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by

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Abstract

The Inequality Process (IP) and the Saved Wealth Model (SW) are particle system models of income distribution. The IP's social science meta-theory requires its stationary distribution to fit the distribution of labor income conditioned on education. The Saved Wealth Model (SW) is an ad hoc modification of the particle system model of the Kinetic Theory of Gases (KTG). The KTG implies the laws of gas thermodynamics. The IP is a particle system similar to the SW and KTG, but less closely related to the KTG than the SW. This paper shows that the IP passes the key empirical test required of it by its social science meta-theory better than the SW. The IP's advantage increases as the U.S. labor force becomes more educated. The IP is the more likely of the two particle systems to underlie an analogue of gas thermodynamics in social science as the KTG underlies gas thermodynamics.

The Particle System Model of Income and Wealth More Likely to Imply an Analogue of Thermodynamics in Social Science

1. Introduction

The Inequality Process (IP) (Angle, 1983-2009) and the Saved Wealth Model (SW) [Chakraborti and Chakrabarti, 2000; Chatterjee and Chakrabarti, 2004; Patriarca, Chakraborti, and Kaski, 2004, discussed in Lux, 2005, 2008; Patriarca, Heinsalu, and Chakraborti. 2006; Scalas, Gallegati, Guerci, Mas, and Tedeschi, 2006; Yakovenko and Rosser, 2009; Sinha, Chatterjee, Chakraborti, and Chakrabarti, 2011] are two interacting particle systems with stationary distributions intended to model income and wealth distribution. The two particle systems are isomorphic to each other up to their stochastic drivers of exchange between particles and a consequence of that difference. The IP, derived from an old verbal theory in economic anthropology, was published in 1983. The IP has been used to discover stable empirical patterns in income and wealth. See Table 1.

Table 1 about here

The SW, published in 2000, is an ad hoc modification of the stochastic particle system model of the Kinetic Theory of Gases (KTG) (Whitney, 1990). The SW is also known in the literature as the kinetic wealth exchange model, but sometimes the IP is classified the same way so that name is not useful to distinguish the SW from the IP. The KTG is the mechanical basis of gas thermodynamics, explaining the macro-level phenomenon of heat in terms of the mechanics of molecules in collision, and implying laws of gas thermodynamics like Boyle's Law or Charles' Law. Boyle's Law relates the volume of a gas to its pressure; Charles' Law relate its volume to its temperature (Fischer-Cripps, 2004). The SW and a related model, Dragulescu and Yakovenko (2000), were presaged in the particle system modelling of income distribution by a broad survey of particle systems involving the exchange of a positive quantity between particles as possible models of economic phenomena (Ispolatov, Krapivsky, and Redner, 1998). Ispolatov et al. examine the stationary distributions that follow from different types of exchange but with scant attention to what stationary distribution might be empirically relevant or specific equations of exchange. Ispolatov et al. discuss general properties of the equations of exchange, whether they are "additive", "multiplicative", and "greedy". As the present paper shows, details of the exchange, finer grained than those categories, matter. Nevertheless, Ispolatov et al. succeeded in introducing the idea of tinkering with a particle system model to the statistical physics community interested in econophysics, the application of classes of models used in statistical physics, such as the particle system, to economics and sociology.

The IP has a social science meta-theory that assigns empirical referents to its parameters and obliges it to pass certain empirical tests. One of these tests is particularly important. The SW has no social science meta-theory. The SW has not been as tested against data as the IP, prompting the questions of whether the SW can pass the key empirical test that the IP's meta-theory requires the IP to pass, and if so, which particle system performs better on the test. This key empirical test is how well the stationary distribution of each model fits the distribution of labor income conditioned on education (item #4 in Table 1).

Verbal description of the difference between the IP and the SW may seem deceptively insignificant. The difference between the IP and the SW is their stochastic driver of exchange of a positive quantity between particles in particle encounters. The IP's stochastic driver is a discrete 0,1 uniform random variable (r.v.). The SW's stochastic driver is a continuous [0,1] uniform random variable. This paper addresses the question whether the IP and SW are, if not mathematically equivalent, interchangeable for practical purposes in empirical work. Data collected by the U.S. Bureau of the Census over four decades are used in this test of two theories (Current Population Surveys, 1962-2004).

1.1 Is the IP or the SW More Likely to Underlie an Analogue of Thermodynamics in Social Science?

A consequence of the comparison of the IP and SW in terms of the key empirical test that the IP must pass and has passed (See item #4 in Table 1) is the likelihood that the better model in this comparison is the more likely to imply an analogue of gas thermodynamics in labor income distribution and perhaps other areas of economics and sociology. Farfetched to think a model like the IP or the SW could imply an analogue of Boyle's or Charles' Laws in social science? The claim of such a discovery would probably encounter verisimilitude issues among social scientists, but empirical relevance and parsimony are more important criteria of a model than verisimilitude. For example, although the KTG has verisimilitude now, i.e., the model seems obvious, when the KTG was first proposed in the 18th century, it didn't. Back then, a molecule was not an imaged object but a hard-to-believe theoretical construct. The KTG contributed to the verisimilitude of the concept of the molecule rather than vice versa. Model verisimilitude trails behind the acceptance of a model's empirical relevance.

If an algebraic structure in social science analogous to gas thermodynamics seems farfetched, consider the following facts. The IP and SW are isomorphic to each other up to their stochastic drivers of exchange of a positive quantity (wealth) between particles. Both the IP and SW are particle systems isomorphic to that of the KTG (in which the positive quantity exchanged between particles is labeled 'kinetic energy') up to two specific differences in the case of the IP and one specific difference in the case of the SW. Since laws such as Boyle's and Charles' follow from the KTG, questions prompted by the similarity of the IP and the SW to the KTG are 1) whether the IP or the SW imply similar algebraic structures in socio-economic phenomena, and, if so, 2) whether the IP or the SW is to be found underlying such structures. The hard part of the derivation is identifying the socio-economic analogues of temperature, pressure, and volume. Angle (2007a) speculates about the sociological analogue of temperature. Table 1 shows there is nothing farfetched about the empirical relevance of the Inequality Process. The comparable table for the SW would contain a part of item #2 in Table 1. The possibility of thermodynamic analogues in socio-economic phenomean is exciting. Imagine opening a thermodynamics textbook like Bejan (1997) or Gyftopoulos and Beretta (2005), and finding an socio-economic analogue of one of its laws or equations. The present paper bears on the question of whether the IP or the SW is the more likely particle system underlying socio-economic analogues of Boyle's or Charles' Law. The more likely candidate particle system to imply an empirical relevant thermodynamic analogue is plausibly the more empirically adequate to pass the critical test that the IP's meta-theory requires of it. While item #4 of Table 1 cites studies showing that the IP passes the test, the present shows whether the SW can pass the same test, and if so, whether the IP or the SW has the superior score on this test and why.

1.2 The SW, an Elaboration of the Kinetic Theory of Gases (KTG)

The stochastic particle system model of the Kinetic Theory of Gases (KTG) randomly pairs particles for random exchanges of a positive quantity modeling the exchange of kinetic energy between the molecules of a dilute gas in collision. The equations of the exchange are (Whitney, 1990):

$$\begin{aligned} x_{it} &= \varepsilon_t \left(x_{i(t-1)} + x_{j(t-1)} \right) \\ x_{jt} &= (1-\varepsilon_t)\left(x_{i(t-1)} + x_{j(t-1)} \right) \end{aligned} \qquad (1a,b)$$

where,

$$\begin{aligned} x_{i(t-1)} &= \text{particle } i\text{'s kinetic energy at time} - \text{step } (t-1) \\ x_{jt} &= \text{particle } j\text{'s kinetic energy at time} - \text{step } t \\ \varepsilon_t &= \text{an i.i.d. } [0,1] \text{ uniform continuous r.v. at time} - \text{step } t \end{aligned}$$

In the KTG the sum of kinetic energy of particles i and j after a collision equals the sum before. Given that the population of particles is isolated in a reflecting container, the sum of kinetic energy over all particles does not change. The stationary distribution of particle kinetic energy in the KTG is a negative exponential distribution. Dragulescu and Yakovenko (2000) re-label the KTG. Re-labeled, its particles represent people instead of gas molecules; the positive quantity exchanged by particles becomes wealth rather than kinetic energy and the mean of wealth becomes the analogue of temperature. Dragulescu and Yakovenko (2000) have to argue that the stationary distribution of kinetic energy in the KTG, the negative exponential, is also that of the empirical distributions of income and wealth. They perceive a fit between the negative exponential distribution and the distribution of adjusted gross income reported by the U.S. Internal Revenue Service. Dragulescu and Yakovenko (2001), however, are not satisfied with the right tail of the negative exponential as a model of the right tail of the distribution of adjusted gross income. They propose a model that sutures a heavier than exponential right tail to the negative exponential central mass and left tail. The present paper shows that a negative exponential distribution is not a good model of the distribution of labor income of workers with post-secondary educations.

1.3 An Elaboration of the Kinetic Theory of Gases (KTG), the One Parameter Saved Wealth Model (OPSW)

The Kinetic Theory of Gases (KTG) is historically the first success of statistical physics. It is the first model that students of statistical physics are taught. It is perhaps natural for a statistical physicist to view an economy in terms of pairwise transactions between parsimoniously simplified agents, particles, and to tinker with the particle system model of the KTG to yield a stationary distribution that might have socio-economic relevance.

1.3.1 Chakraborti and Chakrabarti (2000), the One Parameter Saved Wealth Model (OPSW)

Chakraborti and Chakrabarti (2000) cite Dragulescu and Yakovenko (2000) and use the KTG as Dragulescu and Yakovenko re-label it, but Chakraborti and Chakrabarti also modify the mathematics of the KTG. They write "...no economic agent trades with the entire money he or she possesses without saving a part of it;", i.e., while the entire kinetic energy of a molecule is available for transfer in a collision with another particle, they introduce the fraction of the positive quantity possessed by a particle that a particle cannot lose to another particle. Chakraborti and Chakrabarti introduce this fraction, λ, the proportion of a particle's wealth not at risk of loss in any one encounter with another particle, as a parameter of the population of particles. When λ = 0, the Chakraborti and Chakrabarti model, (2a,b) is equivalent to the KTG, (1a,b). Chakraborti and Chakrabarti call λ "savings". The justification they give for thinking the model relevant to income distribution is a mental image of market transactions between agents in which λ is called an agent's "savings". The label is not apt. λ is not the share of a gain from an encounter that is saved, but rather the fraction of a particle's wealth not at risk of loss in any one encounter with another particle. λ is like the fraction of chips a gambler holds in reserve in any one poker hand or spin of a rouletter wheel. Chakraborti and Chakrabarti (2000) do not discuss the empirical relevance of their model's stationary distribution.

