Nonlinear dynamics in a model of financial development with a risk premium

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Abstract: The relation between the degree of financial development of an economy (measured by the extent in which constraints to credit exist) and fluctuations affecting the trend of economic growth, is a relevant theme of discussion in macroeconomics. Some of the literature on this field argues that the cyclical behaviour is generated endogenously, under the model's assumptions, for specific levels of credit availability. Following this line of reasoning, the paper develops a theoretical framework that places a risk premium over the international interest rate as the centre piece of the explanation for the occurrence of endogenous business cycles, under particular levels of financial development. The risk premium penalizes the borrowing capacity of the less wealth endowed countries. The analysis explores both local and global dynamics.

Keywords: Financial development, Credit constraints, Risk premia, Endogenous business cycles, Nonlinear dynamics, Chaos

JEL classification: O16, E32, C61.

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1. Introduction

From an empirical point of view, it is well accepted that the level of financial development is strongly correlated with economic growth, at least in the short run. This result is highlighted, for instance, by Levine (1997, 2005), Demirguç-Kunt and Levine (2001) and Christopoulos and Tsionas (2004); these last authors present evidence that allow to conclude that the causality runs from finance to growth, and not the other way around. This result seems logical: developed financial markets, in which barriers to credit are not too relevant, contribute to an efficient allocation of productive inputs across economic agents and also in a temporal perspective, allowing for a potentially higher level of generated income.

This simple and intuitive result is relevant when trying to establish a link between the long term growth trend and short run fluctuations. Somehow surprisingly, theories on business cycles and on economic growth have evolved under separate paradigms that rarely intersect each other. Aghion, Angeletos, Banerjee and Manova (2005) stress this odd fact, by recognizing that the modern theory of cycles gives relevance to the degree of financial development as a source of propagation of productivity shocks, but in business cycles analysis these disturbances frequently arise as exogenous; in the opposite field, the modern growth theory emphasizes the central role of productivity and often considers it as the outcome of an endogenous production process, in order to explain growth trends, but it neglects any mechanism of propagation that eventually generates short run fluctuations.

This paper intends to contribute to the literature on the integrated approach to growth and cycles in environments where the degree of financial development may vary. Such literature has benefited from important contributions, starting with the work of Bernanke and Gertler (1989), King and Levine (1993) and Kyotaki and Moore (1997).

Recently, the subject has gained a new impulse with the work of Philippe Aghion and his co-authors. Aghion, Banerjee and Piketty (1999) presented the benchmark model; in this model, the macroeconomic setup is characterized by the existence of an agency problem that limits the access of firms to credit, an element that is modelled by assuming a credit multiplier as the one initially proposed by Bernanke and Gertler (1989). The capital market imperfections generate endogenous fluctuations that persist in the long run. The main economic aggregates (output, investment and the interest rate) will exhibit cycles that are well explained from an economic intuition point of view: there is a multiplier effect of investment that conflicts with increasing interest rates that higher investment levels produce, that is, the tension of two forces that push investment in opposite directions generates and allows to sustain endogenous cycles over time.

The previous work has been extended by Aghion, Baccheta and Banerjee (2004), who study the effects of financial development over small open economies, and conclude that unstable dynamics (in the case, just period two cycles) arise for intermediate levels of financial development, while stability prevails in economies that have credit systems that are either underdeveloped or significantly developed.

In Aghion, Baccheta and Banerjee (2000, 2001), a similar type of analysis is undertaken, but in these studies the focus lies in monetary economies characterized by the presence of nominal price rigidities. The main conclusions are: (i) in credit constrained economies in which debt is issued both in domestic and foreign currencies, currency crises are likely to arise; (ii) currency crises generated by the interplay between credit constraints and price sluggishness are associated with the presence of multiple equilibria.

Finally, the work by Aghion, Howitt and Mayer-Foulkes (2005) establishes the bridge between financial development and growth convergence. Their model proposes an explanation of growth where the growth rate of an economy with a high level of financial development will converge to the rate of growth of the world technology frontier, while all the other economies will systematically grow at a lower rate.

The framework that we propose in the following sections concerns to an endogenous growth setup. This has been the main theoretical structure in which the problem under discussion has been addressed. This is the case of the previously discussed references, as well as of other studies, like Amable, Chatelain and Ralf (2004), who analyze how credit rationing affects endogenous growth when debt is related to the firm’s internal net worth taken as collateral, Blackburn and Hung (1998) and Morales (2003) who concentrate in growth models based on innovation to study the relationship between finance and growth, and also Harrison, Sussman and Zeira (1999) and Khan (2001) who integrate financial intermediation and growth under an AK growth model. The model that we intend to analyze takes as well an AK production function.

Besides the literature on the link between financial development and endogenous growth, the paper also connects to the literature on endogenous business cycles that was initially developed by Benhabib and Day (1981), Day (1982) and Grandmont (1985), among many others, and that was recovered later essentially by two strands: first, the one that searches for non linear dynamics in optimal growth models under competitive markets. Here, we can include Nishimura, Sorger and Yano (1994), Boldrin, Nishimura, Shigoka and Yano (2001), and related literature. These authors search for extreme conditions under which endogenous fluctuations arise in competitive frameworks (e.g., too low discount factors or peculiar types of production functions).

Second, it is relevant to mention the work that has adapted the Real Business Cycle model to a completely deterministic setup able to produce long run business cycles. This work was initiated by Christiano and Harrison (1999), and further developed by Schmitt-Grohé (2000) and Guo and Lansing (2002), among others. These endogenous growth models take the structure of the Real Business Cycle models, namely, a setup where the representative agent has two types of choices to make (between consumption and savings, on one hand, and between leisure and work time, on the other hand), and add to it a production function exhibiting increasing returns to scale (that can result, for instance, from a positive externality on the production of final goods). This framework, that can be contested by the evidence that only too high externality levels are able to produce endogenous cycles, is able to generate cycles of various periodicities including chaotic motion. See Gomes (2006) for a survey on macroeconomic models capable of reproducing cycles as a result just of the non linear relation between economic aggregates.

The model to be presented and discussed is essentially based on the theoretical structure developed by Caballé, Jarque and Michetti (2006) [hereafter CJM]. These authors propose to present a model of financial development pointing to a set of results close to the ones by Aghion, Baccheta and Banerjee (2004), that is, growth related instability eventually arises for intermediate levels of financial development and stability prevails for both low and high levels of credit worthiness. Nevertheless, the type of fluctuations found in the CJM model is much more comprehensive, in the sense that it is not limited to period two cycles, but higher order cycles and complete a-periodicity (including chaos) are obtainable.

