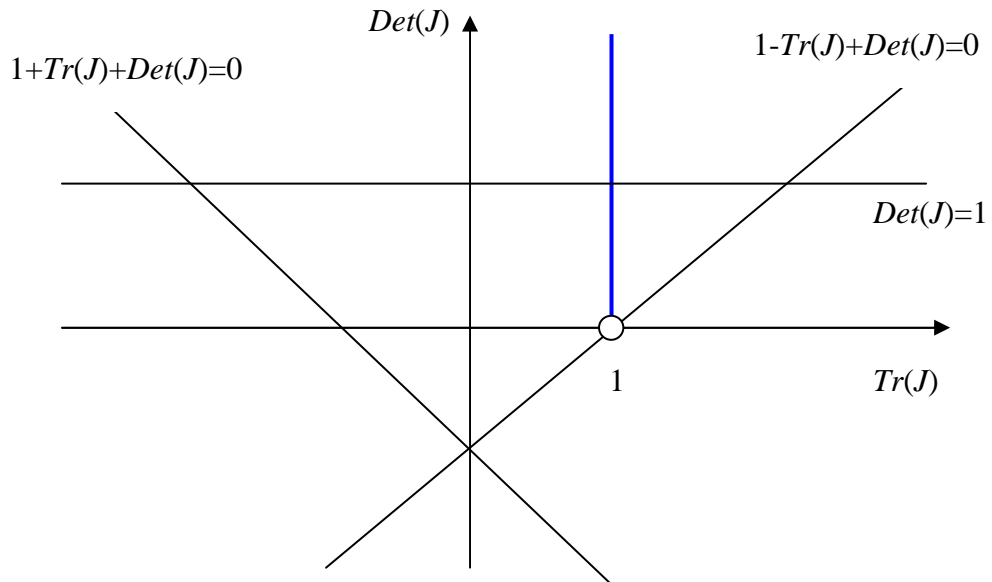


Figure 1 – Local dynamics in the model without optimization ( $n=1$ ).

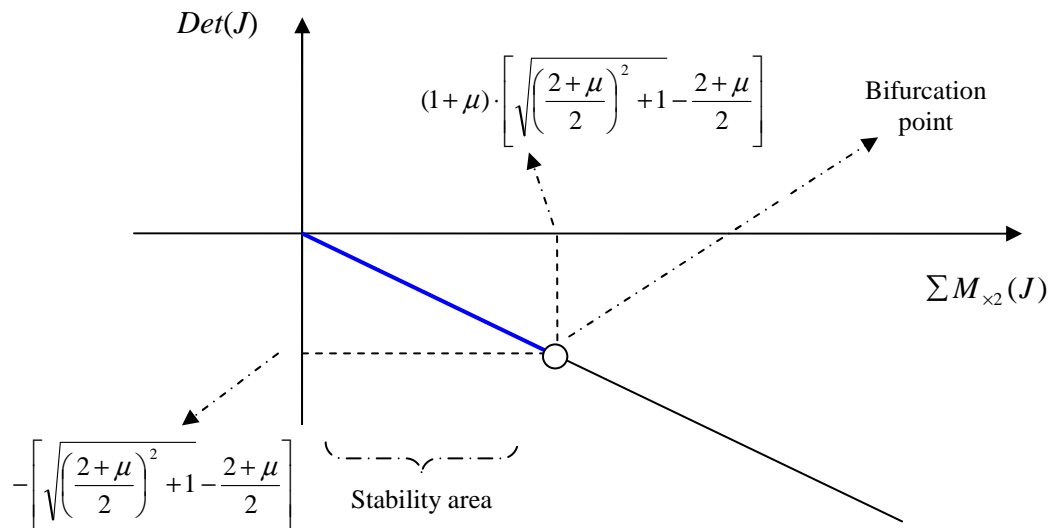


Figure 2 – Local dynamics in the model without optimization ( $n=2$ ).

