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Gomes, Orlando

Escola Superior de Comunicação Social - Instituto Politécnico de  
Lisboa

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# Stability Analysis in a Monetary Model with a Varying Intertemporal Elasticity of Substitution

Orlando Gomes<sup>\*</sup>

Escola Superior de Comunicação Social [Instituto Politécnico de Lisboa] and  
Unidade de Investigação em Desenvolvimento Empresarial [UNIDE/ISCTE].

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**Abstract:** Models dealing with monetary policy are generally based on microfoundations that characterize the behaviour of representative agents (households and firms). To explain the representative consumer behaviour, it is generally assumed a utility function in which the intertemporal elasticity of substitution is constant. Recent literature casts some doubts about the relevance of considering such a constant elasticity value. In this note, we explore the new Keynesian monetary policy model under the assumption that the elasticity of substitution changes with expectations regarding real economic performance. As a result, one observes that some combinations of parameter values allow for a stable fixed point outcome, while other combinations of parameters are compatible with cycles of various periodicities and even a-periodic fluctuations.

**Keywords:** Monetary policy, Intertemporal elasticity of substitution, Stability, Nonlinear dynamics.

**JEL classification:** E52, E32, C62.

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\* Orlando Gomes; address: Escola Superior de Comunicação Social, Campus de Benfica do IPL, 1549-014 Lisbon, Portugal. Phone number: + 351 93 342 09 15; fax: + 351 217 162 540. E-mail: [ogomes@escs.ipl.pt](mailto:ogomes@escs.ipl.pt).

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## 1. Introduction

The intertemporal elasticity of substitution (IES) measures the willingness of consumers in what concerns the substitution between future consumption and present consumption. If the IES is low (close to zero), households reveal their preference for a uniform pattern of consumption over time; otherwise, a high IES (close to unity) implies that households are indifferent to the timing of consumption. There is no consensus in the literature relatively to the extent of the value of this elasticity; relevant empirical estimates reveal that the IES is a value close to zero [Hall (1988), Campbell and Mankiw (1989) and Attanasio and Weber (1995), among others]; however, other authors argue in favour of an IES close to unity [e.g., Beaudry and Wincoop (1996)], a value that has analytical advantages, in the sense that it allows to represent utility through a logarithmic function.

Doubts exist not only in relation to the true value of the IES, but also with respect to the constancy of this value over time. From an analytical point of view, it is convenient to assume a constant IES; nevertheless, recent literature has revealed to be uncomfortable with such assumption. Attanasio and Browning (1995) have laid the foundations of this discussion by showing that the analysis of aggregate consumption data points to the sensitiveness of the IES to some endogenous factors, in particular households' wealth. The conclusion is essentially the one that the rich have a higher IES than the poor. This evidence is confirmed by other authors, like Atkinson and Ogaki (1996, 1997) and Álvarez-Peláez and Díaz (2005), who also stress the significant differences in the IES across rich and poor households.

The previous evidence may be explained and modelled through the idea of a minimum consumption requirement. This requirement introduces a positive dependence between the IES and household wealth, that can be modeled in the context of a macro model, e.g. a growth model. Such an exercise is proposed in Chaterjee and Ravikumar (1999). Theoretical implications of a variable IES are also explored in Bliss (2004) and Guvenen (2006); in both cases, the positive relation between wealth and the IES is emphasized, and it is strongly recommended that this evidence become central in modelling household behaviour in order to avoid misleading outcomes and to achieve a higher level of richness and accuracy of the analysis.

In this note, we inquire about the implications of a non constant IES over the new Keynesian monetary model as proposed by Clarida, Gali and Gertler (1999), and

thoroughly discussed in the literature, e.g. in Walsh (2003) and Woodford (2003). Our assumption is that the IES responds to changes in the expected output gap. That is, we assume not a direct relation between the elasticity and wealth, but between the elasticity and the sentiment of consumers about the future performance of the economy. A low, eventually negative, expected output gap (measured as the difference between effective and potential output, with both variables in logs) implies a pessimistic feeling that leads consumers to adopt a more uniform pattern of consumption over time (a low IES); on the other hand, a largely positive expected output gap gives to the consumer a higher confidence in the way she faces future economic performance, and this leads to the adoption of a less restrictive attitude concerning the timing of consumption (a higher IES).