The Chakraborti and Chakrabarti (2000) particle system is referred to here as the One Parameter Saved Wealth Model (OPSW). The development of the OPSW is chronicled in Yakovenko and Rosser (2009). The equations of the exchange of wealth between two particles in the OPSW, are:

$$\begin{aligned} x_{it} &= \lambda x_{i(t-1)} + \varepsilon_t (1-\lambda)(x_{j(t-1)} + x_{i(t-1)}) \\ x_{jt} &= \lambda x_{j(t-1)} + (1-\varepsilon_t)(1-\lambda)(x_{j(t-1)} + x_{i(t-1)}) \end{aligned} \qquad \text{(2a,b)}$$

where:

$$\begin{aligned} x_{i(t-1)} &= \textit{particle } i\text{'s wealth at time step } (t-1) \\ x_{jt} &= \textit{particle } j\text{'s wealth at time step } t \\ \varepsilon_t &= \textit{an i.i.d. } [0,1] \textit{ uniform continuous r.v. at time} - \textit{step } t \\ 0 &\le \lambda < 1 \\ &\textit{All particles have equal } \lambda. \end{aligned}$$

Note that if λ= 0, 2(a,b), the OPSW becomes 1(a,b), the KTG.

Apart from λ, other features of the KTG particle system remain in place, such as random binary matching of particles for exchanges, the sum of the wealth of two paired particles before an encounter equaling the sum after, and the isolation of the population of particles and their immortality. Patriarca, Chakraborti, and Kaski (2004) report that the stationary distribution of the SW is a gamma probability density function (pdf):

$$f(x) \equiv \frac{1}{\beta^{\alpha}\,\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-x/\beta} \tag{3}$$

where, $x > 0$; x is interpreted as income; α is the shape parameter and β is the scale parameter. The RHS of (3) is denoted GAM(α,β). The negative exponential distribution is a member of the gamma pdf family where α = 1.0. Patriarca et al. (2004) find the shape parameter, α, of the gamma pdf of the stationary distribution of (2a,b) to be:

$$\alpha = \frac{1+2\lambda}{1-\lambda} \tag{4}$$

Substituting 1.0 for α implies that λ = 0, i.e., the KTG, so the Patriarca et al. formula for α is consistent with the OPSW subsuming the KTG. Since mean particle wealth is pre-determined in the OPSW, and the expression for the mean of GAM(α,β) is α/β, (3)'s parameters can be expressed in terms of λ and mean particle wealth.

1.4 The Inequality Process (IP)

The Inequality Process (Angle, 1983-2009) was abstracted from the Surplus Theory of Social Stratification, economic anthropology's explanation of an invariant: the pairing of the earliest evidence of extreme economic inequality in the same archeological strata with the earliest evidence of abundant stored food (Angle, 1983, 1986). This verbal theory asserts that:

a) people compete for surplus, storeable food, a form of wealth,
b) competition distributes wealth and concentrates it,

and,

c) when wealth in the form of storeable food appears among a hunter/gatherer people, usually via the acquisition of agriculture, its concentration among a subset of the population overwhelms the apparent egalitarianism of subsisting without much stored food.

The society that emerges out of a hunter/gatherer population when it acquires a storeable food surplus is called a "chiefdom" by anthropologists, who view it as the most inegalitarian societal form. The chiefdom is otherwise known as the society of the god-king.

The Surplus Theory has a prominent flaw: no answer to the question of why inequality of wealth decreases over the course of techno-cultural evolution beyond the chiefdom when more wealth is produced per capita than in the chiefdom. Gerhard Lenski (1966) amends this flaw in the Surplus Theory speculatively. His speculation explains why inequality of wealth, defined as concentration of wealth, decreased over the course of techno-cultural evolution. The speculation is that the production of more wealth per capita requires workers who are more skilled and that a more skilled worker retains a larger fraction of the wealth that worker produces. The Inequality Process (IP) operationalizes and tests this speculation. Wherever verbal theory offered no help in specifying a mathematical model, the principle of parsimony was used in the specification of the IP.

1.4.1 The One Parameter Inequality Process (OPIP)

A two parameter version of the IP appeared in Angle (1983, 1986). This model was later simplified where one parameter is adequate to explain income and wealth phenomena (Angle, 1993). The one parameter Inequality Process (OPIP) is isomorphic to the SW as defined in (2a,b) up to the stochastic driver of wealth exchange and a consequent difference in the intervals on which the parameters of the two particle systems are defined. Angle (1990) shows that the IP is a particle system similar to the KTG.

The IP's meta-theory makes (1 – ω), where ω is the fraction of wealth lost by a particle in an encounter with another particle, a measure of worker skill, a semi-permanent trait. A particle's ω is apparent when it loses an encounter. A coin toss determines which of two particles randomly paired for competition loses. The share of wealth a loser transfers to a winner, its ω, is pre-determined and to some degree permanent (like a worker's skills). ω is the OPIP particle parameter. In the OPIP, all particles have equal ω's.

The equations for the exchange of wealth between two particles in the OPIP are:

$$\begin{aligned} x_{it} &= x_{i(t-1)} + d_t \omega x_{j(t-1)} - (1-d_t)\omega x_{i(t-1)} \\ x_{jt} &= x_{j(t-1)} - d_t \omega x_{j(t-1)} + (1-d_t)\omega x_{i(t-1)} \end{aligned} \qquad (5a,b)$$

where:

$$\begin{aligned} d_t &= \begin{pmatrix} 1 \textit{ with probability } .5 \\ 0 \textit{ otherwise} \end{pmatrix} \\ \omega &= (1-\lambda) \textit{ where } \lambda \textit{ is the SW parameter} \\ &\quad \textit{except that } 0 < \omega < 1 \\ &\quad \textit{while } 0 \le \lambda < 1 \end{aligned}$$

All particles are randomly paired at each time-step. There are no particle deaths, births, or migration in or out of the population. As in the OPSW, the sum of two particles' wealth before an encounter equals the sum after. To see this fact add (5a) to (5b). The number of particles is an even integer.

(5a,b) can be re-expressed to resemble (2a,b), the transition equations for particle wealth in the OPSW:

$$\begin{aligned} x_{it} &= (1-\omega) x_{i(t-1)} + d_t \omega (x_{j(t-1)} + x_{i(t-1)}) \\ x_{jt} &= (1-\omega) x_{j(t-1)} + (1-d_t)\omega (x_{j(t-1)} + x_{i(t-1)}) \end{aligned} \qquad (5c,d)$$

The OPIP maps into the OPSW, (2a,b), if (1-λ) is substituted for ω and the continuous [0,1] uniform random variate, ε_t, of the OPSW is substituted for the discrete 0,1 uniform random variable, d_t, of the OPIP. This substitution may seem subtle and unimportant, but, as this paper shows, the substitution gives the OPIP, defined by (5a,b), properties distinct from those of (2a,b) although the OPSW is isomorphic to the IP up to that substitution. The substitution requires a different interval on which the model's parameter is defined. The OPSW's particle parameter, λ, can be mapped into the complement of the parameter of the OPIP, (1-ω), one-to-one, except for λ = 0 (where the OPSW becomes the KTG). The IP is not ergodic at ω = 1.0 .

1.4.1.1 Implications of the Different Stochastic Drivers of Wealth Exchange in the OPIP and OPSW

The difference between the OPIP of (5c,d) and the OPSW of (2a,b) is clear in (5e,f), the OPIP expressed in terms of SW notation:

$$\begin{aligned} x_{it} \;&=\; \lambda x_{i(t-1)} + \varepsilon_t (1-\lambda)(x_{j(t-1)} + x_{i(t-1)}) \\ &\qquad + (1-\varepsilon_t)(1-\lambda)(x_{j(t-1)} + x_{i(t-1)})(\varepsilon_t > .5) \\ &\qquad - \varepsilon_t (1-\lambda)(x_{j(t-1)} + x_{i(t-1)})(1-(\varepsilon_t > .5)) \\ x_{jt} \;&=\; \lambda x_{j(t-1)} + (1-\varepsilon_t)(1-\lambda)(x_{j(t-1)} + x_{i(t-1)}) \\ &\qquad - (1-\varepsilon_t)(1-\lambda)(x_{j(t-1)} + x_{i(t-1)})(\varepsilon_t > .5) \\ &\qquad + \varepsilon_t (1-\lambda)(x_{j(t-1)} + x_{i(t-1)})(1-(\varepsilon_t > .5)) \end{aligned} \tag{5e,f}$$

where (ε_t > .5) equals 1.0 if the condition is true, 0.0 otherwise. (5e,f) shows that the one parameter Inequality Process (OPIP), expressed in SW notation, is the OPSW particle system (2a,b) with gains and losses maximally exaggerated. The OPIP is the OPSW with ε_t rounded up to 1.0 or down to 0.0, expressed by the logical term (ε_t > .5), except at λ = 0 (ω = 1.0).

The OPIP of (5e,f) reduces to:

$$\begin{aligned} x_{it} \;&=\; \lambda x_{i(t-1)} + (\varepsilon_t > .5)(1-\lambda)(x_{j(t-1)} + x_{i(t-1)}) \\ x_{jt} \;&=\; \lambda x_{j(t-1)} + (1-((\varepsilon_t > .5)(1-\lambda)(x_{j(t-1)} + x_{i(t-1)}) \end{aligned} \tag{5g,h}$$

In (5e,f) and (5g,h), if (ε_t > .5) is not true (i.e., equals 0.0), particle i of the OPIP loses a (1-λ) fraction of its wealth. However, in the OPSW (2a,b), ε_t > .5 merely means a greater than even probability of gain for particle i <u>ceteris paribus</u>. Particle i actually gains wealth in the SW (2a,b) when ε_t > .5 if:

$$\begin{aligned} \varepsilon_t (1-\lambda)(x_{j(t-1)} + x_{i(t-1)}) \;&>\; (1-\lambda)\, x_{i(t-1)} \\ \varepsilon_t (x_{j(t-1)} + x_{i(t-1)}) \;&>\; x_{i(t-1)} \\ \varepsilon_t\, x_{j(t-1)} \;&>\; x_{i(t-1)} - \varepsilon_t x_{i(t-1)} \\ \varepsilon_t\, x_{j(t-1)} \;&>\; (1-\varepsilon_t)\, x_{i(t-1)} \end{aligned} \tag{6}$$

i.e., the question of whether there is a gain or loss experienced by particle i, a general particle, in the OPSW (2a,b) depends on three variables: ε_t, , $x_{j(t-1)}$, and , $x_{i(t-1)}$, its own wealth. The magnitude of particle i's gain in the OPSW, if particle i has a gain, depends on three variables and the parameter:

$$\begin{aligned} \lambda x_{i(t-1)} + \varepsilon_t (1-\lambda)(x_{i(t-1)} + x_{j(t-1)}) - x_{i(t-1)} \;&>\; 0 \\ -(1-\lambda)(1-\varepsilon_t)\, x_{i(t-1)} + \varepsilon_t (1-\lambda)\, x_{j(t-1)} \;&>\; 0 \\ (1-\lambda)\left[\varepsilon_t\, x_{j(t-1)} - (1-\varepsilon_t)\, x_{i(t-1)}\right] \;&>\; 0 \end{aligned} \tag{7}$$

In the OPSW, particle i loses wealth if the inequalities of (6) are reversed. The magnitude of particle i's loss in the OPSW, if particle i has a loss, is (7) with the inequalities reversed.