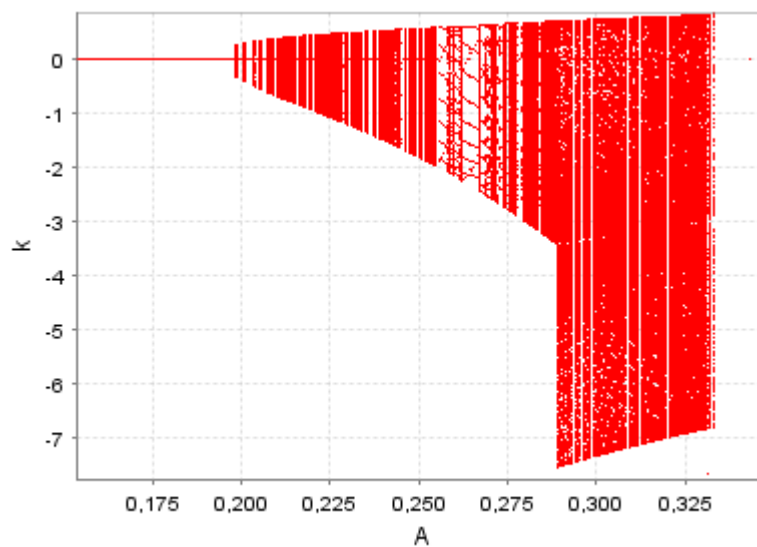


Figure 3 – Bifurcation diagram  $(A, \tilde{k}_t)$  for the model without optimization ( $n=1$ ).

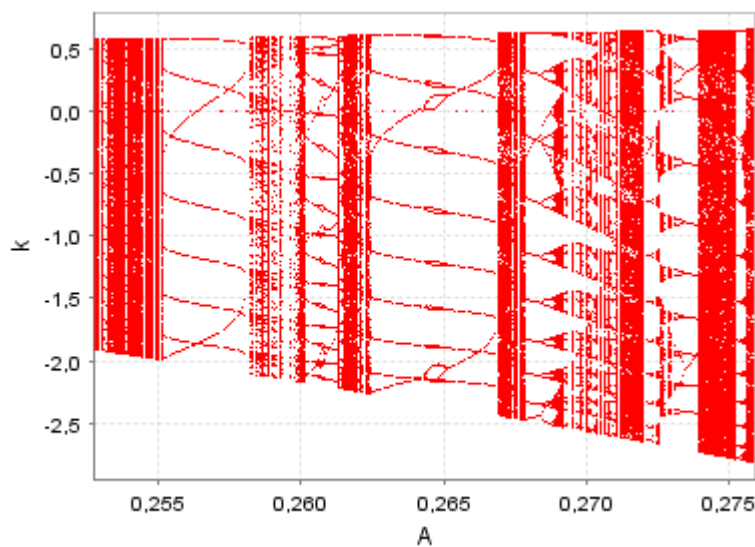


Figure 4 – Bifurcation diagram  $(A, \tilde{k}_t)$  for the model without optimization.

A closer look, for  $0.253 < A < 0.276$  ( $n=1$ ).

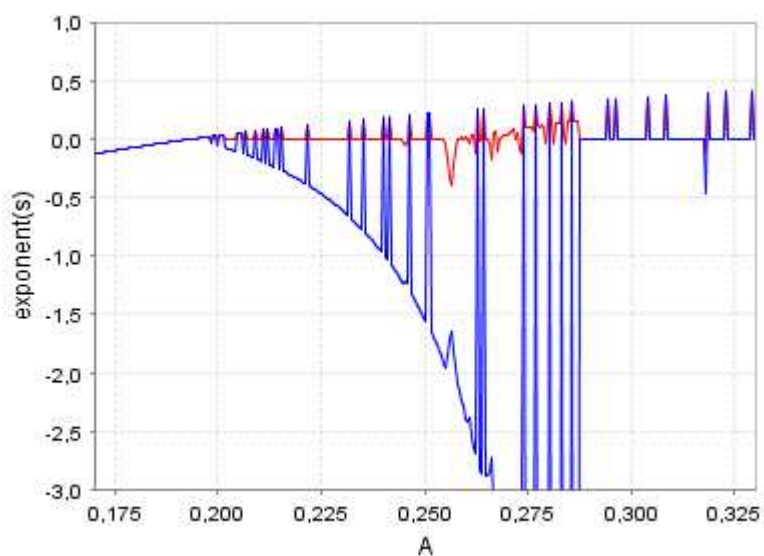


Figure 5 – Lyapunov exponents in the model without optimization ( $n=1$ ).