The framework to develop below is based on the structure of the CJM model, with two important differences: first, we consider a unique input that is internationally available (the CJM model takes a second production factor, which is country specific); second, besides a constraint on credit, we include a second limitation that firms in less developed countries have when searching for credit in international markets: a risk premium is charged over countries that are less endowed in terms of accumulated wealth. This alternative structure is able to generate, for specific values of some meaningful parameters, endogenous cycles of various periodicities, including chaotic motion.

The model is analyzed both in terms of local and global dynamics. Locally, one identifies the points where bifurcations separate regions of stability (or saddle-path stability) from regions of instability; globally, one confirms that stability truly prevails on the areas identified locally as such, while in the locally unstable areas, we find a region of cyclical behaviour before instability becomes dominant (here, we identify instability with the notion of variables diverging to infinity).

Synthesizing, the paper takes the a model of financial development and growth, simplifies its structure in terms of production conditions (a one input AK production function is taken), introduces a risk premium over the interest rate, and it proposes to analyze this alternative framework about finance and growth. As in the CJM study, completely a-periodic cycles are generated.

The remainder of the paper is organized as follows. Section 2 presents the structure of the model. Section 3 studies local dynamics and section 4 global dynamics. Section 5 presents an additional feature by introducing endogenous technical progress. Finally, conclusions are left to section 6.

2. The Model

We consider a small open economy where a large number of households and firms interact. In this economy, population does not grow and, hence, all aggregate variables may, indistinctively, be considered level or per capital variables. We assume that households consume a constant share of the economy’s income, that is, we take $c\in(0,1)$ as the marginal propensity to consume. Firms generate wealth through the production of goods given the resources available for investment.

Basically, we will work with an endogenous growth setup, since the aggregate production function to consider is of the AK type, i.e., $y_t=Ak_t$, with $A>0$ an index of technological capabilities and y_t and k_t the levels of output and physical capital in a given time moment t. Assuming that capital fully depreciates after one period, there is a coincidence between investment and the stock of capital, $i_t=k_t$, and thus output grows proportionally with the increase in the resources invested in the productive activity.

Firms can borrow funds in the domestic financial markets in order to finance their productive projects. The international nominal rate of interest is r>0, but firms have access to loans at this rate only if the level of accumulated wealth of the economy is not below a given threshold value (w_t^*) imposed by monetary authorities. When the economy's level of wealth, w_t, is below the benchmark level, financial markets perceive a risk associated to loans and therefore they will charge a higher interest, which is as much higher as the larger is the difference between w_t and w_t^*. Consequently, for low levels of development (relatively low levels of accumulated wealth), the interest rate becomes a decreasing function of wealth. Formally, the domestic interest rate on productive loans will be

$$r_{t+1} = \begin{cases} r \cdot f\left(\dfrac{w_t}{w_t^*}\right) \text{ if } w_t < w_t^* \\ r \text{ if } w_t \geq w_t^* \end{cases} \tag{1}$$

In (1), we introduce a time lag by assuming that today's wealth levels will be reflected on tomorrow's interest rate. Function f is continuous and differentiable, with f_w<0, $\lim\limits_{w \to w^*} f\left(\dfrac{w_t}{w_t^*}\right) = 1$ and $f(0) \to +\infty$, i.e., an infinite interest rate is hypothetically applied over loans when the economy is hypothetically endowed with no resources. Furthermore, we assume that if $f\left(\dfrac{w_t}{w_t^*}\right) = \xi$, with ξ some positive constant, we can compute an inverse function f^{-1}, such that $w_t = f^{-1}(\xi) \cdot w_t^*$; the following properties should apply: $f_\xi^{-1} < 0$ and $f^{-1}(1) = 1$.

Function f reflects the risk premium on loans. Possibilities of profitable investment require considering that the marginal productivity of capital exceeds the lowest possible interest rate, i.e., $A>r$.

Besides the imposition of a risk premium, the financial sector will be characterized as well by placing quantitative constraints on credit. These may reflect, for instance, inefficiencies arising from information asymmetries. The level of wealth serves as collateral to loans, and firms may borrow at most μw_t, with μ>0 a credit multiplier that is supposed to translate the level of financial development of the national economy. Therefore, our setup assumes two obstacles to credit: first, an interest premium that affects the price of credit; second, a level or quantitative boundary that imposes a ceiling on the availability of credit.

Wealth dynamics will be given by a simple rule. We just consider that next period's level of wealth corresponds to the non consumed income, with income given by output less debt payment. Letting b_t be the amount of financial resources that are borrowed by firms in moment t, we have

$$w_{t+1} = (1-c)\cdot(y_t - r_t \cdot b_t)\,,\ w_0 \text{ given.} \tag{2}$$

In difference equation (2), the output level may be replaced by an expression reflecting the level of investment. Noticing that the economy will invest, on aggregate, the level of available wealth plus the borrowed resources, then $i_t=w_t+b_t$, and therefore output comes $y_t = A\cdot(w_t + b_t)$. Equation (2) is, thus, equivalent to $w_{t+1} = (1-c)\cdot\left[Aw_t + (A-r_t)\cdot b_t\right]$.

Two cases are clearly distinct, in what concerns firms' behaviour. First, if $A \geq r_t$ then it is profitable to invest in production the largest amount that it is possible to borrow. With a marginal productivity above the financial return, firms choose $b_t=\mu w_t$, and thus equation (2) becomes

$$w_{t+1} = (1-c)\cdot\left[A\cdot(1+\mu) - \mu\cdot r_t\right]\cdot w_t\,,\ (A \geq r_t). \tag{3}$$

Second, when $A<r_t$ firms borrow only until the point where the productive marginal return is equal to the interest rate, that is, $y_t - r_t\cdot b_t = r_t \cdot w_t$. Therefore,

$$w_{t+1} = (1-c)\cdot r_t \cdot w_t\,,\ (A<r_t). \tag{4}$$

It is important to associate the previous wealth expressions with our interest condition (1). Observe that under $A \geq r_t$ we have $f\left(\frac{w_{t-1}}{w^*_{t-1}}\right) \geq \frac{A}{r}$, which is equivalent to $w_{t-1} \geq f^{-1}\left(\frac{A}{r}\right)\cdot w^*_{t-1}$. Likewise, $A<r_t$ implies the relation $w_{t-1} < f^{-1}\left(\frac{A}{r}\right)\cdot w^*_{t-1}$. Remind that the inverse function f^{-1} obeys to the properties previously stated in this section;

particularly, it is true that $f^{-1}\left(\frac{A}{r}\right) < 1$, and this condition allows to distinguish among three different states of the assumed economy, according to the following diagram:

$A<r_t$	$A \geq r_t$	
	$w_{t-1} < w_{t-1}^*$	$w_{t-1} \geq w_{t-1}^*$

The previous scheme reveals that the wealth dynamics equation is a piecewise function with three segments, as follows

$$w_{t+1} = \begin{cases} (1-c)\cdot r \cdot f\left(\frac{w_{t-1}}{w_{t-1}^*}\right)\cdot w_t \text{ if } w_{t-1} < f^{-1}\left(\frac{A}{r}\right)\cdot w_{t-1}^* \\ (1-c)\cdot\left[A\cdot(1+\mu)-\mu\cdot r\cdot f\left(\frac{w_{t-1}}{w_{t-1}^*}\right)\right]\cdot w_t \text{ if } f^{-1}\left(\frac{A}{r}\right)\cdot w_{t-1}^* \leq w_{t-1} < w_{t-1}^* \\ (1-c)\cdot\left[A\cdot(1+\mu)-\mu\cdot r\right]\cdot w_t \text{ if } w_{t-1} \geq w_{t-1}^* \end{cases} \quad (5)$$

The assumption of an AK production function implies that our setup is an endogenous growth framework, in the sense that all the mentioned aggregates (y_t, k_t, i_t, w_t) grow in the steady state at a positive and constant rate. Let this rate be $\gamma>0$ and assume that the benchmark level of wealth, w_t^*, represents a trend of accumulated wealth, such that it grows at rate γ for all t. We define constant $\hat{w}^* \equiv \frac{w_t^*}{(1+\gamma)^t}$ and variable $\hat{w}_t \equiv \frac{w_t}{(1+\gamma)^t}$. System (5) is now rewritten for the detrended variable:

$$\hat{w}_{t+1} = \begin{cases} \frac{1-c}{1+\gamma}\cdot r\cdot f\left(\frac{\hat{w}_{t-1}}{\hat{w}^*}\right)\cdot \hat{w}_t \text{ if } \hat{w}_{t-1} < f^{-1}\left(\frac{A}{r}\right)\cdot \hat{w}^* \\ \frac{1-c}{1+\gamma}\cdot\left[A\cdot(1+\mu)-\mu\cdot r\cdot f\left(\frac{\hat{w}_{t-1}}{\hat{w}^*}\right)\right]\cdot \hat{w}_t \text{ if } f^{-1}\left(\frac{A}{r}\right)\cdot \hat{w}^* \leq \hat{w}_{t-1} < \hat{w}^* \\ \frac{1-c}{1+\gamma}\cdot\left[A\cdot(1+\mu)-\mu\cdot r\right]\cdot \hat{w}_t \text{ if } \hat{w}_{t-1} \geq \hat{w}^* \end{cases} \quad (6)$$

The dynamic analysis of system (6) requires transforming the one equation / two time lags expression into a two equations / one time lag system. This may be done by defining variables $\tilde{w}_t \equiv \hat{w}_t - \overline{w}$ and $\tilde{z}_t \equiv \hat{w}_{t-1} - \overline{w}$, with $\overline{w}$ an equilibrium point of system (6). The system that will be subject to analysis is, thus, the one in expression (7).

$$\begin{cases} \hat{w}_{t+1} = \begin{cases} \dfrac{1-c}{1+\gamma} \cdot r \cdot f\left(\dfrac{\tilde{z}_t + \overline{w}}{\hat{w}^*}\right) \cdot (\hat{w}_t + \overline{w}) - \overline{w} \text{ if } \tilde{z}_t < f^{-1}\left(\dfrac{A}{r}\right) \cdot \hat{w}^* - \overline{w} \\ \dfrac{1-c}{1+\gamma} \cdot \left[A \cdot (1+\mu) - \mu \cdot r \cdot f\left(\dfrac{\tilde{z}_t + \overline{w}}{\hat{w}^*}\right)\right] \cdot (\hat{w}_t + \overline{w}) - \overline{w} \text{ if } f^{-1}\left(\dfrac{A}{r}\right) \cdot \hat{w}^* - \overline{w} \le \tilde{z}_t < \hat{w}^* - \overline{w} \\ \dfrac{1-c}{1+\gamma} \cdot \left[A \cdot (1+\mu) - \mu \cdot r\right] \cdot (\hat{w}_t + \overline{w}) - \overline{w} \text{ if } \tilde{z}_t \ge \hat{w}^* - \overline{w} \end{cases} \\ \tilde{z}_{t+1} = \tilde{w}_t \end{cases} \tag{7}$$

Note that the steady state values of variables $\tilde{w}_t$ and $\tilde{z}_t$ are, both, zero.

The first step to analyze (7) consists in determining the steady state. Two steady state points are feasible. The first is valid for $\overline{w} < f^{-1}\left(\dfrac{A}{r}\right) \cdot \hat{w}^*$; the second for $\overline{w} \ge f^{-1}\left(\dfrac{A}{r}\right) \cdot \hat{w}^*$. They are, respectively, $\overline{w}_1 = f^{-1}\left(\dfrac{1+\gamma}{(1-c) \cdot r}\right) \cdot \hat{w}^*$ and $\overline{w}_2 = f^{-1}\left(\dfrac{A \cdot (1+\mu) - \dfrac{1+\gamma}{1-c}}{\mu \cdot r}\right) \cdot \hat{w}^*$.

Two steady state points exist under the assumption that $f^{-1}(\cdot)$ is a constant value. The following condition is essential to guarantee a positive long run level of wealth: $A \cdot (1+\mu) > \dfrac{1+\gamma}{1-c}$.

3. The Analysis of Local Bifurcations

Because system (7) has two equilibrium points, local dynamics must be dissected in the vicinity of each one of these points. Let us start by taking $\overline{w}_1$. The equilibrium point exists for the first equation of the system. Linearizing the system in the vicinity of this point, one obtains

$$\begin{bmatrix}\tilde{w}_{t+1}\\ \tilde{z}_{t+1}\end{bmatrix}=\begin{bmatrix}1 & \frac{1-c}{1+\gamma}\cdot r\cdot f_z\cdot \overline{w}_1\\ 1 & 0\end{bmatrix}\cdot\begin{bmatrix}\tilde{w}_t\\ \tilde{z}_t\end{bmatrix} \quad (8)$$

In (8), f_z represents the derivative of function f in order to $\tilde{z}_t$. The sign of f_z determines the type of dynamics underlying the system. Because $\tilde{z}_t$ is just a linear transformation of $\hat{w}_t$, we must have $f_z<0$. The dynamic behaviour is characterized in proposition 1.

Proposition 1. *Local dynamics in the vicinity of the first equilibrium point, $\overline{w}_1$, are expressed on the following conditions:*

i) If $\frac{1-c}{1+\gamma}\cdot r\cdot f_z\cdot \overline{w}_1>-1$, then the system is stable in the neighbourhood of $\overline{w}_1$;

ii) If $\frac{1-c}{1+\gamma}\cdot r\cdot f_z\cdot \overline{w}_1<-1$, then the system is unstable in the neighbourhood of $\overline{w}_1$;

iii) If $\frac{1-c}{1+\gamma}\cdot r\cdot f_z\cdot \overline{w}_1=-1$, then a Neimark-Sacker bifurcation occurs.