In the OPIP, [for example in (5e,f) and (5g,h), in the SW notation], particle i experiences a gain if $\varepsilon_t > .5$, a loss otherwise, i.e., gain or loss depends on a single variable, ε_t. If particle i gains wealth that amount is, in SW notation, $(1-\lambda)\ x_{j(t-1)}$; i.e., an OPIP gain depends on the parameter and one variable. From particle i's point of view the wealth of its competitors, e.g., $x_{j(t-1)}$, is a random variable. If particle i loses wealth that amount is $(1-\lambda)\ x_{i(t-1)}$, i.e., dependent on the parameter and one variable. From particle i's point of view, however, the loss is just dependent on the parameter since, if it is like a person, it knows its own wealth, i.e., the wealth lost in a loss is not a random variable. Fewer variables in the OPIP determine whether a particle wins or loses and the amount won or lost than in the SW. Not only is gain and loss dependent on fewer variables in the OPIP, there is an asymmetry between gain and loss from the point of view of the general particle, say particle i. Gains are random from the point of view of particle i, whereas losses from the point of view of particle i are determined by the parameter and its wealth going into the encounter with another particle: $(1-\lambda)\ x_{i(t-1)}$.

1.4.1.2 Information on the Parameter in the OPIP

A vector containing a particle's wealth at each time step, in both the OPSW (2a,b) and the OPIP (5a,b), contains information about the parameter of the process. This information is more easily estimated from the OPIP wealth vector than the OPSW's because of the exaggeration of gain and loss in the OPIP. In the OPIP a particle's parameter is so clear that it can be calculated without error from the first instance of a decrease in any particle's wealth, if the direction of time is known. If it is not, the clarity of the OPIP parameter is such that the direction of time can be inferred from a vector of a single particle's wealth amounts: time flows in the direction of the first two equal proportional decreases from an adjacent wealth amount in the sequence. It takes two such decreases, given one of the two hypotheses about which way the vector of particle wealth is oriented in time, because the first such decrease might be an increase if the hypothesis is wrong. Such an inference from the OPSW (2a,b) is not similarly deterministic. It requires many vectors of particle wealth histories with the number of such vectors needed for an estimate of a given precision dependent on λ, i.e., the smaller λ, the more information is needed to estimate λ and the more clouded the asymmetry of gain and loss in the OPSW is.

1.4.1.3 An Approximation to the Stationary Distribution of the OPIP

Angle (2002, 2006) uses the run-like (generalized runs) character of the solution of the OPIP to specify, via the relationship of the gamma pdf to the negative binomial pf (the probability distribution of generalized runs of independent binary events), a shape parameter of a gamma pdf , α, (3), approximating the OPIP's stationary distribution. It is:

$$\alpha \approx \frac{1-\omega}{\omega} \tag{8a}$$

(4) is the expression for the shape parameter of the gamma pdf approximating the stationary distribution of the OPSW (Patriarca et al., 2004). The Patriarca et al. (2004) expression for the shape

parameter of the gamma pdf approximating the stationary distribution of the OPSW is on the RHS of (8b) in IP notation, that is, after the substitution $\lambda = 1 - \omega$ is made. The OPIP expression for the shape parameter of the gamma pdf approximating its stationary distribution, (8a), is on the LHS of (8b):

$$\frac{1-\omega}{\omega} \neq \frac{3-2\omega}{\omega} \qquad (8b)$$

The LHS and RHS of (8b) are not equal. (8b) implies that equal particle parameters, ω, in the OPSW and OPIP yield different stationary distributions.

Angle (2002, 2006) makes no claim that the OPIP has an exactly gamma stationary distribution. He gives a proof that no conservative particle system scattering a positive quantity via binary particle interactions, a class that includes the OPIP and the OPSW, has an exactly gamma stationary distribution. Patriarca et al.'s (2004) finding that the OPSW's stationary distribution is a gamma pdf is a numerical finding, unable to distinguish among an exactly gamma stationary distribution, an asymptotically gamma stationary distribution, or a gammoidal distribution.

1.4.1.4 A Mis-statement of the OPIP

The OPIP (or any other published version of the IP) is mis-stated in Patriarca, Heinsalu, and Chakraborti (2006) as:

$$\begin{aligned} x_{it} &= x_{i(t-1)} + d_t \varepsilon_t\, \omega x_{j(t-1)} - (1-d_t)\, \varepsilon_t \omega x_{i(t-1)} \\ x_{jt} &= x_{j(t-1)} - d_t \varepsilon_t\, \omega x_{j(t-1)} + (1-d_t)\, \varepsilon_t \omega x_{i(t-1)} \end{aligned} \qquad (9a,b)$$

where ε_t is an i.i.d. [0,1] continuous uniform random variable and d_t is 1 with probability p if $x_i > x_j$, , 0 otherwise, and $0 < \omega < 1$.

2.0 Setting Up the Key Empirical Test of the Inequality Process Versus the Saved Wealth Model

The Inequality Process (IP) must explain the distribution of wage income conditioned on education since it is derived from verbal theory that asserts that more skilled workers lose less in the competition for wealth, identifying (1-ω) as a measure of worker skill. This mandatory test for the IP requires identifying the stationary distribution of the wealth of particles in the ω_ψ equivalence class with the distribution of wage income at the ψ^{th} level of education. The fraction that the particles in the ω_ψ equivalence class of particles forms of the whole population of particles is set equal to the fraction workers at the ψ^{th} level of education form of the labor force. This version of the Inequality Process differs from the OPIP in that there is a distribution of values of the particle parameter ω_ψ in the population of particles. In the OPIP particles with equal values of ω compete for each other's wealth. In the version of the IP that must pass the test of fitting the distribution of labor income conditioned on education, particles compete with others with different ω_ψ's. This version of the IP is the Inequality Process with distributed omega (IPDO) (Angle, 2002, 2006). It has the following equations for the exchange of wealth between particles i and j:

$$\begin{aligned} x_{i\psi t} &= x_{i\psi(t-1)} + d_t \omega_\theta x_{j\theta(t-1)} - (1-d_t)\omega_\psi x_{i\psi(t-1)} \\ x_{j\theta t} &= x_{j\theta(t-1)} - d_t \omega_\theta x_{j\theta(t-1)} + (1-d_t)\omega_\psi x_{i\psi(t-1)} \end{aligned} \qquad (10a,b)$$

The IPDO is isomorphic to (5a,b) except that particle i is in the ψ^{th} ω equivalence class (all particles whose parameter is ω_ψ), while particle j is in the θ^{th} ω equivalence class. Particles i and j are distinct although they may be drawn from the same equivalence class, i.e. it is possible that $\omega_\psi = \omega_\theta$. The stationary distribution of wealth in each IPDO ω_ψ equivalence class is not in general equal to that of the OPIP with equal ω_ψ unless the ω_ψ equivalence class includes the entire particle population, in which case the IPDO is identical to the OPIP. Another difference between the OPIP and IPDO is that in the OPIP mean wealth in its sole ω_ψ equivalence class is exactly 1.0, whereas in the IPDO, only the unconditional mean of wealth, μ, is exactly 1.0 . In the IPDO, mean wealth in the ω_ψ equivalence class, μ_ψ, is not constrained except that the weighted mean of the μ_ψ must add to the unconditional mean, μ.

2.1 The SW Analogue of the IPDO, the SWDO

While the Inequality Process (IP) must explain the distribution of wage income conditioned on education since its verbal social science meta-theory asserts that more skilled workers lose less in the competition for wealth, the meta-theory of the Saved Wealth Model (SW) imposes no such constraint on it. That is because there is none. This paper raises the question of whether the SW can pass the mandatory empirical test that the IP must and has (item #4, Table 1) passed: fitting the distribution of labor income conditioned on level of worker education. But no publication has appeared in the literature with the Saved Wealth Model (SW) analogue of the IPDO. To compare the IPDO to its SW analogue, a Saved Wealth Model with distributed omega (SWDO) has to be specified. Since the difference between the IP and SW is well defined, the way to specify the SW analogue of the IPDO is also well defined. The equations for the exchange of wealth between particles of the SW analogue of the IPDO, the Saved Wealth Model with Distributed Omega (SWDO), expressed in the IPDO's notation, are:

$$\begin{aligned} x_{i\psi t} &= x_{i\psi(t-1)} + \varepsilon_t \omega_\theta x_{j\theta(t-1)} - (1-\varepsilon_t)\omega_\psi x_{i\psi(t-1)} \\ x_{j\theta t} &= x_{j\theta(t-1)} - \varepsilon_t \omega_\theta x_{j\theta(t-1)} + (1-\varepsilon_t)\omega_\psi x_{i\psi(t-1)} \end{aligned} \qquad (11a,b)$$

where ε_t is an i.i.d. [0,1] continuous uniform random variate and $0 < \omega_\psi \leq 1$.

The SWDO should not be confused with the model in Chatterjee, Chakrabarti, and Manna (2004) whose equations for the exchange of wealth between two particles are:

$$\begin{aligned} x_{i\psi t} &= x_{i\psi(t-1)} + \varepsilon_t \omega_{\theta t} x_{j\theta(t-1)} - (1-\varepsilon_t)\omega_{\psi t} x_{i\psi(t-1)} \\ x_{j\theta t} &= x_{j\theta(t-1)} - \varepsilon_t \omega_{\theta t} x_{j\theta(t-1)} + (1-\varepsilon_t)\omega_{\psi t} x_{i\psi(t-1)} \end{aligned} \qquad (12a,b)$$

where ε_t is an i.i.d. [0,1] continuous uniform random variate, and $\omega_{\psi t}$ is an i.i.d. (0,1] continuous uniform random variate, as is $\omega_{\theta t}$. This SW model violates the Inequality Process' (IP's) meta-theory, which asserts that a particle's ω is semi-permanent in the same way that a person's education or a worker's skill level is semi-permanent, i.e., in (10a,b) the particle's parameter, ω_ψ, is constant and not a random variate at each time step as in (12a,b). As with all SW models (12a,b) is the result of numerical tinkering. It is intended to yield a stationary distribution that has a gammoidal left tail and central mass as well as a heavier than exponential right tail.

Making ω_ψ a random variable to thicken the tail of the unconditional distribution of wealth, which is what (12a,b) does, is not necessary in the IPDO. Angle (1996, 2003a) shows that the unconditional distribution of wealth in the IPDO, with ω_ψ's estimated from the distribution of labor income conditioned on education and ω_ψ equivalence classes forming the same fraction of the population of IPDO particles as groups of workers with the corresponding level of education, has a heavier than exponential right tail, one heavy enough to account for aggregate labor income in the U.S.' National Income and Product Accounts.

3. Does the Saved Wealth Model with Distributed Omega (SWDO) Pass The Key Test Set by The IP's Meta-Theory?

With the specification of the Saved Wealth Model with Distributed Omega (SWDO) there is an SW analogue of the Inequality Process with Distributed Omega (IPDO) and the comparison of how well the Saved Wealth Model (SW) does on the empirical test required by the Inequality Process's (IP's) meta-theory can proceed. The IP's meta-theory designates the empirical referent of $(1-\omega_\psi)$ as worker productivity, operationalized as worker education. Consequently, the stationary distribution of wealth of the IPDO, the IP in which particles have possibly differing values of ω (as workers have possibly differing educations) is obliged to fit the distribution of labor income conditioned on education. This key test of the IP specifies that <u>when</u> a) the stationary distribution of wealth in the ψ^{th} equivalence class of particles is fitted to the distribution of labor income of workers at the ψ^{th} level of education, and b) the fraction of particles in the ψ^{th} equivalence class equals the fraction of workers at the ψ^{th} level of education, <u>then</u> c) the model's stationary distributions fit the corresponding empirical distributions of labor income of workers at each level of education distinguished, and d) estimated $(1-\omega_\psi)$ increases with level of education. This is a sharp test of the Inequality Process since the IP predicts that the ω_ψ's be ordered in a particular way: inversely by level of employment, i.e., smaller ω_ψ associated with more education. This paper distinguishes six levels of education. See Table 2. Thus, the IP predicts that the ω_ψ's be ordered, when estimated from a year's labor income by six levels of worker education, in one way out of 6! ways. 6! = 720.