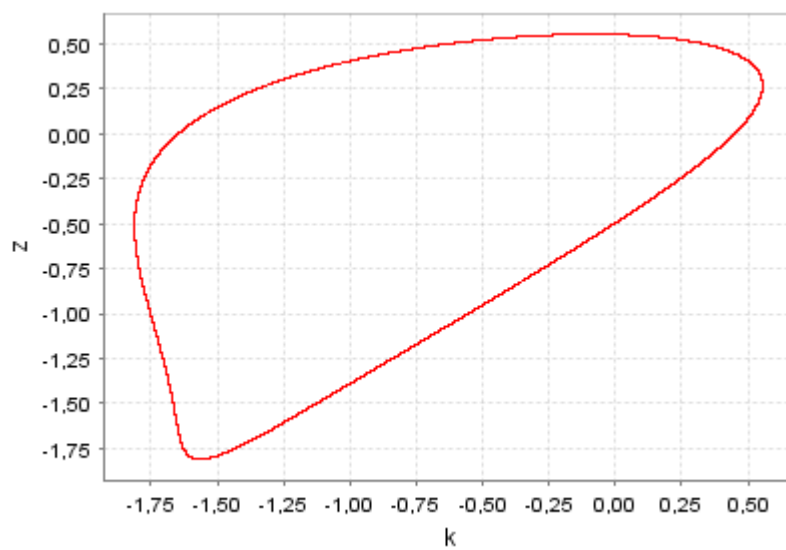


Figure 6 – Attractor for the model without optimization ( $A=0.25$ ;  $n=1$ ).

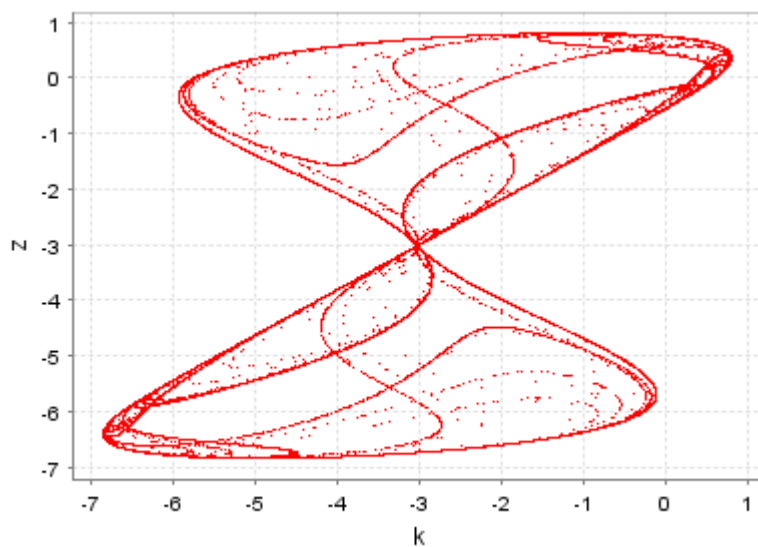


Figure 7 – Attractor for the model without optimization ( $A=0.33$ ;  $n=1$ ).

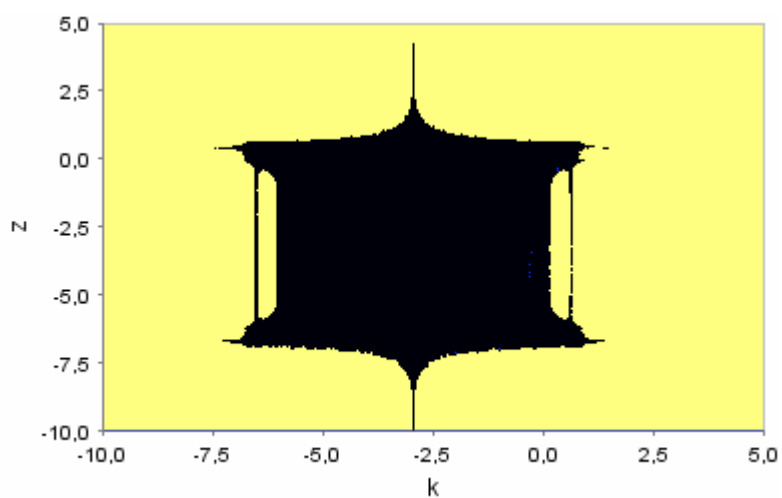


Figure 8 – Basin of attraction for the model without optimization ( $A=0.33$ ;  $n=1$ ).