Therefore, we will be mainly concerned with the impact over the monetary policy model of relaxing the assumption of a constant IES, replacing this assumption by the idea that the elasticity increases with the expected output gap. We will regard that over a framework that is linear in its nature, and therefore only capable of revealing fixed point stability or instability outcomes, the new assumption is capable of generating a significant set of new results, for different combinations of parameter values. Fixed point stability may give place to nonlinear motion, i.e. to cycles of distinct periodicities and even chaotic motion. This happens, for instance, when we change the value of the parameter that determines the type of monetary policy (active or passive) that the Central Bank follows, when it chooses to adopt a simple interest rate rule.

Nonlinear motion is found for a low value of the discount factor concerning future utility. It is known, from the growth literature, that a low discount factor may be consistent with nonlinear motion [see, for instance, Nishimura and Yano (1995) and Nishimura, Shigoka and Yano (1998)]; this result is also present in our monetary framework.

The note is organized as follows. Section 2 presents the model; section 3 develops its main dynamic properties and section 4 is destined to some conclusions.

## **2. The New-Keynesian Monetary Policy Model**

In the presentation of the New Keynesian monetary policy model, we follow Walsh (2003) and Woodford (2003). First, a deterministic IS curve is considered. This establishes a relation of opposite sign between the real interest rate and the output gap, as follows,

$$x_t = -\varphi \cdot (i_t - E_t \pi_{t+1}) + E_t x_{t+1}, \quad x_0 \text{ given.} \quad (1)$$

The output gap is defined as  $x_t = \ln y_t - \ln \hat{y}_t$ , with  $y_t$  the effective level of output and  $\hat{y}_t$  the potential output. Variable  $i_t$  stands for the nominal interest rate and  $E_t \pi_{t+1}$  and  $E_t x_{t+1}$  correspond to expectations concerning next period inflation and output gap, respectively;  $\varphi$  is an elasticity parameter.

The new Keynesian Phillips curve respects to a contemporaneous relation between the output gap and the inflation rate. This relation is obtained from the mechanics of a micro model where firms update prices sluggishly. The equation takes the shape,

$$\pi_t = \lambda x_t + \beta \cdot E_t \pi_{t+1}, \quad \pi_0 \text{ given.} \quad (2)$$

Parameter  $0 < \beta < 1$  is the intertemporal discount factor and  $\lambda$  reflects the degree of price flexibility. Generally, this is taken as a constant and equal to  $(\sigma + \eta) \cdot \frac{(1 - \alpha) \cdot (1 - \alpha\beta)}{\alpha}$ , with  $\alpha$  the share of firms that do not adjust their prices in each time period (this parameter can also be thought as the probability associated with maintaining prices from one period to the next, for some representative firm);  $\sigma$  represents the inverse of the intertemporal elasticity of substitution of consumption and  $\eta$  is the inverse of the intertemporal elasticity of substitution of leisure. To focus our analysis in the first intertemporal elasticity, we take  $\eta$  as constant.

Relatively to  $\sigma$ , we consider that this is dependent on expectations about short run economic performance. Note that a high  $\sigma$  implies that households want a uniform pattern of consumption over time (the utility function has a high degree of concavity), while a low  $\sigma$  means that households are indifferent to the timing of consumption (the utility function is concave, but the degree of concavity is low). Our assumption is that expectations about a large output gap in the future increase the intertemporal elasticity of substitution, that is, the increased optimism of households is translated on a smaller relevance attributed to the timing of consumption, meaning a low value of  $\sigma$ . In the opposite case, when a low (eventually negative) output gap is evidenced, a precautionary behaviour of consumers will prevail and they will prefer a uniform pattern of consumption, i.e., a low intertemporal elasticity of substitution for consumption (a higher value of  $\sigma$ ). To translate analytically the previous reasoning, we

consider upper and lower bounds for  $\sigma$ ,  $\sigma_{min}$  and  $\sigma_{max}$ , and consider the following function:

$$\sigma_t = \frac{\sigma_{min} + \sigma_{max}}{2} - \frac{\sigma_{max} - \sigma_{min}}{\pi} \cdot arctg\left(\frac{E_t x_{t+1} - x^*}{x^*}\right) \quad (3)$$

with  $x^*$  the steady state value of  $x_t$ . Function (3) establishes the discussed opposite sign relation between the expected output gap and  $\sigma$ . Figure 1 presents graphically this sigmoid-shaped function.