Table 2 about here.

3.1 The Data for the Key Empirical Test of the IPDO and SWDO

The present paper performs the empirical test of the IPDO, required by its meta-theory, on both the IPDO and the SWDO in identical ways and then compares the performance of the IPDO and SWDO on this test. The test fits the stationary distribution of each model to 43 years of data on labor income (annual wage and salary income) in the U.S. from 1961 through 2003. These data are from the March Current Population Survey (CPS) conducted by the U.S. Bureau of the Census. A large number of households in the U.S. civilian, non-institutionalized population are sampled. The March

CPS is also called the Annual Social and Economic Survey. The March CPS asks for the annual wage and salary income of members of each household sampled in the previous calendar year. For example, data on calendar 1986 labor income were collected in the March 1987 CPS. March CPS data from the 1962 through 2004 surveys supply data on annual labor income in calendar 1961 through 2003. The data were acquired from the Unicon Research Corporation (Current Population Surveys, March 1962-2004). Unicon Research formats the data in a readily accessed database, cleans the data, and documents the consistency of variable definitions from year to year and other aspects of the procedural history of the March CPS. Figure 1 displays the empirical target to which the stationary distributions of the IPDO and SWDO will be fitted for 1986. Figure 1 displays histograms of labor income in nine bins $10,000 wide from {$1 to $10,000} through {$80,001 to $90,000} for each of six levels of worker education. Figure 1 leaves out the tenth relative frequency fitted, {$90,001 to $100,000}. Dollar values have been adjusted to constant 2003 dollars using the Council of Economic Advisers' PCE (personal consumption expenditure) price index numbers from Table B-7 Chain-type price indexes for gross domestic product 1959-2004, ***Economic Report of the President***, February 2005 (Council of Economic Advisers, 2005). The population studied is people with at least $1 in annual wage and salary income and who are 25+ in age.

Figure 1 about here

3.1.1 Ordering Categories of Education

Comparing how well the IPDO and the SWDO fit the distribution of labor income conditioned on education requires an ordered set of education categories. The order of a set of education categories is clear if education is coarsely categorized. The U.S. Bureau of the Census changed its education categories in 1990. A single set of education categories for the period 1962-2004 has to be sufficiently coarse to be insensitive to the change of Census Bureau education categories in 1990. Another consideration is the amount of information in a set of ordered categories. The amount of information is at a maximum if, subject to the coarseness requirement to assure order and insensitivity to the change in Census Bureau categories in 1990, the number of categories is as large as possible, the distribution of observations falling into the categories uniform, and the number of observations falling into all categories sufficient to estimate the category's relative frequency with a small standard error of estimate. The highest level of education in 1962 had a small relative frequency while the lowest level of education had a small relative frequency in 2004. The top and bottom categories are "open end". The minimum age of 25 of workers in the study allows them to complete advanced educations. Table 2 categorizes the education of U.S. workers 1962-2004 in a way that takes these constraints into account.

Table 2 about here

4.0 Estimating the Vector of IPDO ω_ψ's and SWDO ω_ψ's that Minimizes the Sum of Weighted Squared Errors

The IPDO's stationary distribution and the SWDO's are fitted to the distribution of wage income in 1986 conditioned on education (the six categories of table 2) by searching over the parameter vector of six ω_ψ's, one for each of the ω_ψ equivalence classes of particle matched to a category of education. The number of particles in each ω_ψ equivalence class is proportional to the fraction workers with the corresponding level of education in the labor force in a particular year. The fits are done year by year separately for the IPDO and the SWDO. The vector of ω_ψ's providing the best fit for a model becomes the final estimate for that model for a particular year. The stochastic search algorithm is a modified simulated annealing algorithm (Nemhauser and Woolsey, 1988).

4.1 Overview of Fit Via Searching the Parameter Vector

The steps in the fitting of the IPDO or the SWDO to six relative frequency distributions per year in each of 43 years (6 X 43 = 258 histograms) are done in a cycle. The steps for fitting one model (e.g. in terms of the IPDO) are:

a) The fits are done year by year.
b) The number of particles in each of the six ω_ψ equivalence classes is made proportional to the fraction of the labor force with the corresponding level of education.
c) The vector of six ω_ψ's is initiated to .5's.
d) The initial vector of six ω_ψ's is perturbed by multiplying it by the product of a vector of continuous [0,1] uniform random variates and a scalar damping factor. In the first iteration of fits the damping factor is 1.0. In the seven succeeding iterations the damping factor is multiplied by .5. A different vector of uniform random variates is used in each iteration of fits. The smaller damping factor is the "cooling" aspect of a fit.
e) Each perturbed vector of six ω_ψ's is used to generate an IPDO stationary distribution of wealth.
f) The scale of the wealth amounts in all six ω_ψ equivalence classes is multiplied by the estimate of the unconditional mean of labor income in the year whose labor income distribution conditioned on education is being fitted. See Section 8, the Appendix for the estimation method of the unconditional mean of labor income.
g) The wealth distribution in each of the ω_ψ equivalence classes is fitted to the labor income distribution of the corresponding education group of the year being fitted.
h) Errors in the fit are squared and weighted by the fraction of the labor force at each education level in that year and summed.
i) If the perturbed vector fits better than the optimum parameter vector, the mean of the two is taken, and that mean vector becomes the current optimum parameter vector.
j) The damping factor is re-set to 1.0 after eight independently perturbed fits with the initial vector or current optimum parameter vector have been performed, a "re-heating" or "annealing".
k) There are 100 iterations of the 8 successively damped perturbations of the parameter vector regardless of how closely the fitted relative frequencies approximate the empirical relative frequencies, i.e., 800 IPDO stationary distributions are generated and fitted in each one of the 36 independent fits of the IPDO to a particular year's data. 43 years' data are fitted.

The fitting of the SWDO to the same data set is done identically but for the line of GAUSS code in the generation of its stationary distribution that distinguishes it from the IPDO and the address to which the output files are directed. The IPDO program rounds up or down the stochastic driver of wealth exchange between particles in the SWDO, a [0,1] continuous uniform random variate, to a 0,1 discrete uniform random variate, the IPDO's stochastic driver of wealth exchange between paired particles. GAUSS is a matrix-oriented programming language used to simulated the IPDO and the SWDO and to fit their stationary distributions to the data (Aptech Systems, 2009).

4.1.1 Details of Search over the Model Parameter Vector

There are constraints on the search over the parameter vector. A perturbed value of ω_ψ smaller than .001 is replaced by .001. Similarly, a perturbed value of ω_ψ greater than .999 is replaced by .999. The start vector of ω_ψ's is six .5's, the midpoint of the interval on which the ω_ψ's are defined. The number of particles in each ω_ψ equivalence class is the rounding of ($w_{\psi t}$ x 1,000) where $w_{\psi t}$ is the relative frequency of workers at the ψ^{th} level of education in year t. Note that 'w' is chosen because it is a weight in a weighted sum and should not be confused with 'ω' omega. Each model (IPDO or SWDO) simulation is run for 300 iterations before sampling. Then at the 301st the wealth of each particle is recorded. Each particle's membership in an ω_ψ equivalence class is known. The simulation runs for another twenty-five iterations. At the 326th simulation, the wealth of each particle is recorded again, and so on to the 401st iteration, at which point there are 5 observations on the wealth of each of 1,000 particles for 5,000 observations altogether. Particle wealth in each ω_ψ equivalence class is adjusted via (13) so that mean wealth in each ω_ψ equivalence class equals mean labor income of workers with a given level of education. Particle wealth is then aggregated into ten relative frequency bins, i.e., {$1 to $10,000}, {$10,001 to $20,000}, up to {$90,001 to $100,000} in 2003 dollars, the same bins empirical incomes are aggregated into. The percentage of labor incomes under $100,000 in 2003 dollars is over 95% in every year from 1961 through 2003. Over-estimation of the frequency of large incomes is a common misperception.

4.3 An Example of How Fit is Done in One Year, 1986

The fitting procedure employed in this paper fits the model's (IPDO or SWDO) stationary distribution of wealth conditioned on the particle parameter, ω_ψ, to the distribution of U.S. labor income conditioned on education is most easily explained by how it works in a particular year. 1986 is chosen as the example. Figure 4 shows that both models reach their best fit in the mid-1990's. Figure 1 displays the empirical target of the IPDO and SWDO fits in 1986.

4.3.1 The 1986 IPDO Estimates of ω_ψ and μ_ψ, the Mean of Wealth in the ω_ψ Equivalence Class

There are 36 weighted sums of squared errors produced by the 36 independent fits of the IPDO to the 1986 data. The weights are the fraction of the sample in each education category. The mean of the 36 sums of weighted squared errors for 1986 is 0.0040189. Their standard error of estimate is 0.0001495 . This statistic is estimatable because there are 36 independent fits of the IPDO to the histograms of Figure 1. The 36 $\hat{\omega}_{\psi t}$ vectors estimated in these fits to 1986 data are all ordered as predicted by the IP's meta-theory. $\hat{\omega}_{\psi t}$ is used in lieu of ω_ψ to indicate that ω_ψ is being replaced by an estimate of it (the caret or "hat") in a particular year (by the subscript 't', in this instance 1986). The means of the 36 1986 IPDO estimates of $\hat{\omega}_{\psi t}$'s are shown in Table 3 along with their standard errors of estimate. The estimated means of wealth, $\hat{\mu}_{\psi t}$, in each of the six $\hat{\omega}_{\psi t}$ equivalence class stationary distributions are also given in Table 3 along with their standard errors of estimate. The unconditional mean of wealth in the OPIP and the IPDO is a constant. Without loss of generality it is assigned the value 1.0 for reasons of stability in numerical calculation. $\hat{\mu}_{\psi t}$ is the estimated mean of wealth of the stationary distribution of wealth of IPDO particles in the $\hat{\omega}_{\psi t}$ equivalence class. Each $\hat{\mu}_{\psi t}$ is the mean of the 36 estimates of $\mu_{\psi t}$ generated by the 36 independent fits of the IPDO to the data of Figure 1. The 36 $\hat{\mu}_{\psi t}$ vectors are all ordered from small to large with (1-$\hat{\omega}_{\psi t}$), i.e., wealth in the $\hat{\omega}_{\psi t}$ equivalence class identified by the IP's meta-theory as corresponding to more productive workers, with productivity operationalized as workers with more education. The estimation of the IPDO's relative frequencies in terms of 2003 dollars is presented requires converting $\hat{\mu}_{\psi t}$ into 2003 dollars and is presented in Section 7, the Appendix. The small standard errors of the $\hat{\omega}_{\psi t}$'s and the $\hat{\mu}_{\psi t}$'s show the unlikelihood of a different ordering of each. Nothing in the fitting and estimation procedure forces the $\hat{\omega}_{\psi t}$'s to scale inversely with level of education or the $\hat{\mu}_{\psi t}$'s to scale with education, the orders predicted by the IP's meta-theory.