Proof: The trace and the determinant of the Jacobian matrix in (8) are, respectively, $Tr(J)=1$ and $Det(J)=-\frac{1-c}{1+\gamma}\cdot r\cdot f_z\cdot \overline{w}_1>0$. Conditions for stability are $1-Tr(J)+Det(J)>0 \Rightarrow -\frac{1-c}{1+\gamma}\cdot r\cdot f_z\cdot \overline{w}_1>0$, which is an universal condition; $1+Tr(J)+Det(J)>0 \Rightarrow 2-\frac{1-c}{1+\gamma}\cdot r\cdot f_z\cdot \overline{w}_1>0$, which is also an universal condition; and $1-Det(J)>0 \Rightarrow \frac{1-c}{1+\gamma}\cdot r\cdot f_z\cdot \overline{w}_1>-1$, a condition that applies only for certain combinations of parameter values. Thus, stability can only break down in the circumstance in which the eigenvalues of J become a pair of complex conjugate eigenvalues, i.e., when $Det(J)=1$, or, yet, a Neimark-Sacker bifurcation occurs. Condition $Det(J)>1$ implies instability. Figure 1 depicts graphically this stability result.

- Figure 1 here -

□

Consider now $\overline{w}_2$. This equilibrium relates to the second equation of (7). Once again, we linearize the system, to obtain

$$\begin{bmatrix} \tilde{w}_{t+1} \\ \tilde{z}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1-c}{1+\gamma}\cdot\mu\cdot r\cdot f_z\cdot\overline{w}_2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \tilde{w}_t \\ \tilde{z}_t \end{bmatrix} \quad (9)$$

Proposition 2. *In the vicinity of* $\overline{w}_2$, *the dynamics of the financial model are given by the following conditions:*

i) If $\frac{1-c}{1+\gamma}\cdot\mu\cdot r\cdot f_z\cdot\overline{w}_2 > -2$, *then the system is saddle-path stable in the neighbourhood of* $\overline{w}_2$*;*

ii) If $\frac{1-c}{1+\gamma}\cdot\mu\cdot r\cdot f_z\cdot\overline{w}_2 < -2$, *then the system is unstable in the neighbourhood of* $\overline{w}_2$*;*

iii) If $\frac{1-c}{1+\gamma}\cdot\mu\cdot r\cdot f_z\cdot\overline{w}_2 = -2$, *then a bifurcation occurs.*

Proof: Once again, we look at conditions 1-*Tr*(*J*)+*Det*(*J*)>0, 1+*Tr*(*J*)+*Det*(*J*)>0 and 1-*Det*(*J*)>0 to characterize stability. The first is never satisfied, thus stability (stable node or stable focus) cannot hold; because *Det*(*J*)<0, the third condition is always verified. Thus, it is through the analysis of the sign of 1+*Tr*(*J*)+*Det*(*J*) that we can distinguish between stability outcomes. When *Det*(*J*)<-2, we will have a saddle-path stable equilibrium (one of the eigenvalues of the Jacobian lies inside the unit circle, while the other does not); *Det*(*J*)>-2 implies instability (both eigenvalues outside the unit circle). A bifurcation separates the two referred regions, for *Det*(*J*)=-2. The three previous conditions are equivalent to the ones in the proposition.

Relatively to the bifurcation observe that for the presented determinant value, the two eigenvalues of *J* are -1 and 2. Note that this cannot be considered a flip bifurcation, because although this kind of bifurcation implies that one of the eigenvalues must be equal to -1, it also requires that *Tr*(*J*)$\in$(-2,0) and *Det*(*J*)$\in$(-1,1), which is not the case. Graphically, we have

- Figure 2 here -

□

Specific *f* function. Consider now the following particular function *f*: $f\left(\frac{w_t}{w_t^*}\right)=\left(\frac{w_t}{w_t^*}\right)^{-\theta}$, θ>0. This function obeys to the properties previously postulated. With a particular functional form, one is able to present explicit expressions for the steady states. They are

$$\overline{w}_1=\left(\frac{(1-c)\cdot r}{1+\gamma}\right)^{1/\theta}\cdot\hat{w}^* \text{ and } \overline{w}_2=\left(\frac{\mu\cdot r}{A\cdot(1+\mu)-\frac{1+\gamma}{1-c}}\right)^{1/\theta}\cdot\hat{w}^*.$$

According to system (7), we must have $\overline{w}_2>\overline{w}_1$, which is equivalent, under the specific case in consideration, to $A<\frac{1+\gamma}{1-c}$. Combining this relation with the constraint that allows for a positive $\overline{w}_2$, we can present the following boundary values for the economy's growth rate: $\gamma\in\left(A\cdot(1-c)-1;(1+\mu)\cdot A\cdot(1-c)-1\right)$. The growth rate is bounded given the level of technology, the marginal propensity to consume and the level of financial development. This last parameter is particularly relevant, because it establishes a relation between constraints on credit and growth: the lower are the constraints, the higher is the potential pace of growth.

Local dynamics can be addressed under the specific risk premium function. The Jacobian matrices are, for each one of the equilibrium points:

$$J_1=\begin{bmatrix}1 & -\theta\\ 1 & 0\end{bmatrix} \text{ for } \overline{w}_1;\ J_2=\begin{bmatrix}1 & \theta\cdot\left[A\cdot(1+\mu)\cdot\frac{1-c}{1+\gamma}-1\right]\\ 1 & 0\end{bmatrix} \text{ for } \overline{w}_2.$$

Matrices J_1 and J_2 are particular cases of the matrices in (8) and (9). In the first case, a unique bifurcation parameter exists: θ<1 implies stability and θ>1 instability. A Neimark-Sacker bifurcation occurs at θ=1. For $\overline{w}_2$, the combination of parameters separating the region of saddle-path stability from instability is $A\cdot(1+\mu)\cdot\frac{1-c}{1+\gamma}=\frac{\theta+2}{\theta}$; we conclude that the higher the degree of financial development, the more likely will be the situation in which the system falls into the instability region.

The analysis of global dynamics will allow to clarify the apparent paradox that the previous arguments enclose: first, we have stated that a higher degree of financial development allows for a potentially higher growth rate, which seems an intuitive result; second, high values of parameter μ are associated with instability (this result clearly arises in the global analysis of the section that follows). This result may be justified under the idea that a too high level of μ means too few constraints on credit, or too low collateral requirements on loans. This, in turn, can increase the risk of failure in paying the loans by the borrowers, what can lead to situations of strong decline in the confidence underlying the financial system that may culminate in financial crises. Instability for high values of μ can therefore be associated to credit availability that is not constrained by any precautionary measures.