The area in black is the basin.

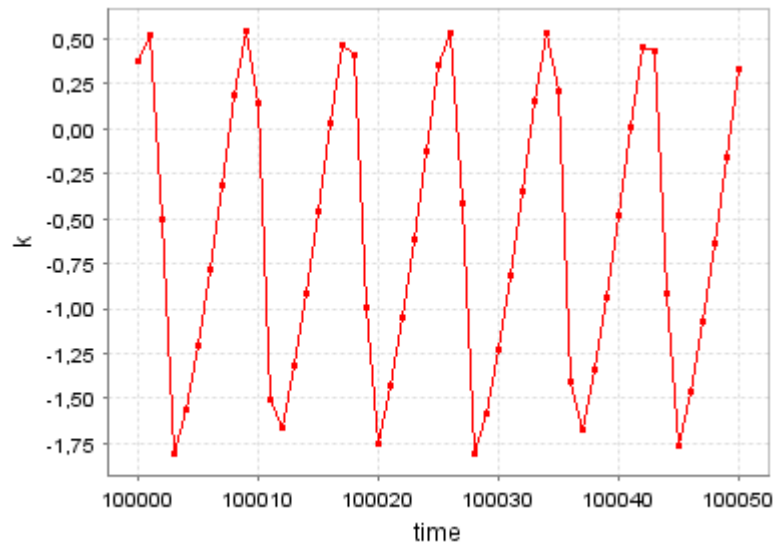


Figure 9 – Time series of  $\tilde{k}_t$  for the model without optimization ( $A=0.25$ ;  $n=1$ ).

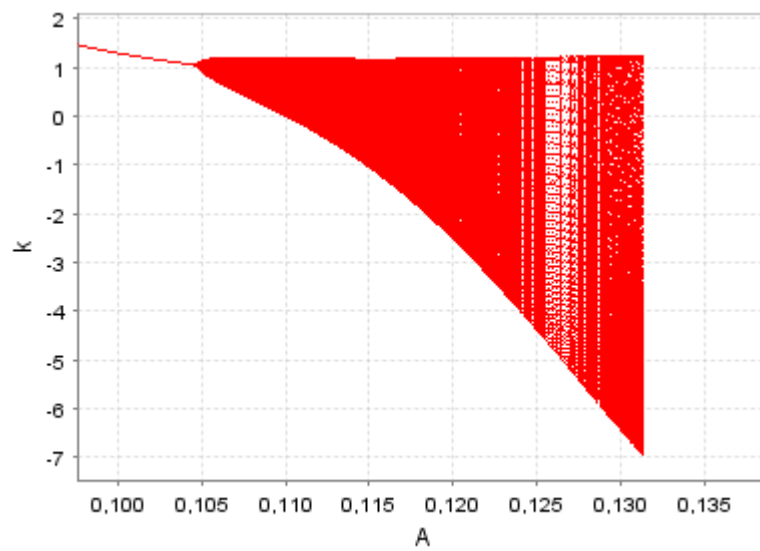


Figure 10 – Bifurcation diagram ( $A, \tilde{k}_t$ ) for the model without optimization ( $n=4$ ).