Table 3 about here

4.3.2 SWDO Fits in 1986

The SWDO relative frequencies fit are the mean of the 36 sets of relative frequencies estimated in the 36 fits of the SWDO's stationary distribution to 1986 data. 34 out of the 36 SWDO $\hat{\omega}_{\psi t}$ vectors estimated in this fit to 1986 data are ordered as predicted by the IP's meta-theory. Each SWDO fit has a weighted sum of squared errors. The mean of the 36 SWDO sums is 0.0057261, 42% larger than that of the IPDO. The standard deviation of the SWDO sums of squared errors is much greater than that of the IPDO. The SWDO's standard deviation, .00159842, is over ten times that of the IPDO. The SWDO fits the six empirical partial distributions but not as well as the IPDO nor as reliably.

Table 4 about here.

Table 4 shows that the SWDO passes the tests set by IPDO meta-theory, that is, its estimated vector of $\omega_{\psi t}$'s, its $\hat{\omega}_{\psi t}$, (where $\hat{\omega}_{\psi t} = 1 - \hat{\lambda}_{\psi t}$) varies inversely with level of education and its estimated vector of $\mu_{\psi t}$'s, its $\hat{\mu}_{\psi t}$. Estimates of the parameters of the fitted stationary distributions of the SWDO and IPDO do not approximate each other: compare the SWDO's $\hat{\omega}_{\psi t}$'s (data column 1 of Table 4) to those of the IPDO (data column 1 of Table 4). The SWDO's $\hat{\omega}_{\psi t}$'s are over twice those of the IPDO. The standard error of estimate of the SWDO's $\hat{\omega}_{\psi t}$ are larger than the IPDO's particularly for the more educated. The SWDO's estimate of the unconditional mean of annual wage and salary income approximates the IPDO's. The SWDO's estimate is \$30,921 in 2003 dollars whereas the IPDO's is \$31,043. The s.e.e. of the IPDO's estimate of the unconditional mean of annual wage and salary income in 1986 is \$234 while the SWDO's is \$286.

Figure 2 about here.

4.3.3 Comparing IPDO and SWDO Fits to the 1986 Histograms in Figure 1

Figure 2 displays the IPDO (solid curves) and SWDO (dashed curves) partial stationary distributions fitted to the partial distribution of 1986 labor income of people with an elementary education or less (red curves), to the partial distribution of people with some college (green curves), and to the partial distribution of people with post-graduate educations (purple curves). The fitted IPDO partial stationary distributions transition are more peaked at their mode than the SWDO's in the case of the least educated to less peaked at the mode than the SWDO's in the case of the most educated. The IPDO stationary distributions (those with ω_ψ equivalence classes with smaller ω_ψ) fitted to the labor income distributions of the more educated have larger variances than the comparable SWDO stationary distributions fitted to the same data. Consequently, the IPDO partial stationary distributions fitted to the labor income distributions of people with at least some college have heavier tails than the fitted SWDO partial stationary distributions. The comparisons of the fitted partial stationary distributions not shown in Figure 2 are intermediate between the comparisons that are.

Figure 3 about here.

5. Comparing IPDO and SWDO Fits 1961-2003

The findings of the previous section of the paper about IPDO and SWDO fits to 1986 data generalize to March CPS data from 1961 through 2003. In each year the IPDO and the SWDO are fitted in the same way as in 1986. In each year of the 43 years of data, each model is independently fitted 36 times. Figure 3 displays the (43 years X 6 levels of education =) 258 IPDO and SWDO $\hat{\omega}_{\psi t}$ estimates from 1961 through 2003. The $\hat{\omega}_{\psi t}$'s displayed in Figure 3 are the mean of 36 estimates in each year. Both the mean IPDO and SWDO $\hat{\omega}_{\psi t}$ vectors in every year are ordered as the IPDO's meta-theory requires its ω_ψ's to be ordered, i.e., inversely with level of education. However, the IPDO's and the SWDO's $\hat{\omega}_{\psi t}$ do not equal or even approximate each other. The SWDO's $\hat{\omega}_{\psi t}$ are over twice as large as the IPDO's. The trajectories of the IPDO's and SWDO's $\hat{\omega}_{\psi t}$ from 1961 through 2003 are nearly flat, although the estimates in each year are independently arrived at. There appears to be more noise in the SWDO's estimates than the IPDO's.

Figure 4 about here

In not a single year is the mean IPDO fit (the mean of 36 independent IPDO fits) poorer than the mean SWDO fit. Figure 4 graphs the 36 IPDO and SWDO fits in each year. Figure 4 is the piecewise linear curve formed by connecting fit k in year t and fit k in year t+1. These two fits were estimated independently of each other. Both the IPDO and the SWDO fit the data well. However, the IPDO fits are in all but a few cases better, so much reliably better that Figure 4 shows at a glance that it is redundant to do 43 two sample difference of means tests. The 36 IPDO and the 36 SWDO fits in each year hardly overlap and the difference between their means is over three times the standard error of estimate of the mean fit of each model (i.e., the mean of 36 fits). The standard error of estimate of the mean of 36 IPDO fits is smaller than that of the 36 SWDO fits. The IPDO grand total of the 36 sums of weighted squared errors in each of 43 years, 1,548 fits altogether, is 6.4559; the SWDO's is 8.9676. The sum of IPDO weighted squared errors is 72% of that of the SWDO.

Out of the 43 x 36 = 1,548 fits of the IPDO to the distribution of annual wage and salary income conditioned on education, 1961-2003, with 36 independent fits per year, in only 31 instances (2%) did the $\hat{\omega}_{\psi t}$ vector fail to be ordered exactly as the IP's meta-theory requires. The SWDO conformed less closely to the IP's meta-theory requirement. Out of the 43 x 36 = 1,548 fits of the SWDO to the 43 years of data on the distribution of annual wage and salary income conditioned on education with 36 independent fits per year, there are 87 instances of the ω_{ψ} vector failing to be ordered exactly as the IP's meta-theory predicts (about 5.6%). The IPDO had 31 such failures out of its 1,548 fits (2%).

Figure 5 about here.

5.1 Why the IPDO Fits the Data Better Than the SWDO

Figure 5 shows the ratio of the IPDO weighted sum of squared errors to the SWDO weighted sum of squared errors at each level of worker education in each year. The IPDO is often a better fit than the SWDO to the labor income distributions of the two least educated groups, but its superiority is not great or uniform over time. However, among workers with at least some college education, the IPDO provides a distinctly superior fit to annual wage and salary income distribution, a superiority that grows over the decades. The education level of the U.S. labor force steadily rose, so the IPDO's advantage overt the SWDO grew over time. See figure 6.

Figure 6 here.

6. Conclusions

The Inequality Process (IP) and the Saved Wealth Model (SW) are stochastic binary interacting systems. Both models randomly pair particles for interaction. In both models the population of particles is isolated and the positive quantity, called wealth, exchanged between particles when they are paired and interact is neither created nor destroyed. Every time particles are paired, they exchange wealth. Since the populations of particles in both models are isolated, the sum of wealth over all particles does not vary over time. Since the encounters are zero-sum, they can be construed as competition for wealth. The IP was abstracted in the early 1980's from an old theory of economic anthropology about the origin of substantial economic inequality in competition for stored food, as speculatively extended by a sociologist to account for decreasing inequality, in the sense of concentration, over the course of techno-cultural evolution. This speculative extension is that more skilled workers retain a larger share of the wealth they create. This verbal meta-theory assigns an empirical referent to the IP's parameter, a characteristic of particles. This parameter is denoted ω, $0.0 < \omega < 1.0$, and is the share of wealth a particle loses when it loses to another particle. This meta-theory designates the empirical referent of $(1-\omega_\psi)$ as worker productivity, operationalized as the ψ^{th} level of worker education.

Consequently, the stationary distribution of wealth of the Inequality Process with Distributed Omega (IPDO), the IP in which particles can have different values of ω (as workers can have different educations), is obliged to fit the distribution of labor income conditioned on education, taking labor income as the index to the primary form of capital in industrial societies, human capital, operationalized here as a worker's education. This obligation is tested by showing that when a) the stationary distribution of wealth in the IPDO ψ^{th} equivalence class of particles is fitted to the distribution of labor income of workers at the ψ^{th} level of education, and b) the fraction of particles in the ψ^{th} equivalence class equals the fraction of workers at the ψ^{th} level of education, then c) the IPDO's stationary distribution in the ψ^{th} equivalence class fits the income distribution of workers at the ψ^{th} level of education, and d) the estimated $(1-\omega_\psi)$'s increase with level of education. The IPDO passes these tests.

6.1 How does the Saved Wealth Model Do on the Empirical Test the Inequality Process' Meta-Theory Requires it to Pass?

Although published after the Inequality Process was shown to be a particle system model similar to that of the Kinetic Theory of Gases (Angle, 1990), Ispolatov, Krapivsky, and Redner (1998) raise the possibility for statistical physicists of modifying a particle system so that it models socio-economic phenomena. The Kinetic Theory of Gases (KTG) is the best known particle system among statistical physicists. Dragulescu and Yakovenko (2000) relabel the KTG. The Saved Wealth Model (SW) (Chakraborti and Chakrabarti, 2000) accept their relabeling of the KTG and tinker with its mathematics to yield the Saved Wealth Model (SW).

The SW has had many fewer and less demanding empirical tests than the IP because it a) appeared 17 years later, b) was not abstracted from a social science meta-theory assigning an interpretation to its parameter and setting tests that it must pass, c) was proposed by physicists whose educations were not in socio-economic phenomena, and, d), possibly surprisingly for sociologists, do not view the empirical testing and enlarging of the empirical explanandum of a mathematical model of labor income as a priority[2]. For example, Chakraborti and Chakrabarti (2000) does not broach socio-economic theory or data. Despite a radical difference in provenance, the SW is

[2] Econophysicists are not averse to testing models on stock market data.

isomorphic to the IP up to the stochastic driver of wealth exchange between particles and the end point of the interval on which its parameter is defined, a consequence of the different stochastic driver. The IP's driver of wealth exchange is a 0,1 discrete random variate; the SW's is a continuous [0,1] uniform random variate. The IP's particle parameter, ω, is the fraction of wealth a particle loses to another particle; the SW's parameter, denoted λ, is the complement of ω, $\lambda = 1 - \omega$, with an exception at the endpoint of the interval on which the parameters are defined. The SW's particle parameter, λ, can be mapped into the complement ($1-\omega$) of the IP's parameter, ω, one to one, except at $\lambda = 0$ (the special case of SW equal to the KTG) because the IP with $\omega = 1.0$ is not ergodic.

While there is no published SW analogue of the IPDO, an SWDO, it is clear how to define an SW version that is isomorphic to the IPDO (the Inequality Process with Distributed Omega) up to the difference in definition between the OPIP (the One Parameter Inequality Process) and the OPSW (the One Parameter Saved Wealth Model). The SWDO passes the key empirical test required of the IP by its social science meta-theory. The stationary distribution of both the IPDO and the SWDO provide a moderately close fit to the 258 partial distributions of the distribution of labor income in 43 years conditioned on 6 levels of education, as does the IPDO. In every one of the 43 years from 1961 through 2003 the mean of the 36 independently estimated ω_ψ SWDO vectors is ordered as required by the IPDO's meta-theory (inversely with level of education). The same is true of the IPDO. See Figure 3.