4. Global Dynamics

The study of global dynamics requires the consideration of a specific form of the system (we consider a same *f* function as in the final part of the previous section) and to assume some benchmark values for parameters. The following are chosen as reasonable values: c=0.75, γ=0.04 and r=0.03; we take as well the indexes $\hat{w}^*=1$ and A=3. The remaining two parameters, θ and μ, will assume several different values in the analysis.

Note that although system (7) may be analyzed in terms of global dynamics, it is a different system for different equilibrium values. Hence, we should study dynamics taking, alternatively, $\overline{w}_1$ and $\overline{w}_2$. Let us start by considering $\overline{w}_1$. In this case, local dynamics has pointed to a Neimark-Sacker bifurcation occurring at θ=1. For values of θ below one, stability prevails (in this case, the global dynamics result is coincidental with the result found locally), while for values of θ above 1 a region of endogenous cycles will arise before instability sets in.[1]

One of the values of θ for which cycles are present is θ=1.3. For this value, we draw a bifurcation diagram concerning parameter μ. One observes that fluctuations indeed prevail for a given set of values of the credit multiplier. Note that μ is bounded

[1] A bifurcation diagram for parameter θ would confirm these dynamic properties. Since we are essentially concerned with the role of the credit multiplier, we omit the presentation of this diagram.

from above, given condition $A\cdot(1+\mu)>\frac{1+\gamma}{1-c}$. Figure 3 displays the bifurcation diagram.[2]

- Figure 3 here -

Recall that $\tilde{w}_t$ is a variable that is modified twice. First, it was detrended and then it was normalized to a zero steady state. The original variable has a positive detrended equilibrium value and follows an upward sloping trend. The modified variable follows, for a specific value of μ for which fluctuations are evident (μ=1.5), the time path displayed in figure 4. Figure 5 presents, for the same value of μ, the long term attracting set of the relation between $\tilde{w}_t$ and $\tilde{z}_t$.

- Figures 4 and 5 here -

The presence of chaotic motion is well demonstrated through the graphical examples, but we can reemphasize the idea by computing Lyapunov characteristic exponents (LCEs). These are a measure of chaos and they indicate the presence of this type of dynamic behaviour if, in a two dimensional system as the one we consider, at least one of the two LCEs is positive. LCEs evaluate the exponential divergence of nearby orbits, that is, they search for sensitive dependence on initial conditions, a property that is accepted to characterize the presence of chaotic motion. Sensitive dependence basically means that if a same deterministic system initialized in two distinct points (even though these may be located very close to each other) it produces long term time series that have no identifiable common features.

Table 1 presents the computation of LCEs and the fractal dimension of the attractor, for various values of parameters θ and μ.[3] The fractal dimension of the attractor is given by the formula $D=1+\frac{\lambda_2}{|\lambda_1|}$, according to the definition by Kaplan and Yorke (1979), where λ_1 is the negative LCE and λ_2 the positive one. If both LCEs are

[2] All the figures concerning global dynamics presented in this paper are drawn using IDMC software (interactive Dynamical Model Calculator). This is a free software program available at www.dss.uniud.it/nonlinear, and copyright of Marji Lines and Alfredo Medio.
[3] iDMC software is also used to compute LCEs.

positive, a circumstance that eventually occurs and that is generally designated by hyper chaos, then the attractor dimension is $D = 2 + \lambda_1 + \lambda_2$, with both λ_1 and λ_2 above zero.[4]

Evidently, the fractal dimension can only be computed for chaotic systems; otherwise, the dimension of the attractor in an order 2 system is equal to 1, that is, the fractal dimension has correspondence on the Euclidean dimension. The non integer dimension that one finds when chaos exists can be thought of as a measure of the degree of chaos. We will have $D>1$, and the higher is D, the stronger is the chaotic nature of the system, in the sense that the divergence of nearby orbits is more intense.

μ	θ	**LCEs**	**Fractal dimension**
1.5	1.3	0.08; 0.20	2.28
1	1.3	0.02; 0.16	2.18
0.5	1.3	-0.01; 0	1
1.5	1.27	0.09; 0.18	2.28
1	1.27	0.03; 0.14	2.17
0.5	1.27	-0.02; -0.02	1
1.5	1.32	0.08; 0.22	2.30
1	1.32	0.01; 0.17	2.18
0.5	1.32	-0.02; 0	1

Table 1 – LCEs and fractal dimensions for system (7), with $\overline{w} = \overline{w}_1$.

In table 1, we consider three possible values for μ and θ. The dynamics are very sensitive to the value of θ; and therefore we consider three values of this parameter that are close together and that involve the presence of chaotic motion. We observe that for μ=0.5, chaos is ruled out, independently of the value of θ, while μ=1 and μ=1.5 correspond to cases of hyper chaos for the selected values of the parameter θ. In these cases we compute a fractal dimension higher than 2.

Consider now the alternative case, where $\overline{w} = \overline{w}_2$. As one has observed through the local analysis, the system now undergoes a different type of bifurcation. Thus, we will certainly obtain distinct dynamic results. In this case, we consider θ=1.1, a value that leads us directly to the region of endogenous fluctuations. Figure 6 respects to the bifurcation diagram regarding the credit boundary variable,

[4] See Medio and Lines (2001), chapter 7, about definitions on LCEs and attractor dimension.

- Figure 6 here -

Once more, cycles of no identifiable periodicity are observed for a given interval of values of the credit parameter. The wealth variable is subject to business cycles, as it can be confirmed by looking at the diagrams in figures 7 and 8 (observe the similarity between the strange attractors in figures 5 and 8).

- Figures 7 and 8 here -

Chaotic motion is confirmed through the computation of LCEs and presentation of table 2, which has a same type of contents as table 1.

μ	θ	**LCEs**	**Fractal dimension**
3	1.1	0.09; 0.14	2.23
2	1.1	0.06; 0.12	2.18
4	1.1	0.09; 0.16	2.25
3	1.09	0.08; 0.13	2.21
2	1.09	0.06; 0.11	2.17
4	1.09	0.11; 0.14	2.25
3	1.12	0.10; 0.15	2.25
2	1.12	0.06; 0.13	2.19
4	1.12	0; --	Instability

Table 2 – LCEs and fractal dimensions for system (7), with $\overline{w} = \overline{w}_2$.

The analysis of table 2 indicates the presence of different 'degrees' of chaos for several values of the parameters, with θ above but close to unity. Note that cases of hyper chaos, that is, attractors with dimensions higher than two are, once again, observed.

As regarded, assuming one or the other equilibrium value, implies getting different dynamic results, but in both cases we find regions of chaotic motion for some values of the level of financial development, meaning that endogenous business cycles may arise as the result of a combination of quantitative constraints on credit and a risk premium that injures the capacity of poorer countries to access credit.

