On the stability of endogenous growth models: an evaluation of the agents' response to output fluctuations

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April 2007

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- April, 2007 -

Abstract: This paper presents three modified versions of the simple AK endogenous growth model. Such frameworks stress the role of consumers' sentiment, the impact of fiscal policy and the effect of non-optimal investment decisions made by firms. In all the cases, today's decisions take into consideration the economic performance of the previous period; in the first case, households react pro-cyclically to the output path; in the second case, a counter-cyclical fiscal policy is considered; and in the third case, firms adopt a pro-cyclical behaviour concerning investment choices. We study the stability properties of the three models and conclude that, on each one of them, a saddle-path stable equilibrium exists.

Keywords: Endogenous growth, Consumers' sentiment, Fiscal policy, Investment, Output gap, Saddle-path stability.

JEL classification: O41, C62.

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Acknowledgements: Financial support from the Fundação Ciência e Tecnologia, Lisbon, is grateful acknowledged, under the contract No POCTI/ECO/48628/2002, partially funded by the European Regional Development Fund (ERDF).

1. Introduction

The simple version of the AK growth model, analyzed, for instance, in Rebelo (1991), considers a capital accumulation process that is intrinsically unstable. The steady state is not reached, unless the representative agent provokes a discontinuous jump in the level of consumption placing it on its long term equilibrium value. In this paper, we ask how this outcome is modified when economic agents react to business cycles. We assume three types of agents, who adopt a pro-cyclical or a counter-cyclical behaviour concerning the deviations of effective output relatively to its potential level. First, households respond pro-cyclically to output fluctuations; at this level, we follow empirical evidence related to consumers' sentiment. Second, government action is introduced through the idea that fiscal policy is counter-cyclical: periods of recession are fought with expansionary measures, while periods of expansion are attenuated through restrictive measures concerning public expenditures. Third, firms, as consumers, tend to act pro-cyclically; for a positive output gap, firms will expect demand to rise, and thus they invest more, while negative output gaps will imply a contraction of the amount and extent of undertaken investment projects.

Each one of the above topics has been thoroughly discussed in the literature, through multiple view points. In what follows, we just refer to some relevant works on these areas. We begin by addressing consumers' sentiment.

In planning how to allocate their disposable income, households tend to observe and to take in consideration macroeconomic fluctuations. Sentiments of optimism / pessimism are the result of the way GDP performance is perceived, implying that a higher / lower share of consumers' income is effectively directed to consumption. This logical reasoning is confirmed empirically in the studies of Bram and Ludvigson (1998), McNabb and Taylor (2002), Goh (2003), Doms and Morin (2004), Souleles (2004) and Dion (2006), among others. These authors find evidence of two essential links: first, business cycles influence consumers' sentiment; this can be easily confirmed by looking at reports on consumer confidence throughout the world, namely in the developed countries. Second, consumers' sentiment triggers changes on the savings rate. It is well accepted, as demonstrated in the studies referred above, that the precautionary motive to save is dependent on sentiments about short term economic growth. Thus, it seems unquestionable, from an empirical point of view, that the marginal propensity to consume rises with the expansionary phase of the cycle and falls otherwise, under this two step mechanism: first, agents form sentiments by looking at the behaviour of the national income time series and, second, these sentiments are reflected in the consumption-savings decisions.

From a theoretical point of view, it is worth citing the model by Westerhoff (2007), who uses the above empirical evidence to support a framework where changes on consumption expenditures resulting from varying levels of confidence will lead to a perpetuation of business cycles in the long term. These endogenously generated business cycles confirm the Keynesian view of self-fulfilling prophecies: because households take seriously the information overall economic fluctuations transmit, fluctuations will persist in time.

In what concerns fiscal policy, this is a theme largely discussed in its relation to growth and fluctuations. The logical principle is simple: governments have the power to use their budget to stabilize the economic system. In periods of recession, public expenditures rise faster to stimulate growth, while in periods of expansion, the growth rate of public consumption is attenuated (this implies, of course, the assumption that the government has always the flexibility to rise and lower their expenditures to respond to fluctuations).

Empirical evidence seems to support the above argument. For instance, Gali and Perotti (2003) find evidence of a fiscal policy that is becoming more counter-cyclical over time in the countries that form the European Monetary Union, and they remark that this is a trend that other developed countries are also following. In turn, Hein and Truger (2006) have some doubts concerning the ability of the European governments in maintaining such kind of policy, since evidence points to some countries being not capable to keep public deficits below the 3 percent of the GDP target, in a moment where the European economy performs poorly. These authors contrast the inability of the Euro area governments with the well succeeded North-American fiscal policy, which has been in fact strongly counter-cyclical. Other evidence on counter-cyclical fiscal policy worldwide can be found in Budina and Wijnbergen (1997), Perry (2003), Afonso, Nickel and Rother (2006), Mackiewicz (2006) and Staehr (2007), among others.