6.2 The Difference Made by Similar Sounding Stochastic Drivers of Wealth Exchange

The IP employs a 0,1 discrete uniform random variate to drive wealth exchange between particles. It is a coin toss to see which wins wealth from the other. The amount of wealth is predetermined. The SW employs a similar sounding stochastic driver of wealth exchange between particles, a [0,1] continuous uniform random variate. The IP's driver can be considered a rounding up or down of the SW's, an exaggeration of winning or losing. While similar sounding, the difference of stochastic driver creates differences in how well the IPDO perform on the test that the IPDO's social science meta-theory sets for it. While the mean estimated ω_ψ vectors in each year of both the IPDO and SWDO are ordered as required by the IP's meta-theory, Figure 3 shows that the IPDO's ω_ψ's are quite different from the SWDO's ω_ψ's estimated from the <u>same relative frequency distribution</u>, for exsmple, 1986' six relative frequency distributions shown in Figure 1. Not only is there a gross difference between the IPDO and SWDO estimates of ω_ψ , Figure 3 shows that the SWDO estimates are noisier than the IPDO's. The greater variability in the SWDO's estimates of ω_ψ also is seen when one looks at the ordering of the SWDO's ω_ψ's by size in each of the 36 independent estimates obtained in each of the 43 years of data. In any one year the mean of the SWDO's estimates of ω_ψ, like the IPDO's, are ordered by size as the IPDO's meta-theory requires the IPDO estimates of ω_ψ to be ordered. However, in the 1,548 (36 independent estimates X 43 years = 1,548) SWDO ω_ψ vectors estimated there are 87 instances of ω_ψ vectors not ordered exactly as the IPDO's meta-theory requires, i.e., a 5.6% error rate according to the IPDO's meta-theory. The comparable error rate for the 1,548 estimated IPDO ω_ψ vectors is 31 estimates failing to be ordered exactly as required by the

IPDO's meta-theory, a 2% error rate. If ω_ψ is a signal, then the exaggeration of winning and losing in the IP makes that signal less obstructed by noise than in the SW.

Although the fit of the SWDO's stationary distribution to the distribution of labor income conditioned on education in the U.S. 1961 through 2003 is good, it is not as good as that of the IPDO. Figure 4 shows that the sum of squared error of the IPDO and SWDO fits of the 36 independent fits of each to the distribution of labor income conditioned on education in each year only fail to show the IPDO's sum to be smaller in a few instances out of (1,548 IPDO independent fits + 1,548 SWDO independent fits =) 3,096 fits. The distance between the mean IPDO fit and the SWDO fit in any one year is well over three multiples of the standard error of estimate of these means. The IPDO standard error of estimate of mean fit is smaller in the 43 years of fits than that of the SWDO. Figure 5 shows where the IPDO's advantage in fit is by worker level of education. While the IDPO fits are, averaged over 43 years, better than the SWDO's regardless of education level, the IDPO's advantage in fit is concentrated among the more educated where it increases over time. Figures 2 shows why: the tails of the IPDO's stationary distribution in ω_ψ equivalence classes with smaller ω_ψ are heavier than those of the SWDO. The right tail of the distribution of labor income of the more educated, in particular, is heavier than that of the less educated. See Figure 1. The shapes in Figure1 are typical throughout the time period 1961 to 2003: that's why the IPDO and SWDO estimates of ω_ψ's are flat over time in Figure 3. Figure 6 shows why the fit advantage of the IPDO increased between 1961 and 2003. Figure 6 shows that the fraction of the U.S. labor force with the least education (less than a high school diploma) decreased sharply, while the fraction with more education than a high school diploma increased steadily. The U.S. labor force shifted out of levels of education in which the fit advantage of the IPDO over the SWDO was slight into levels of education in which is more substantial. The two models are not interchangeable as socio-economic models.

6.2 The Particle System that is the More Likely Source of an Empirically Relevant Analogue of Thermodynamics in the Social Sciences

The Inequality Process (IP) has been shown to explain the empirical phenomena of income and wealth enumerated in Table 1. The IP was derived from verbal social science theory and is based on the knowledge incorporated in it. This meta-theory poses tests for the IP the most important of which is the requirement that it fit the distribution of labor income conditioned on education. It does. The Saved Wealth Model (SW) is isomorphic to the IP up to the stochastic driver of wealth exchange and a consequence of that difference. It is the result of tinkering with the particle system model of the Kinetic Theory of Gases (KTG), proposed in the 18th century as a model of gas thermodynamic laws such as Boyle's Law and Charles' Law. Boyle's Law is about the relationship of pressure and volume in a gas in a container. Charles' Law is about the relationship of temperature and volume of a gas in a container. Together they are referred to as the "combined gas law" (Fischer-Cripps, 2004). It is natural to ask whether the IP or the SW is the more likely source of empirically relevant analogues of Boyle's Law and Charles' Law in socio-economic phenomena. The hard part of answering this question is identifying the socio-economic analogues of temperature, pressure, and volume. Angle (2007a) suggests a socio-economic analogue of temperature different from the straight forward analogizing of temperature as a function of mean molecular kinetic energy in the KTG with mean wealth or income. The present paper's demonstration that the IP passes the test its meta-theory requires of it better than the SW suggests that the IP's one difference from the SW, a stochastic driver of wealth exchange that exaggerates the effect of the exchange to a clear win or loss from what it would have been in the SW, is naturally selected and that the IP is perhaps a better bet than the SW to be shown to be the particle system underlying a socio-economic analogue of the combined gas law.

7. Appendix Estimating Mean Labor Income of Workers at the ψ^th^ Level of Education From, $\hat{\mu}_{\psi t}$, Mean Wealth in the $\hat{\omega}_{\psi t}$ Equivalence Class

The estimate of mean 1986 labor income at each level of education, its $\hat{\hat{x}}_{\psi t}$ (in constant 2003 dollars), is estimated as:

$$\hat{\hat{x}}_{\psi t} = \left(\frac{\hat{x}_{(50)\psi t}}{x_{(50)\psi}} \right) \hat{\mu}_{\psi t} \tag{13}$$

where,

$\hat{x}_{(50)\psi t}$ = median annual labor income of workers at ψ^{th} level of education, in constant 2003 dollars, estimated from data;

$x_{(50)\psi}$ = median wealth of particles in the IPDO's $\omega_{\psi t}$ equivalence class;

$\hat{\mu}_{\psi t}$ = mean wealth of particles in the IPDO's $\omega_{\psi t}$ equivalence class.

With the $\hat{\hat{x}}_{\psi t}$'s in hand, the unconditional mean of labor income in 1986 of people aged 25 to 65, $\hat{\hat{x}}_t$, is estimated as:

$$\hat{\hat{x}}_t = \sum_{\psi} w_{\psi t} \hat{\hat{x}}_{\psi t}$$

where,

$w_{\psi t}$ = fraction of sample at the ψ^{th} level of education .

The 1986 IPDO estimate of the unconditional mean of annual labor income, $\hat{\hat{x}}_{1986}$, is calculated as the mean of its estimates in the 36 independent fits of the IPDO to the empirical distribution: $31,043 in 2003 constant dollars. The standard error of this estimate of $\hat{\hat{x}}_{1986}$ is the mean of the 36 estimates, $234.

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Table 1. The Empirical Explanandum of the Inequality Process
1. The universal pairing (all times, all places, all cultures, all races) of the appearance of social inequality and concentration of wealth after hunter/gatherers acquire a storeable food surplus. (Angle, 1983, 1986)
2. Why the gamma family of probability density functions (pdfs) is a useful approximation to labor income distributions conditioned on education and why the unconditional distribution of labor income has a right tail whose heaviness approximates that of a Pareto pdf; (Angle, 1983, 1986, 1996, 2002, 2003, 2006, 2007b)
3. How the unconditional distribution of personal income appears to be gamma distributed at the national level and in successively smaller regions although the gamma distribution is not closed under mixture, i.e., under aggregation by area; (Angle, 1996)
4. The shapes of the distribution of labor income of workers by level of education, why this sequence of shapes changes little over decades, and why it is similar to the sequence of shapes of the unconditional distribution of personal income over the course of techno-cultural evolution; (Angle, 1983, 1986, 2002, 2003, 2006, 2007b)
5. The dynamics of the distribution of labor income conditioned on education as a function of the unconditional mean of labor income and the distribution of education in the labor force; (Angle, 2003a, 2006, 2007b)
6. Why and how the distribution of labor income is different from the distribution of income from tangible assets; (Angle, 1997)
7. Why the IP's parameters estimated from certain statistics of the year to year labor incomes of individual workers are ordered as predicted by the IP's meta-theory and approximate estimates of the same parameters from the fit of the IP's stationary distribution to the distribution of wage income conditioned on education; (Angle, 2002)
8. The Kuznets Curve in the Gini concentration ratio of labor income during the industrialization of an agrarian economy; (Angle, Nielsen, and Scalas, 2009)
9. In an elaboration of the basic IP: if a particle in a coalition of particles has a probability different from 50% of winning a competitive encounter with a particle not in the coalition, this modified IP can reproduce features of the joint distribution of personal income to African-Americans and other Americans: a) the % minority effect on discrimination (the larger the minority, the more severe discrimination on a per capita basis); b) the relationships among: i) % of a U.S. state's population that is non-white; ii) median non-African-American male labor income in a U.S. state; iii) the Gini concentration of non-African-American male labor income in a U.S. state; and iv) the ratio of African-American male to non-African-American male median labor income in a U.S. state. (Angle, 1992)

Table 2. Ordered Set of Education Categories
eight years or fewer years of primary education (including illiteracy); open end category
some high school education
high school graduate (completion of four years of secondary education
some college (some post-secondary education)
college graduate (completion of four years of post-secondary education)
at least some post-graduate education (including academic and professional degree programs); open end category

Table 3. Estimated Particle Parameters $\hat{\omega}_{\psi t}$ and the Estimated Mean of Wealth, $\hat{\mu}_{\psi t}$, in Each IPDO ω_ψ Equivalence Class Based on 31 Fits of the IPDO to the Distribution of Wage Income Conditioned on Education in U.S. in 1986				
education	estimated $\hat{\omega}_{\psi t}$	mean standard error of estimate of $\hat{\omega}_{\psi t}$ in 36 replications of fit to 1986 data	estimated $\hat{\mu}_{\psi t}$ wh ere μ = 1.0 (mean of 36 independent replications of fit to 1986 data)	mean standard error of estimate of $\hat{\mu}_{\psi t}$ in 36 replications of fit to 1986 data
eight years or less	.4733	.0200	0.6571	.0280
some high school	.4261	.0173	0.7826	.0273
high school graduate	.3674	.0096	0.8602	.0130
some college	.3162	.0104	1.0046	.0234
college graduate	.2528	.0090	1.2568	.0353
some post-graduate education or more	.1940	.0078	1.6152	.0402