5. An Extension: Endogenous Technological Progress

The model in the previous sections may be extended in several directions. In what follows, we consider that the generation of technology is endogenous, through two assumptions that do not change significantly the qualitative nature of the model, but that allow to find additional results concerning non linear long run behaviour. The two assumptions are: (i) technology is the only input in the production of additional technology; (ii) decreasing marginal returns are assumed in order to obtain a stable equilibrium point.

The dynamic behaviour of the technology variable is given by

$A_{t+1} = g(A_t)$, with A_0 given, $g>0$, $g'>0$ and $g''<0$. (10)

Endogenous technology growth implies two changes in our framework: equilibrium values $\overline{w}_1$ and $\overline{w}_2$ will depend on the steady state value of *A*, and the conditions that characterize the different states (i.e., $A_t<r_t$ and $A_t \geq r_t$) are now dependent on the evolution of the technology variable.

In what concerns local dynamics, we do not find too pronounced changes. Steady state values are the same as before, with a slight difference in $\overline{w}_2$: *A* is replaced by the corresponding steady state value. Linearized systems in the vicinity of steady states are respectively, for $\overline{w}_1$ and $\overline{w}_2$:

$$\begin{bmatrix} \tilde{w}_{t+1} \\ \tilde{z}_{t+1} \\ A_{t+1}-\overline{A} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1-c}{1+\gamma}\cdot r \cdot f_z \cdot \overline{w}_1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & g'(\overline{A}) \end{bmatrix} \cdot \begin{bmatrix} \tilde{w}_t \\ \tilde{z}_t \\ A_t-\overline{A} \end{bmatrix} \quad (11)$$

$$\begin{bmatrix} \tilde{w}_{t+1} \\ \tilde{z}_{t+1} \\ A_{t+1}-\overline{A} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1-c}{1+\gamma}\cdot \mu \cdot r \cdot f_z \cdot \overline{w}_2 & \frac{1-c}{1+\gamma}\cdot (1+\mu) \cdot \overline{w}_2 \\ 1 & 0 & 0 \\ 0 & 0 & g'(\overline{A}) \end{bmatrix} \cdot \begin{bmatrix} \tilde{w}_t \\ \tilde{z}_t \\ A_t-\overline{A} \end{bmatrix} \quad (12)$$

In both cases, one of the eigenvalues of the Jacobian matrix is $g'(\overline{A})$, and the other two are the same as in the dimension 2 system. If $g'(\overline{A})$ is below unity, local dynamics are characterized precisely in the same way as previously: for both steady states a bifurcation separates a region of stability (or saddle-path stability) from a region of instability, where fluctuations are eventually observed.

Consider the specific f function of previous sections, and take $g(A_t) = B \cdot A_t^{\phi}$, B>0 and $\phi \in (0,1)$. With these functions, we briefly analyze global dynamics. Take the same array of values as before for c, γ, r and $\hat{w}^*$; consider θ=1.3 (for $\overline{w}_1$), θ=1.1 (for $\overline{w}_2$), B=1.05 and ϕ=0.25. Figures 9 and 10 present the bifurcation diagrams for the system considering, respectively, $\overline{w}_1$ and $\overline{w}_2$, and taking μ as the bifurcation parameter.

- Figures 9 and 10 here -

Similar attractors to the ones in figures 5 and 8 can be found in this case. Table 3 discusses the degree of chaoticity that various combinations of parameters allow for.

	μ	θ	B	ϕ	LCEs	Fractal dimension
$\overline{w}_1$	3	1.3	1.05	0.25	-1.39; 0.03; 0.18	2.15
$\overline{w}_2$	8	1.1	1.05	0.25	-1.39; 0.07; 0.11	2.13
$\overline{w}_1$	3	1.3	1.05	0.5	-0.69; 0.03; 0.17	2.29
	3	1.3	1.2	0.25	-1.39; 0.04; 0.19	2.17
	2	1.3	1.05	0.5	-0.69; 0.01; 0.17	2.26
	2	1.3	1.2	0.25	-1.39; 0.01; 0.17	2.13
$\overline{w}_2$	8	1.1	1.05	0.5	-0.69; 0.09; 0.12	2.30
	8	1.1	1.2	0.25	-1.39; 0.08; 0.12	2.14
	6	1.1	1.05	0.5	-0.69; 0.08; 0.12	2.29
	6	1.1	1.2	0.25	-1.39; 0.08; 0.11	2.14

Table 3 – LCEs and fractal dimensions for the system with endogenous technology.

Note that now we are dealing with a three dimensional system, and therefore three LCEs are jointly computed. Note, as well, that the third equation that we have introduced relates to a process of knowledge accumulation under decreasing returns, and thus stability prevails in what concerns the new dimension we add. As a result, one of the LCEs is always negative, while the other two give similar results to the ones found in the exogenous technology case. In table 3, various combinations of parameters to which chaotic motion exists are considered, and we find, for all of them, that despite taking an additional dimension into the system, the dimension of the attractor continues to be given by a value slightly above 2. In this case, the Lyapunov dimension or fractal dimension is given by the formula $D = 2 + \frac{\lambda_1 + \lambda_2}{|\lambda_3|}$, where the LCEs in the numerator are the positive ones and the exponent in the denominator is the negative LCE.

Since the dynamics of technology are independent of wealth and decreasing marginal returns prevail in the accumulation of knowledge, the technology variable converges to a long term fixed point, independently of parameter values. Wealth dynamics will vary with the values of the credit multiplier and other parameters, as before, but the introduction of the technological sector reveals new possibilities for endogenous fluctuations.

6. Conclusions

We have examined a model of financial development where constraints on credit and a risk premium over the less wealth endowed are considered. As a result, we have concluded that a high level of financial development has a favourable effect over the potential to grow; nevertheless, the results also point to a perverse impact of a too loose policy concerning credit availability, because this can lead to instability. In the proposed framework, instability can be interpreted as a state where excess of credit conducts to a failure of the financial system to maintain the mutual confidence in the credit market that allows for loans with low collateral requirements.

For some levels of the credit constraint parameter, endogenous business cycles were found, an observation that confirms the results on other studies in the field (namely, the CJM model). We identify a link between the functioning of the credit market and the volatility of some fundamental economic aggregates, with this link arising from the nonlinear nature of the relation between variables, namely from the piecewise relation between a constant marginal returns value and a varying interest rate.

Introducing an endogenous technology generation process, we have confirmed the richness of possible long term results on a model that never loses its endogenous growth character; the economy's long run growth rate is always constant on average (because constant marginal returns on production hold in every analyzed case), even though some circumstances of the financial markets push the setup to a long term result where the time path of the growth rate fluctuates around a constant mean.