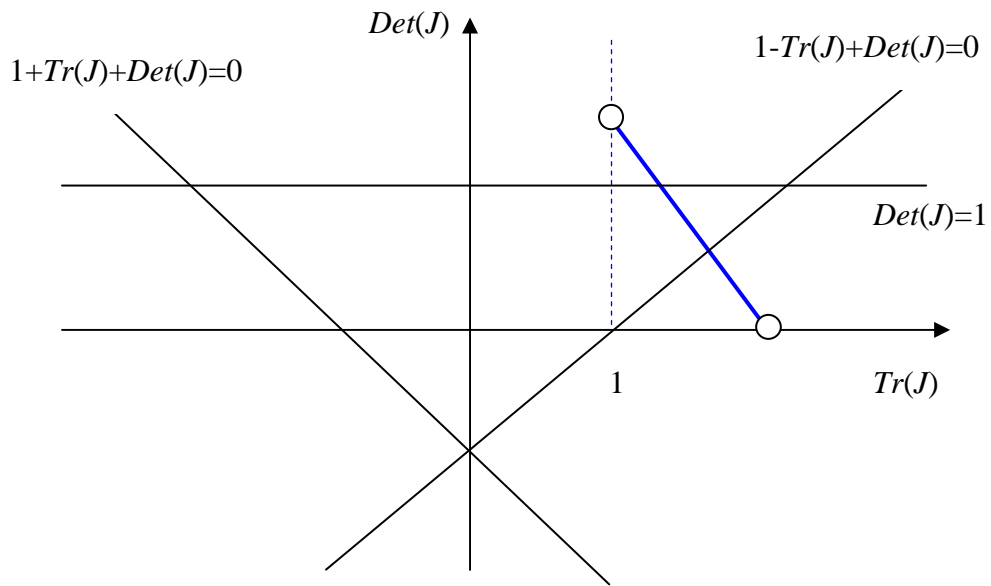


Figure 11 – Local dynamics in the model with optimization ( $n=1$ ).

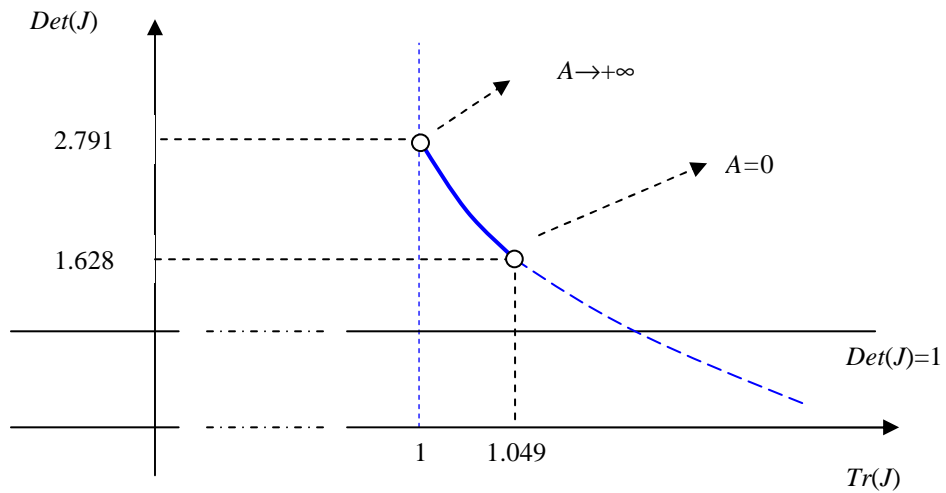


Figure 12 – Local dynamics in the model with optimization, for a selected array of parameter values and a varying  $A$  ( $n=1$ ).

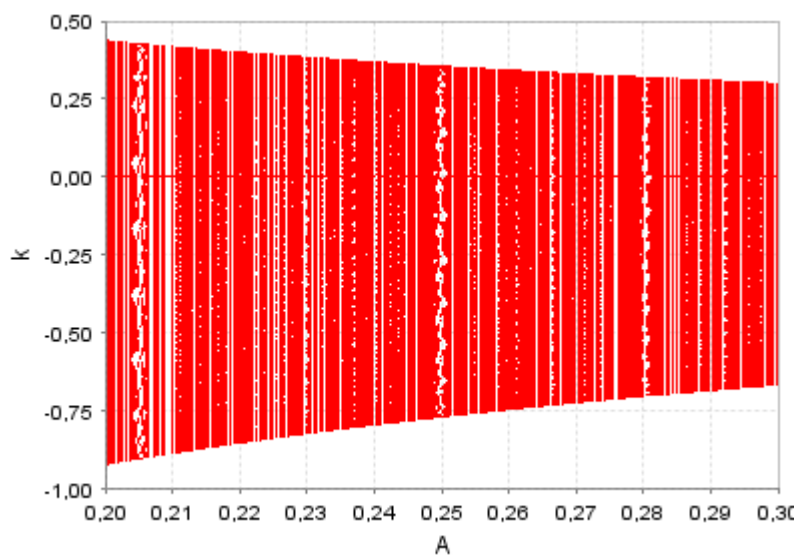


Figure 13 – Bifurcation diagram  $(A, \tilde{k}_t)$  for the model with optimization ( $n=1$ ).