From a theoretical perspective, the standard result consists on finding positive effects of a counter-cyclical fiscal policy. For instance, the paper by Martin and Rogers (1997) highlights the positive effects of this type of stabilization policy over growth, human capital accumulation and welfare. Others, however, find arguments to question the virtues of the referred policy; in particular, Gordon and Leeper (2005) argue that in phases of recession a counter-cyclical policy is likely to change agents’ expectations, since this type of policy increases public indebtedness, which raises future debt service payments; as a result, agents will perceive that taxes must rise in the future and this tends to change savings rates accordingly. Thus, responses to expected future policies may turn recessions deeper and lengthier. In fact, the authors call the attention for the fact that this expectations channel may create business cycles that simply would not exist if the counter-cyclical policy was not adopted.

Concerning fiscal policy, we keep two ideas in mind. First, in an ideal world, where agents' expectations do not distort public authorities' intensions, the counter-cyclical policy has clear stabilization effects; second, although the counter-cyclical policy is favourable in terms of long term economic performance, not always the governments have the ability or the desire to undertake such policy. Alesina and Tabellini (2005) provide an explanation for why some countries, namely the ones in developing regions, follow pro-cyclical fiscal policies; the argument is based on political distortions and appropriation of rents by less-than-benevolent governments. To this explanation, we can add the pressure that many governments suffer (both in developed and developing countries) in times of recession to increase expenditures to mitigate the effects of unemployment and slower growth.

Finally, with respect to firms' investment decisions, these are clearly conditioned by expectations about future demand. If firms associate demand expectations with economic performance, then it is logical that investment decisions react pro-cyclically to output, and thus the growth of investment will depend on the extent of the (positive or negative) output gap.

Evidence on the pro-cyclicity of investment is also easy to find in the economic literature. Such references include, just to cite a few, Backus and Kehoe (1992), Bergman, Bordo and Jonung (1998), Baxter and King (1999) and Dosi, Fagiolo and Roventini (2006). In this last paper, also a theoretical model is constructed; this assumes reasonable hypothesis about firms' behaviour (in particular, that investment decisions are lumpy and that firms are constrained by their financial structures), to reach the result of a pro-cyclical investment movement over time.

Following sections study three alternative endogenous growth models, where each type of agent adopts a rule of reaction to observed business cycles. As a result we abandon the one-equation AK simple model that lacks transitional dynamics, to study three dimensional models with relevant transitional dynamics properties. Namely, we will observe that saddle-path stability holds in all the cases, being possible to derive stable trajectories. These stable trajectories define a convergence to the steady state relation between consumption / public expenditures / investment and capital accumulation.

The remainder of the paper is organized as follows. Section 2 studies the growth model under a households' reaction to fluctuations. Section 3 focuses on the role of counter-cyclical public expenditures. Section 4 studies the version of the model with pro-cyclical investment decisions. In section 5, a numerical illustration of the various models is proposed. Finally, section 6 concludes.

2. Endogenous Growth and Consumers' Sentiment

2.1 The AK Model

Consider a standard AK endogenous growth model. Income is generated through a simple constant returns to scale production function, $y_t=Ak_t$, where y_t denotes income or output, k_t represents physical capital and $A>0$ is a technological index. Note that all variables can be understood as representing levels or per capita units, since no population growth is assumed. The process of capital accumulation is described by a trivial capital accumulation equation, $k_{t+1} = Ak_t - c_t + (1-\delta)\cdot k_t$, with k_0 given, c_t the level of consumption and $\delta>0$ the rate of depreciation of physical capital. A representative agent maximizes function $V_0 = \sum_{t=0}^{+\infty} \beta^t \cdot U(c_t)$, with $0<\beta<1$ the discount factor and $U(c_t) = \ln c_t$, i.e., a simple utility function exhibiting decreasing marginal utility is taken.