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Figures

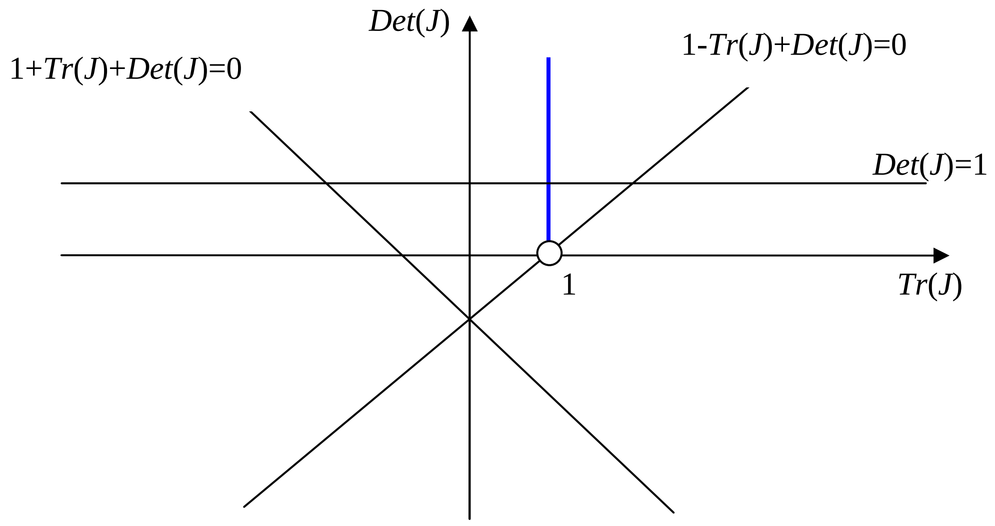

Figure 1 – Local dynamics around $\overline{w}_1$.

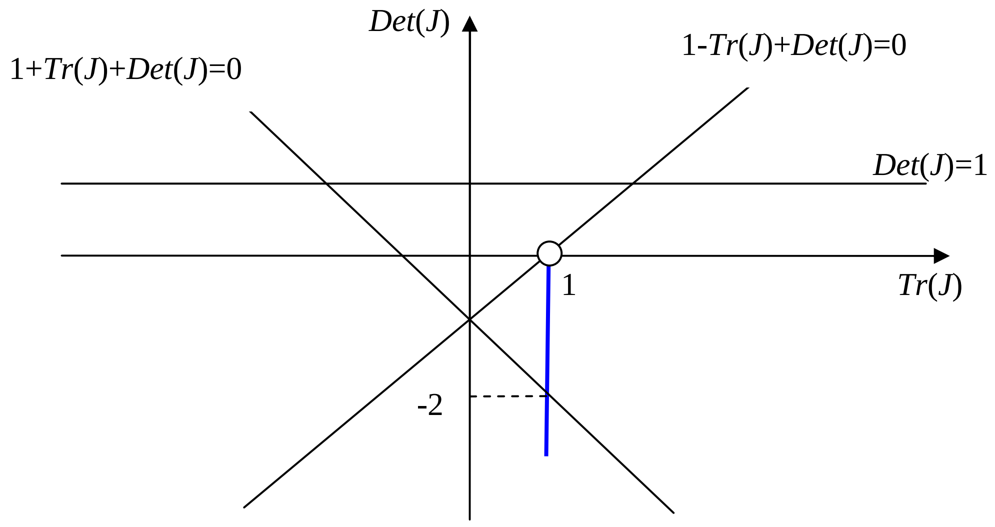

Figure 2 – Local dynamics around $\overline{w}_2$.

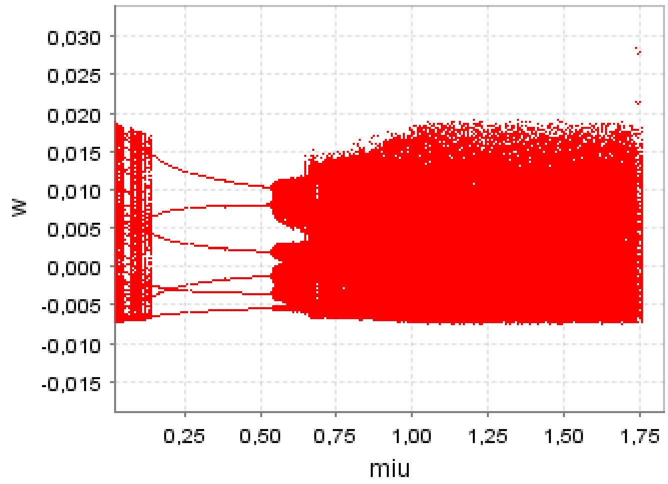

Figure 3 – Bifurcation diagram ($\tilde{w}_t$;μ), for $\overline{w} = \overline{w}_1$.

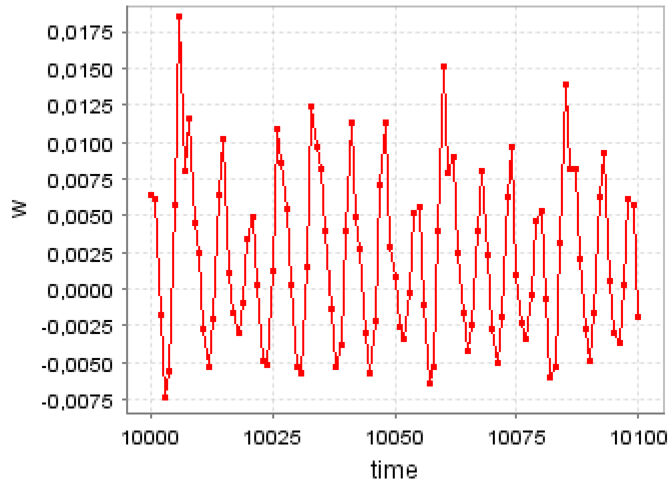

Figure 4 – Time series of $\tilde{w}_t$ (μ=1.5), for $\overline{w} = \overline{w}_1$.

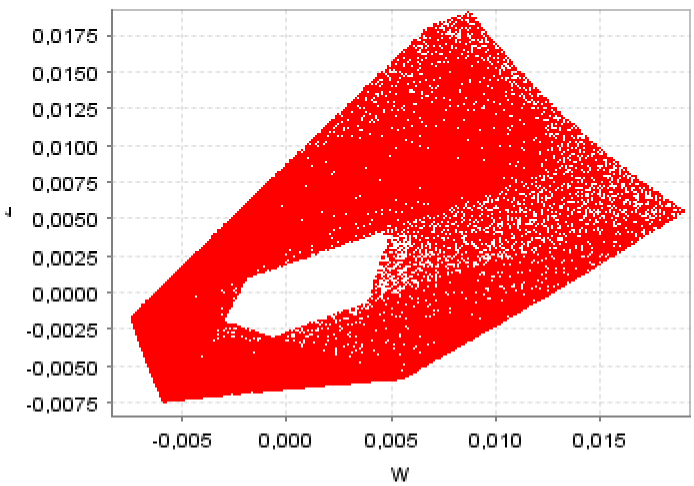

Figure 5 - Attractor $\tilde{w}_t$, $\tilde{z}_t$ (μ=1.5), for $\overline{w} = \overline{w}_1$.