Table 4. Estimated Particle Parameter Vector, $\hat{\omega}_{\psi t}$, and the Estimated Mean of Wealth, $\hat{\mu}_{\psi t}$, in Each SWDO ω_ψ Equivalence Class Based on 36 Fits of SWDO to Distribution of Wage Income Conditioned on Education in U.S. in 1986				
education	estimated $\hat{\omega}_{\psi t}$	standard error of $\hat{\omega}_{\psi t}$ (estimated from 36 replications of fit to 1986 data)	$\hat{\mu}_{\psi t}$ estimated where $\mu_t = 1.0$ (mean of 36 independent replications of fit to 1986 data)	standard error of $\hat{\mu}_{\psi t}$ (estimated from 36 replications of fit to 1986 data)
eight years or less	.9697	.0221	0.6811	0.0269
some high school	.9055	.0350	0.7426	0.0248
high school graduate	.8006	.0214	0.8517	0.0121
some college	.6928	.0298	0.9837	0.0328
college graduate	.5346	.0252	1.2786	0.0406
some post-graduate education or more	.4138	.0191	1.6425	0.0406

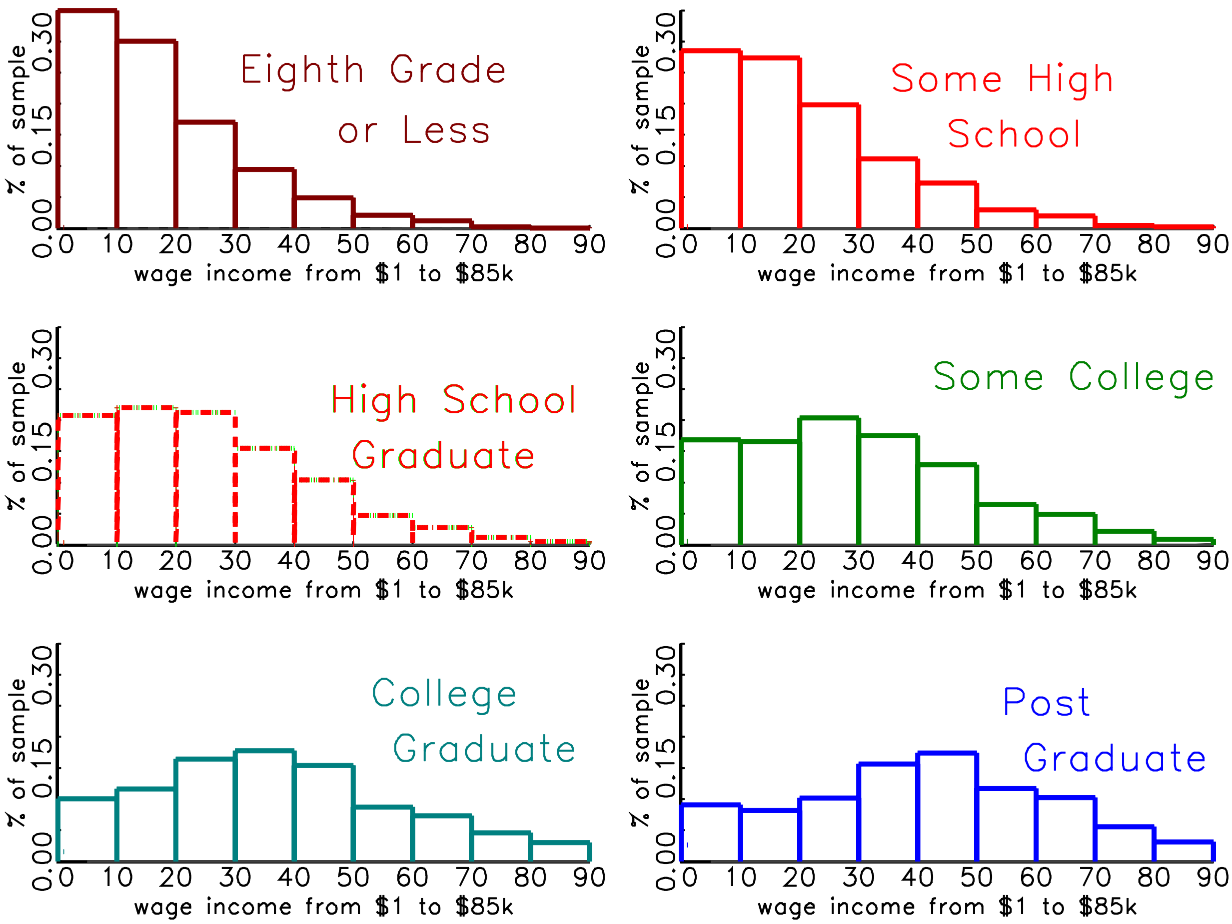

Figure 1. Distribution of U.S. Annual Labor Income in 1986 Conditioned on Education.
Histograms are 1986 relative frequencies. Dollar amounts are in terms of 2003 constant dollars. Sample: People age 25+ with at least $1 in labor income. Source: March Current Population Surveys (Unicon Research).

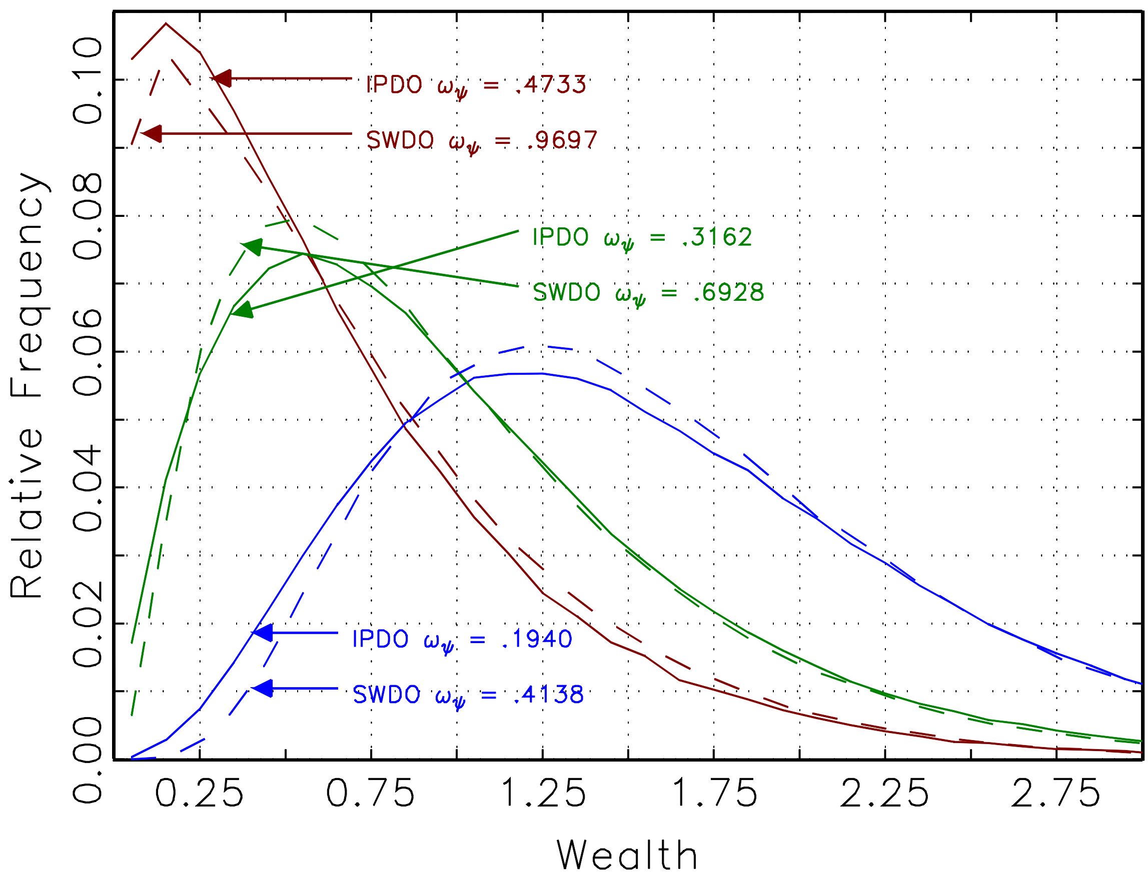

Figure 2. Differences in Peakedness at Mode between IPDO's (solid curve) and SWDO's (dashed curve) Stationary Distributions Fitted to Partial Distributions of Annual Labor Income in 1986 Conditioned on Education.
Each piecewise linear curve is the relative frequency polygon histograms of bins 0.1 wide at wealth midpoint. Unconditional mean of wealth, μ, equals 1.0 .