The dynamics of the previous simple model are straightforward to obtain. The computation of first order conditions leads to a constant over time growth rate of consumption: $\gamma = \frac{c_{t+1}}{c_t} - 1 = \beta \cdot (A+1-\delta) - 1$. Defining $\hat{k}_t \equiv \frac{k_t}{(1+\gamma)^t}$ and $\hat{c}_t \equiv \frac{c_t}{(1+\gamma)^t}$, we may rewrite the capital accumulation constraint as the following difference equation: $\hat{k}_{t+1} = \frac{1}{\beta} \cdot \hat{k}_t - \frac{\hat{c}^*}{\beta \cdot (A+1-\delta)}$, with $\hat{c}^* \equiv \hat{c}_{t+1} = \hat{c}_t$. Noticing that $\frac{d\hat{k}_{t+1}}{d\hat{k}_t} = \frac{1}{\beta}$, we conclude that this one dimensional system is unstable; the steady state point $(\hat{k}_t, \hat{c}_t)$ is attained only if the representative agent has the ability to choose $\hat{c}_0 = \hat{c}^*$.

In what follows, we ask whether a departure from the optimality scenario produces relevant changes on the previous (in)stability result. This departure is, for now, associated with consumers' sentiment concerning last period economic performance. We assume that the consumption growth rate is maintained at level γ only if the effective output coincides with potential output; if the output gap becomes negative, then the growth rate is lowered relatively to the benchmark level, as a result of a precautionary attitude by consumers; in the opposite case, a positive output gap is associated to optimistic beliefs and thus the growth rate of consumption rises above γ. In this way, we might say that the representative agent does not act as a fully rational agent – she uses information about last period's output gap, and reacts accordingly in what concerns the chosen growth rate of consumption.

Let x_t represent the output gap. This is defined as the difference in logs between effective output and potential output, $x_t = \ln y_t - \ln y_t^*$. Potential output, in turn, is considered equal to the steady state level of output, that is, $y_t^* = Ak_t^*$, with k_t^* the long term physical capital level. The following dynamic equation intends to capture the above reasoning,

$$\hat{c}_{t+1} = \left[1 + \frac{2\cdot(\gamma-\gamma_0)}{\pi\cdot(1+\gamma)}\cdot arctg(x_{t-1})\right]\cdot\hat{c}_t \tag{1}$$

with $\gamma_0 > 0$. In equation (1), the growth rate of consumption depends on the output gap of the previous time period. If x_{t-1}=0, then $\hat{c}_{t+1} = \hat{c}_t$ and we are back on the benchmark case. Because the output gap tends to differ from zero, the growth rate of consumption will depart from its reference level γ. Figure 1 displays the relation between the lagged output gap and the consumption growth rate.

*** Figure 1 here ***

2.2 Dynamics

To solve the above model, it is useful to define variables $\tilde{k}_t \equiv \hat{k}_t - \hat{k}^*$ and $\tilde{z}_t \equiv \tilde{k}_{t-1}$. With these definitions, we rewrite equation (1),

$$\hat{c}_{t+1} = \left[1 + \frac{2\cdot(\gamma-\gamma_0)}{\pi\cdot(1+\gamma)}\cdot arctg\left(\ln\frac{\tilde{z}_t + \hat{k}^*}{\hat{k}^*}\right)\right]\cdot\hat{c}_t \tag{2}$$

The capital constraint is equivalent to,

$$\tilde{k}_{t+1} = \frac{1}{\beta}\cdot(\tilde{k}_t + \hat{k}^*) - \frac{\hat{c}_t}{\beta\cdot(A+1-\delta)} - \hat{k}^* \tag{3}$$

A system with three equations, (2), (3) and $\tilde{z}_{t+1} = \tilde{k}_t$ emerges, and from this system one can withdraw relevant information regarding the stability of the endogenous growth problem. First, observe that in the steady state, $\frac{\hat{c}^*}{\hat{k}^*} = (1-\beta)\cdot(A+1-\delta)$ and $\tilde{k}^* = \tilde{z}^* = 0$.

The system may be linearized in the steady state vicinity. The linear version is presentable in matricial form,

$$\begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{z}_{t+1} \\ \hat{c}_{t+1} - \hat{c}^* \end{bmatrix} = \begin{bmatrix} 1/\beta & 0 & -1/(1+\gamma) \\ 1 & 0 & 0 \\ 0 & \frac{2}{\pi}\cdot\frac{1-\beta}{\beta}\cdot(\gamma-\gamma_0) & 1 \end{bmatrix} \cdot \begin{bmatrix} \tilde{k}_t \\ \tilde{z}_t \\ \hat{c}_t - \hat{c}^* \end{bmatrix} \tag{4}$$

The Jacobian matrix in (4) allows to find the following values,

Trace: $Tr(J) = \frac{1+\beta}{\beta}$;

Sum of principal minors: $M(J) = 1/\beta$;

Determinant: $Det(J) = -\frac{2}{\pi}\cdot\frac{1-\beta}{\beta}\cdot\frac{\gamma-\gamma_0}{1+\gamma}$.