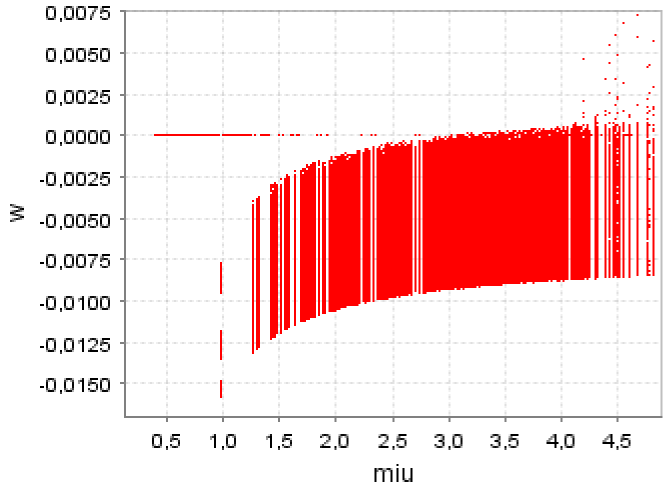

Figure 6 – Bifurcation diagram ($\tilde{w}_t$;μ), for $\overline{w} = \overline{w}_2$.

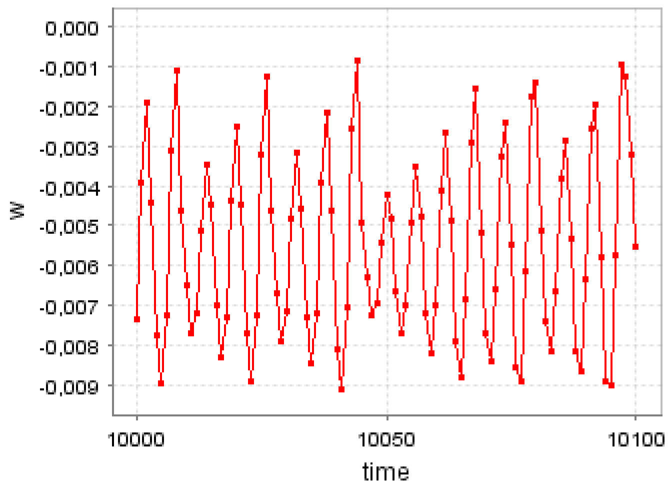

Figure 7 – Time series of $\tilde{w}_t$ (μ=3), for $\overline{w} = \overline{w}_2$.

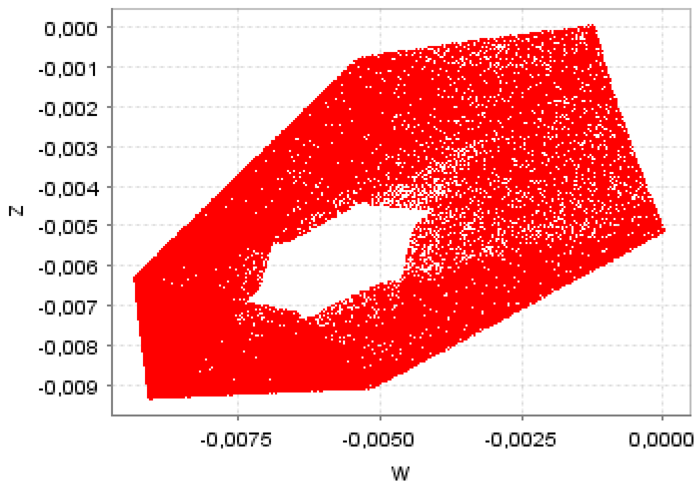

Figure 8 - Attractor $\tilde{w}_t$, $\tilde{z}_t$ (μ=3), for $\overline{w} = \overline{w}_2$.

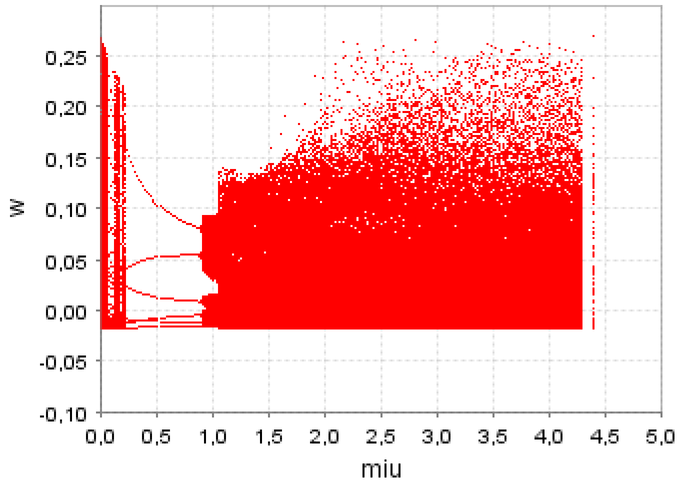

Figure 9 – Bifurcation diagram ($\tilde{w}_t$;μ), for $\overline{w} = \overline{w}_1$ and with endogenous technology.

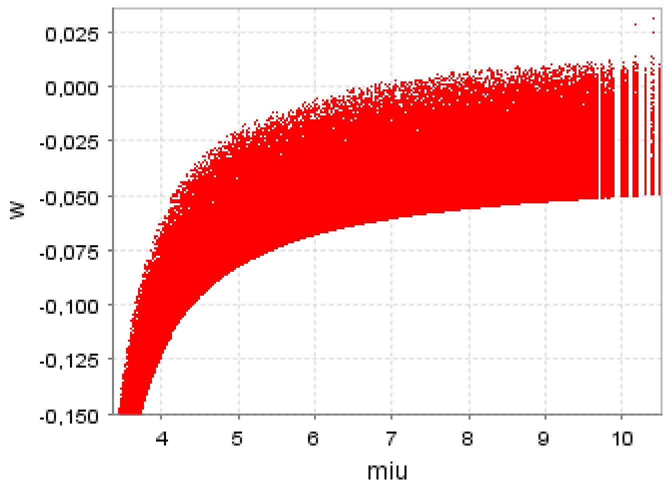

Figure 10 – Bifurcation diagram ($\tilde{w}_t$;μ), for $\overline{w} = \overline{w}_2$ and with endogenous technology.