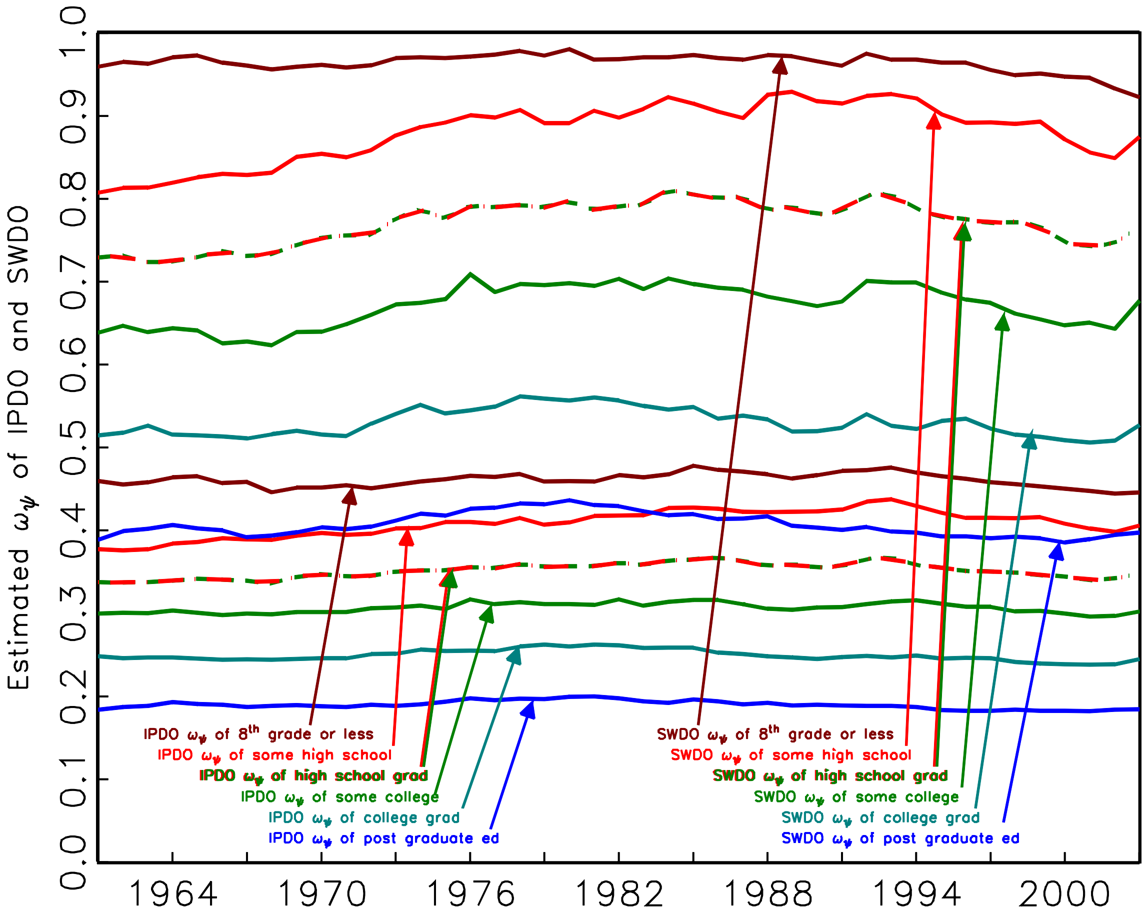

Figure 3. Time-series of Estimates of ω_ψ in IPDO and SWDO

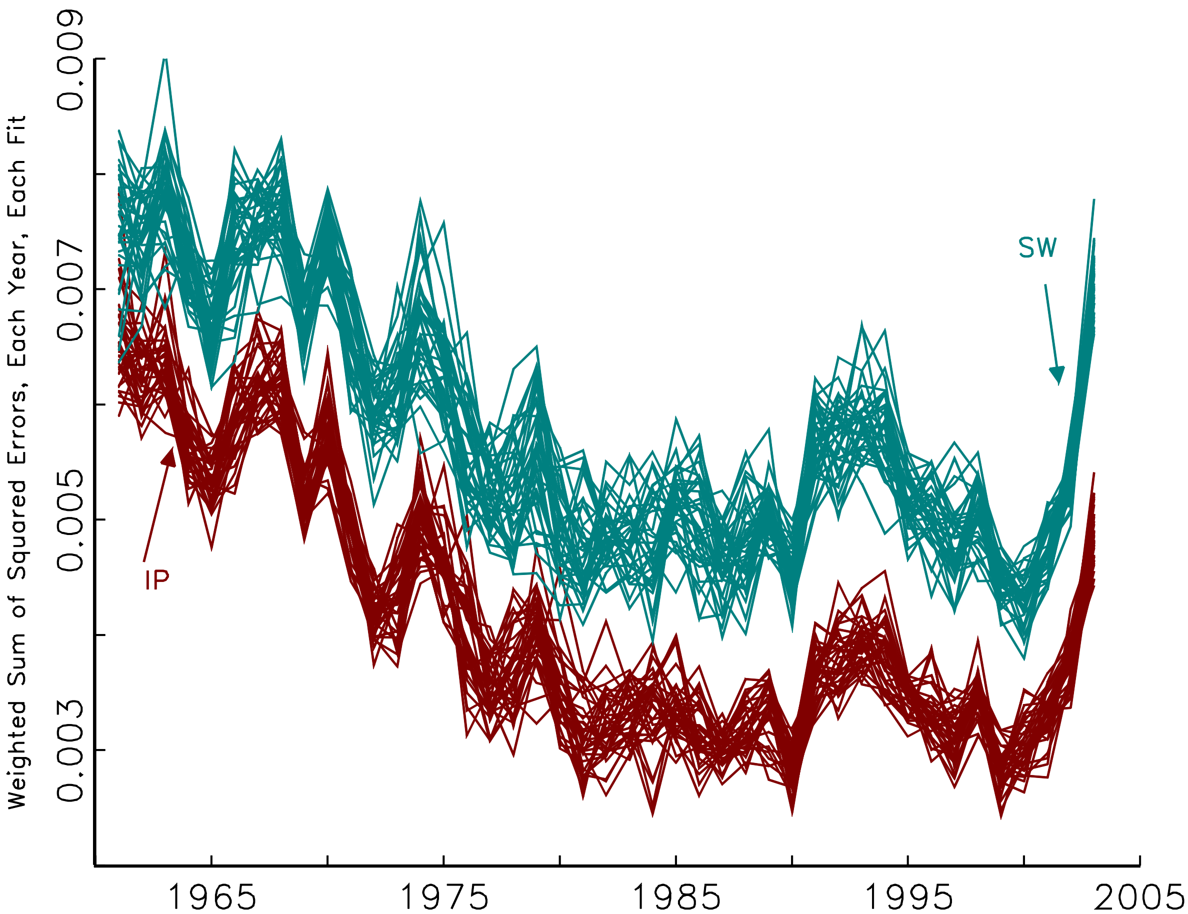

Figure 4. 36 IPDO Fits Per Year v. 36 SWDO Fits Per Year

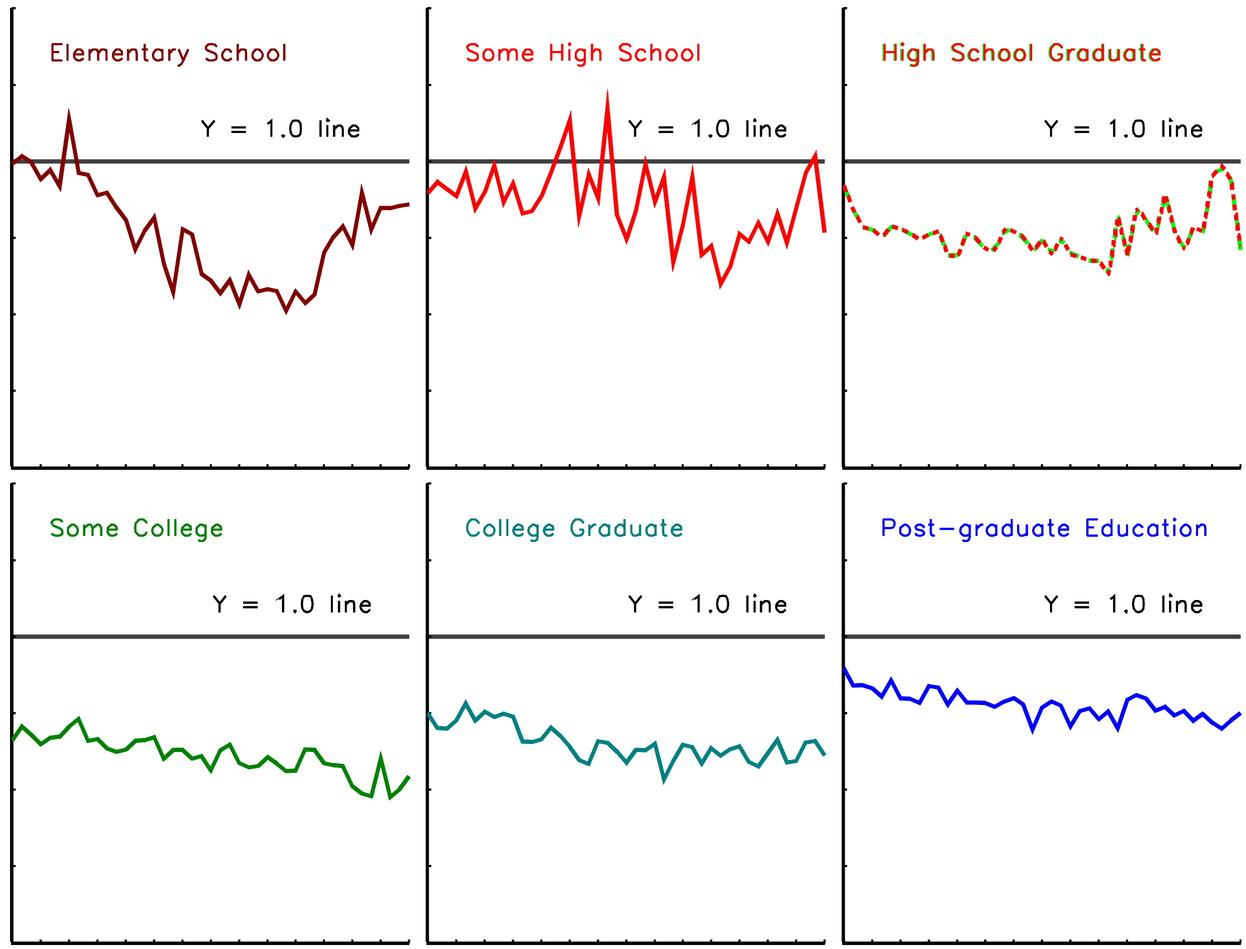

Figure 5. Time-Series of the Ratio of Mean Squared Error of IPDO Fits to Mean Squared Error of SWDO Fits

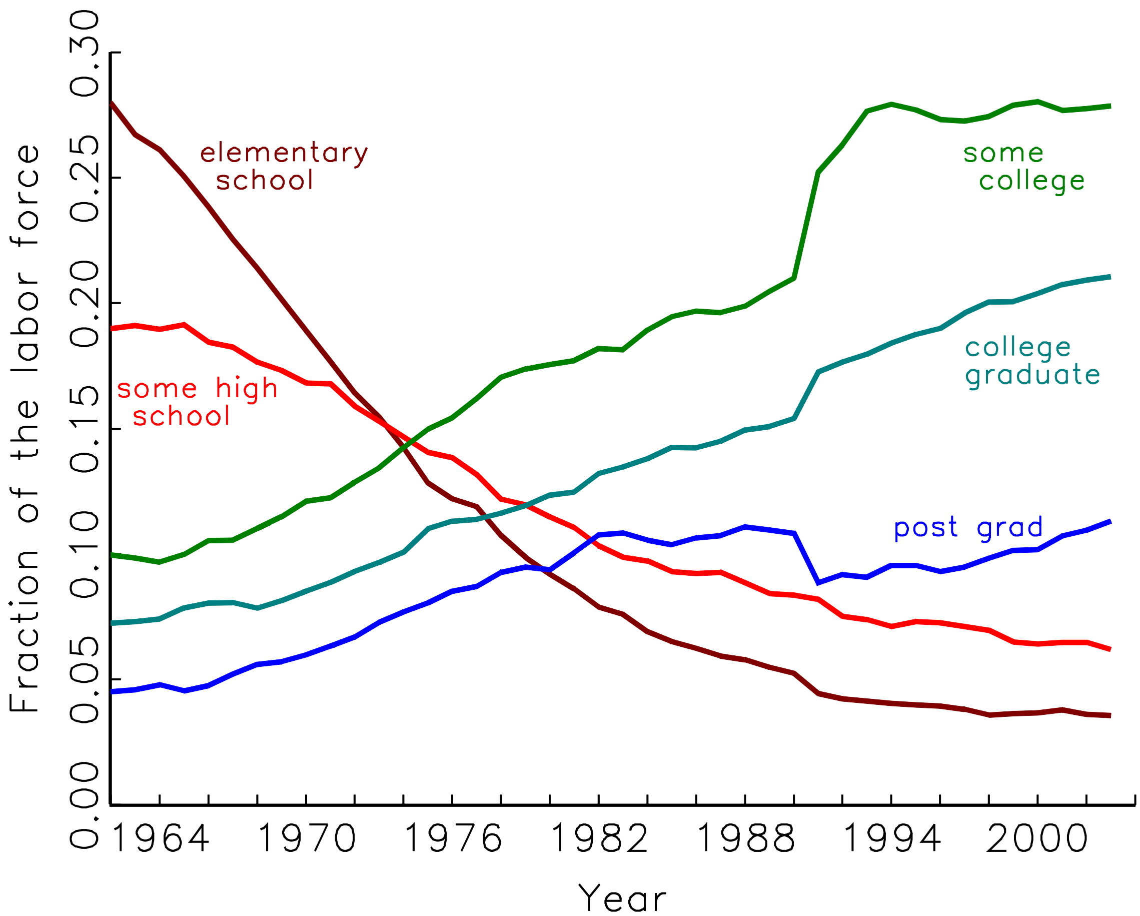

Figure 6. Fraction of Labor Force by Level of Education (except high school graduates), 1962-2004