The stability result is given in proposition 1.

Proposition 1. *The modified endogenous growth model, where consumption growth is determined by the representative consumer response to output deviations relatively to its potential level, is saddle-path stable. Following the one dimensional stable trajectory, the system converges to the steady state.*

Proof: Because *Tr*(*J*)>0 and *Det*(*J*)<0, it is straightforward to conclude that one of the eigenvalues of the Jacobian matrix is a negative value (λ_1<0), while the other two are positive (λ_2,λ_3>0). If inequality $(1+\lambda_1)\cdot(1+\lambda_2)\cdot(1+\lambda_3)>0$ holds, one can immediately conclude that λ_1 lies inside the unit circle. This expression is equivalent to $1+Tr(J)+M(J)+Det(J)>0$, and, given that in our particular case $M(J)=Tr(J)-1$, we can simplify the expression further to write $2Tr(J)+Det(J)>0$; this inequality corresponds, for the computed *J* matrix, to $\frac{1+\beta}{1-\beta}>\frac{1}{\pi}\cdot\frac{\gamma-\gamma_0}{1+\gamma}$, which is a true relation given that the left hand side is higher than 1, while for a reasonable growth rate the right hand side corresponds to a value lower than unity. Thus, we conclude that $-1<\lambda_1<0$.

Relatively to the other two eigenvalues, we may consider relation $(1-\lambda_1)\cdot(1-\lambda_2)\cdot(1-\lambda_3)>0$ to state that if this condition holds then $\lambda_2<1\wedge\lambda_3<1$ or, alternatively, $\lambda_2>1\wedge\lambda_3>1$. The proposed condition effectively holds; to confirm this statement, note that it is equivalent to $1-Tr(J)+M(J)-Det(J)>0$ or yet, on our particular system, $Det(J)<0$, which is, effectively, true. Finally, note that under $(1-\lambda_1\lambda_2)\cdot(1-\lambda_1\lambda_3)\cdot(1-\lambda_2\lambda_3)<0$, we have $\lambda_2\lambda_3>1$, a condition that is compatible only with one of the above pairs of conditions, i.e., with $\lambda_2>1\wedge\lambda_3>1$; thus, if the inequality is satisfied, one can conclude that both eigenvalues lie outside the unit circle. Condition $(1-\lambda_1\lambda_2)\cdot(1-\lambda_1\lambda_3)\cdot(1-\lambda_2\lambda_3)<0$ is equivalent to $1-M(J)+Det(J)\cdot\left[Tr(J)-Det(J)\right]<0$, or, in our particular system, $2-Tr(J)+Det(J)\cdot\left[Tr(J)-Det(J)\right]<0$. This condition holds for every admissible parameter values, and thus the signs of the eigenvalues are fully identified.

One has found that $-1<\lambda_1<0$, $\lambda_2>1$ and $\lambda_3>1$ are the signs of the eigenvalues, that define a saddle-path stable equilibrium, where the stable path is one dimensional in the defined three-dimensional space■

The stability result may be depicted through graphical representation. In a graph relating the trace and the determinant of the Jacobian matrix, one is able to display a set of bifurcation lines (which correspond to the expressions in the proof of proposition 1 turned into equalities, i.e., $2Tr(J)+Det(J)=0$, $Det(J)=0$, $2-Tr(J)+Det(J)\cdot\left[Tr(J)-Det(J)\right]=0$) and to identify the region of stability (the area inside the triangle formed by the three bifurcation lines). Figure 2 presents such diagram.

*** Figure 2 here ***

The model in appreciation does not produce fixed-point stability (all eigenvalues inside the unit circle) but rather a saddle-path equilibrium. Hence, if one attempts to find the location of admissible results in the trace-determinant diagram of figure 2, these will fall outside the presented stable area. In what follows, we confirm this argument by drawing a curve that represents the set of possible outcomes in the trace-determinant referential. With this procedure, we observe that no bifurcation line is crossed and thus only one stability outcome is feasible; as stated in proposition 1, this result is saddle-path stability, with a one dimensional stable arm.