\*\*\* Figure 1 here \*\*\*

According to the expression in (3), the sticky prices parameter  $\lambda$  in equation (2) is no longer a constant value; it depends on the expected output gap.

Two additional remarks are needed in order to discuss the stability properties of our problem. First, we have to define how expectations are formed. We consider that agents are well informed and believe that the output gap and the inflation rate converge in the long run to their equilibrium values; the expectation rules are

$$E_t x_{t+1} = x^* + w \cdot (x_{t-1} - x^*) \quad (4)$$

$$E_t \pi_{t+1} = \pi^* + v \cdot (\pi_{t-1} - \pi^*) \quad (5)$$

Parameters  $0 < w < 1$  and  $0 < v < 1$  reflect the velocity of convergence. Parameter  $\pi^*$  is the steady state inflation rate. Note that in moment  $t$  agents do not know what values the endogenous variables assume, and therefore they use previous period information.

The second item that deserves attention is the value of the interest rate. This is chosen by the monetary authority and may take the form of the well known Taylor rule. The version of the Taylor rule that we take is the one by Clarida, Gali and Gertler (1999),

$$i_t = i^* + \gamma_\pi \cdot (E_t \pi_{t+1} - \pi^*) + \gamma_x \cdot x_t \quad (6)$$

with  $i^*, \gamma_\pi, \gamma_x > 0$ . The Taylor rule leads commonly to a stable outcome when an active monetary policy is followed ( $\gamma_\pi > 0$ ).

Replacing (4), (5) and (6) in (1) and (2), this system becomes

$$x_t = \theta + \frac{w}{1 + \varphi \cdot \gamma_x} \cdot x_{t-1} - \frac{\varphi \cdot (\gamma_\pi - 1) \cdot v}{1 + \varphi \cdot \gamma_x} \cdot \pi_{t-1} \quad (7)$$

$$\pi_t = \beta \cdot (1 - v) \cdot \pi^* + \theta \cdot \lambda_t + \frac{w}{1 + \varphi \cdot \gamma_x} \cdot \lambda_t \cdot x_{t-1} + \left[ \beta v - \frac{\varphi \cdot (\gamma_\pi - 1) \cdot v}{1 + \varphi \cdot \gamma_x} \cdot \lambda_t \right] \cdot \pi_{t-1} \quad (8)$$

$$\text{where } \theta \equiv \frac{\varphi \cdot [1 + (\gamma_\pi - 1) \cdot v] \cdot \pi^* - \varphi \cdot i^* + (1 - w) \cdot x^*}{1 + \varphi \cdot \gamma_x}.$$

Note also that, given the expectations rule (4),

$$\sigma_t = \frac{\sigma_{\min} + \sigma_{\max}}{2} - \frac{\sigma_{\max} - \sigma_{\min}}{\pi} \cdot \arctg \left( w \cdot \frac{x_{t-1} - x^*}{x^*} \right) \quad (9)$$

Relatively to system (7)-(8), it is straightforward to obtain steady state values for the endogenous variables. Imposing  $x_t = x_{t-1} = x^*$  and  $\pi_t = \pi_{t-1} = \pi^*$ , one finds a unique steady state point:  $x^* = \frac{(1 - \beta) \cdot i^*}{\lambda^* - \gamma_x \cdot (1 - \beta)}$  and  $\pi^* = \frac{\lambda^* \cdot i^*}{\lambda^* - \gamma_x \cdot (1 - \beta)}$ , with  $\lambda^* = \left( \frac{\sigma_{\min} + \sigma_{\max}}{2} + \eta \right) \cdot \left( \frac{(1 - \alpha) \cdot (1 - \alpha\beta)}{\alpha} \right)$ .