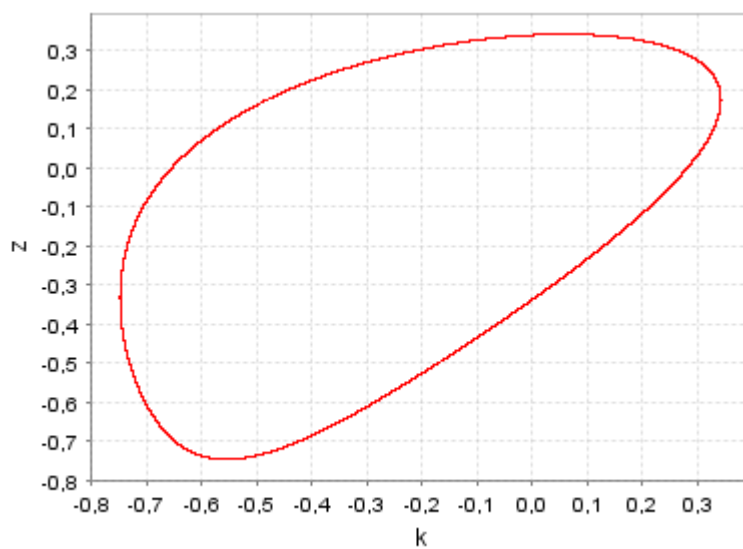


Figure 14 – Attractor for the model with optimization ( $A=0.2593$ ;  $n=1$ ).

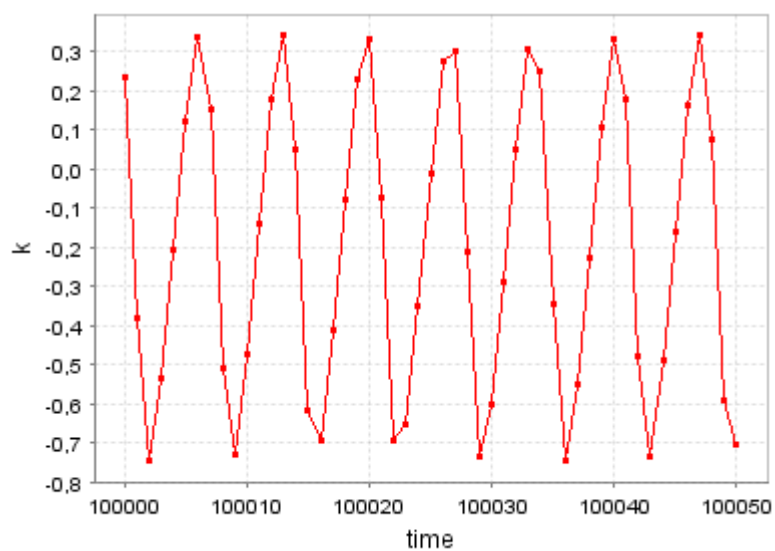


Figure 15 – Time series of  $\tilde{k}_t$  for the model with optimization ( $A=0.2593$ ;  $n=1$ ).

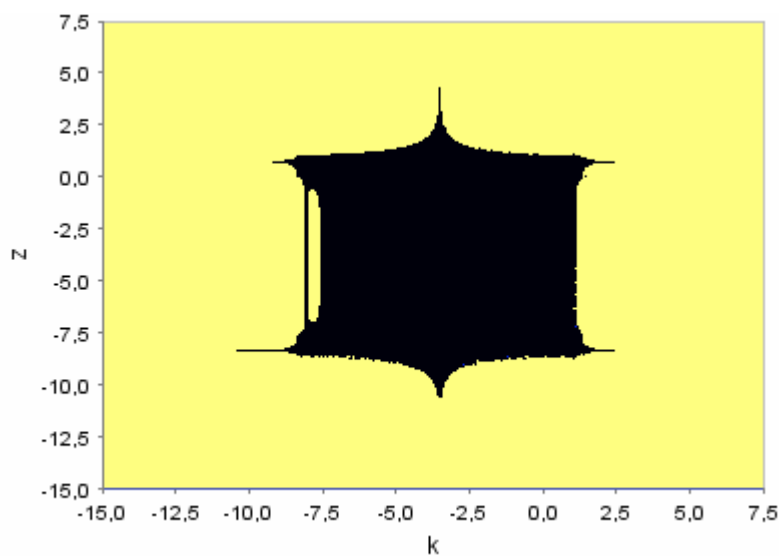


Figure 16 – Basin of attraction for the model with optimization ( $A=0.2593$ ;  $n=1$ ).

The area in black is the basin.