### 3. Stability and Long Term Nonlinear Motion

To analyze the stability properties of system (7)-(8), we can linearize it in the vicinity of  $(x^*, \pi^*)$ . The following expressions correspond to the trace and to the determinant of the Jacobian matrix of the linearized system,<sup>1</sup>

$$Tr(J) = \beta \cdot v + \frac{w - \varphi \cdot (\gamma_\pi - 1) \cdot v \cdot \lambda^*}{1 + \varphi \cdot \gamma_x} \quad (10)$$

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<sup>1</sup> These expressions are derived in appendix.

$$Det(J) = \frac{\beta \cdot v \cdot w}{1 + \varphi \cdot \gamma_x} - \frac{\varphi \cdot (\gamma_\pi - 1) \cdot v \cdot w}{1 + \varphi \cdot \gamma_x} \cdot \frac{(1 - \alpha) \cdot (1 - \alpha\beta)}{\alpha} \cdot \frac{\sigma_{\max} - \sigma_{\min}}{\pi} \quad (11)$$

Values (10) and (11) allow to obtain stability conditions; however, given the complexity of these expressions, few information can be withdrawn concerning long term qualitative outcomes. To accomplish meaningful results, we consider hereafter an array of specific parameter values. We let  $\gamma_\pi$  be the bifurcation parameter and assume the following values:

$$[\varphi, v, w, i^*, \eta, \gamma_x, \beta, \alpha, \sigma_{\min}, \sigma_{\max}] = [0.5; 0.5; 0.5; 0.01; 2; 0.5; 0.2; 0.25; 2; 100]$$

Note that with the above set of parameter values, we allow the IES of consumption to vary between 0.01 and 0.5; note, as well, that the discount factor is a low value, meaning that future utility is heavily discounted (such a value is needed to obtain non linear motion). For the specified parameter values, the following steady state results are achieved:  $\sigma^* = 51$ ;  $\lambda^* = 151.05$ ;  $x^* \approx 5.31 \times 10^{-5}$ ;  $\pi^* \approx 0.01$ .

Given the vector of parameter values, expressions (10) and (11) become:  $Tr(J) = 30.71 - 30.21 \cdot \gamma_\pi$  and  $Det(J) = 8.93 - 8.89 \cdot \gamma_\pi$ . In our two dimensional system, conditions for stability are:

$$1 - Tr(J) + Det(J) > 0 \Rightarrow 21.32 \cdot \gamma_\pi - 20.28 > 0 \Rightarrow \gamma_\pi > 0.951$$

$$1 + Tr(J) + Det(J) > 0 \Rightarrow 40.64 - 39.1 \cdot \gamma_\pi > 0 \Rightarrow \gamma_\pi < 1.039$$

$$1 - Det(J) > 0 \Rightarrow 8.89 \cdot \gamma_\pi - 7.93 > 0 \Rightarrow \gamma_\pi > 0.892$$

Under the above conditions, it is clear that stability is guaranteed only for a small interval of  $\gamma_\pi$ ; in particular, the double inequality  $0.951 < \gamma_\pi < 1.039$  guarantees stability.

The previous result on local dynamics can be confirmed by looking at the global behaviour of the system. Figures 2 and 3 characterize the global behaviour.<sup>2</sup>

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<sup>2</sup> These figures are drawn using IDMC software (interactive Dynamical Model Calculator). This is a free software program available at [www.dss.uniud.it/nonlinear](http://www.dss.uniud.it/nonlinear), and copyright of M. Lines and A. Medio.

\*\*\* Figures 2 and 3 here \*\*\*

The bifurcation diagram in figure 2 reveals that the stability region found through local analysis is indeed a region of fixed point stability. Nevertheless, the flip bifurcation that occurs as  $\gamma_\pi$  rises to assume value 1.039 does not give place immediately to a region of instability; as one observes, a period doubling process leads the system from a fixed point to periodic cycles and, finally, a-periodic or chaotic motion; therefore, one should conclude that an overactive monetary policy (a high  $\gamma_\pi$ ) may give rise to endogenous fluctuations and, thus, a failure of the monetary authority regarding the stabilization of prices.

Figure 3 displays an attractor that describes the long run relation between the output gap and the inflation rate for a value of  $\gamma_\pi$  to which chaotic motion is evidenced ( $\gamma_\pi=1.1$ ). Interestingly, this long term relation seems to have the shape one expects to find when looking at a Phillips curve: for low levels of inflation, a small change in prices has a significant impact over the real economy; for high levels of inflation, to obtain a small gain in terms of the output gap a huge variation on inflation is necessary.

#### **4. Conclusions**

The new Keynesian monetary policy model is an important tool to understand and explain the role of the monetary authorities in guaranteeing price stability. The prototype model is linear in its nature, and therefore if one ignores supply and demand shocks, an active monetary policy Taylor rule is capable of fulfilling the goal of low and stable inflation. In this note, we have relaxed one of the central assumptions regarding the microfoundations of the model; instead of a constant intertemporal elasticity of substitution for consumption, we have considered that the IES changes with expectations regarding real economy performance. The logical argument is that an expansion can be associated with an increase in the IES, while a downturn in economic performance produces a precautionary attitude of households that favours a decline in the IES.

The referred assumption introduces an important degree of nonlinearity into the system and one observes that stability is no longer guaranteed in every circumstance. For some parameter values (in particular, we have analyzed the case of changes in the Taylor rule parameter) there is a transition process from fixed point stability to an

unstable outcome. Cycles of various periodicities and completely a-periodic motion arise, an observation that allows for inferring that an active monetary policy can give place to endogenous fluctuations under some meaningful economic circumstances. Note that the obtained fluctuations arise for a low discount factor, a result that is compatible with the one in growth theory concerning one and two sector optimal control problems.

## Appendix

In this appendix, we derive the Jacobian matrix of the system, in order to compute the trace and the determinant in expressions (11) and (12).

The linearization of system (7)-(8) in the vicinity of  $(x^*, \pi^*)$ , leads to

$$\begin{bmatrix} x_t - x^* \\ \pi_t - \pi^* \end{bmatrix} = J \cdot \begin{bmatrix} x_{t-1} - x^* \\ \pi_{t-1} - \pi^* \end{bmatrix}. \text{ The elements of the Jacobian matrix are:}$$

$$j_{11} = \left. \frac{\partial x_t}{\partial x_{t-1}} \right|_{(x^*, \pi^*)} = \frac{w}{1 + \varphi \cdot \gamma_x}$$

$$j_{12} = \left. \frac{\partial x_t}{\partial \pi_{t-1}} \right|_{(x^*, \pi^*)} = -\frac{\varphi \cdot (\gamma_\pi - 1) \cdot v}{1 + \varphi \cdot \gamma_x}$$

$$j_{21} = \left. \frac{\partial \pi_t}{\partial x_{t-1}} \right|_{(x^*, \pi^*)} = \frac{w \cdot \lambda^*}{1 + \varphi \cdot \gamma_x} + \left[ \theta + \frac{w \cdot x^*}{1 + \varphi \cdot \gamma_x} - \frac{\varphi \cdot (\gamma_\pi - 1) \cdot v \cdot \pi^*}{1 + \varphi \cdot \gamma_x} \right] \cdot \lambda'$$

where  $\lambda'$  represents the derivative of  $\lambda_t$  relatively to  $x_{t-1}$  in the vicinity of the steady state;

$$j_{22} = \left. \frac{\partial \pi_t}{\partial \pi_{t-1}} \right|_{(x^*, \pi^*)} = \beta \cdot v - \frac{\varphi \cdot (\gamma_\pi - 1) \cdot v \cdot \lambda^*}{1 + \varphi \cdot \gamma_x}$$

In what concerns  $j_{21}$ , observe the following:

i) Expression  $\theta + \frac{w \cdot x^*}{1 + \varphi \cdot \gamma_x} - \frac{\varphi \cdot (\gamma_\pi - 1) \cdot v \cdot \pi^*}{1 + \varphi \cdot \gamma_x}$  can be simplified by taking the

steady state relation  $x^* = \frac{\pi^* - i^*}{\gamma_x}$ ; it is straightforward to realize that the above

expression is equal to  $x^*$ ;

ii) The derivative  $\lambda'$  is the following,

$$\left. \frac{d\lambda_t}{dx_{t-1}} \right|_{x^*} = -\frac{(1-\alpha) \cdot (1-\alpha\beta)}{\alpha} \cdot \frac{w}{x^*} \cdot \frac{\sigma_{\max} - \sigma_{\min}}{\pi}$$

Therefore,

$$j_{21} = \frac{w \cdot \lambda^*}{1 + \varphi \cdot \gamma_x} - \frac{(1-\alpha) \cdot (1-\alpha\beta)}{\alpha} \cdot w \cdot \frac{\sigma_{\max} - \sigma_{\min}}{\pi}$$

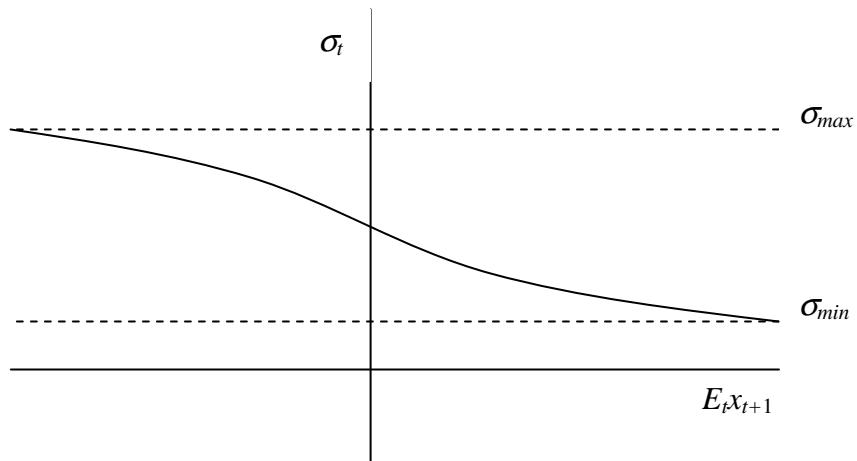
The trace and the determinant in (10) and (11) are obtained directly from the Jacobian matrix:  $Tr(J) = j_{11} + j_{22}$  and  $Det(J) = j_{11} \cdot j_{22} - j_{12} \cdot j_{21}$  ■

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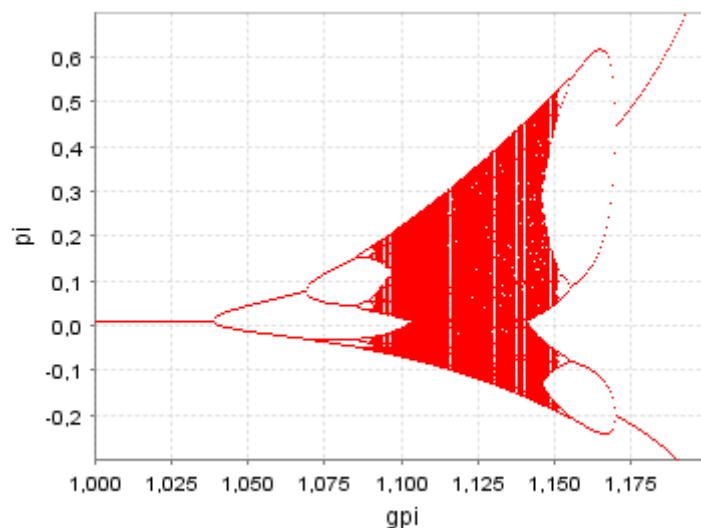
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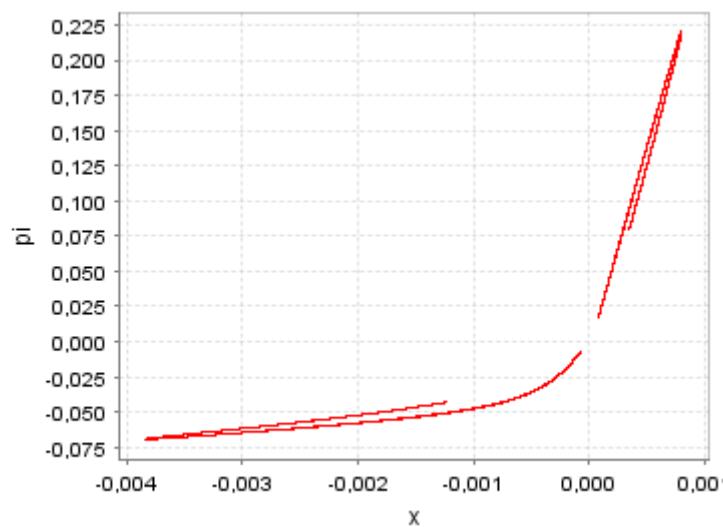
## Figures



**Figure 1 – The IES function.**



**Figure 2 – Bifurcation diagram ( $\pi_i; \gamma_\pi$ ).**



**Figure 3 – Long term attracting set ( $\gamma_\pi=1.1$ ).**