Replace, in the determinant expression, parameter β by its definition in terms of the trace, i.e., $\beta = 1/(Tr(J)-1)$. The outcome is

$$Det(J) = 2 \cdot \frac{2-Tr(J)}{\pi} \cdot \left[1 - \frac{(1+\gamma_0)\cdot(Tr(J)-1)}{A+1-\delta}\right] \tag{5}$$

Expression (5) is valid only for a small interval of values of the trace; this is bounded by β<1, i.e., $Tr(J)$>2, and by $\gamma>\gamma_0$, i.e., $Tr(J) < \dfrac{2+A-\delta+\gamma_0}{A+1-\delta}$. Observe, as well, that computing $\dfrac{dDet(J)}{dTr(J)} = 0$, we find a minimum for (5), which is precisely the average point of the two above boundary values, that is, $Tr(J) = \dfrac{3\cdot(1+\gamma_0)+A+1-\delta}{2\cdot(1+\gamma_0)}$.

Figure 3 draws the admissible pairs trace-determinant of the system; these are all located in a region where saddle-path stability prevails (as we have seen before, this is a region where a one dimensional stable arm exists). Note that the U-shaped curve is (5) and the bold region of this curve is the one where our system's dynamics may be located; the other curve is one of the bifurcation lines of figure 2.

*** Figure 3 here ***

Proposition 2 characterizes how the system behaves when the saddle-trajectory is followed.

Proposition 2. *In the endogenous growth model with consumers' response to output deviations, convergence to the steady state is guaranteed when the following stable trajectory is followed,*

$$\hat{c}_t = \hat{c}^* + \frac{1-\lambda_1\beta}{\beta^2 \cdot (A+1-\delta)} \cdot \tilde{k}_t + \frac{\lambda_1 \cdot (1-\lambda_1\beta)}{\beta^2 \cdot (A+1-\delta)} \cdot \tilde{k}_{t-1}.$$

Proof: The Jacobian matrix in (4) has a unique eigenvalue located inside the unit circle, $-1 < \lambda_1 < 0$. For this, we compute the corresponding eigenvector, by solving the following system,

$$\begin{cases} \left(\frac{1}{\beta} - \lambda_1\right) \cdot p_1 & & -\frac{1}{\beta \cdot (A+1-\delta)} \cdot p_3 & = 0 \\ p_1 & -\lambda_1 \cdot p_2 & & = 0 \\ & \frac{2}{\pi} \cdot \frac{1-\beta}{\beta} \cdot (\gamma - \gamma_0) \cdot p_2 & +(1-\lambda_1) \cdot p_3 & = 0 \end{cases}$$

Letting p_1=1, the eigenvector may take the form $P = \begin{bmatrix} 1 & \frac{1}{\lambda_1} & \frac{1-\lambda_1\beta}{\beta^2 \cdot (A+1-\delta)} \end{bmatrix}'$.

Given the eigenvector, the stable trajectory is presentable as $\hat{c}_t - \hat{c}^* = \frac{p_3}{p_1} \cdot \tilde{k}_t + \frac{p_3}{p_2} \cdot \tilde{z}_t$ or $\hat{c}_t = \hat{c}^* + \frac{1-\lambda_1\beta}{\beta^2 \cdot (A+1-\delta)} \cdot \tilde{k}_t + \frac{\lambda_1 \cdot (1-\lambda_1\beta)}{\beta^2 \cdot (A+1-\delta)} \cdot \tilde{z}_t$, which is equivalent to the expression in the proposition■

According to the stable trajectory in proposition 2, we realize that convergence to the steady state requires consumption to increase with positive variations in the contemporaneous value of the capital variable and with negative variations in the previous period level of capital.

Basically, through the introduction of a households' response to economic fluctuations we were able to change the stability properties of the benchmark endogenous growth model, turning it possible to characterize the dynamic behaviour of economic aggregates through a saddle-path equilibrium.

3. Endogenous Growth and a Counter-Cyclical Fiscal Policy

3.1 The Fiscal Policy Rule

Return to the basic scenario of a rational representative agent that, under the benchmark AK growth framework, chooses an optimal constant growth rate of consumption over time. Into this model, we now introduce the role of government as supplier of public goods that enhance households' utility. We define variable g_t as the level of aggregate public expenditures; these are financed by taxes, and thus, under a balanced budget assumption, g_t represents as well the tax level.

Public expenditures contribute to rise the utility withdrawn from consumption, an idea that is translated on the following utility function: $U(c_t \cdot g_t{}^{\eta})$, with η>0. The resource constraint is now $k_{t+1} = Ak_t - c_t - g_t + (1-\delta)\cdot k_t$, k_0 given. Solving the optimization problem, it is straightforward to find $\frac{c_{t+1}}{c_t} = \frac{g_{t+1}}{g_t} = \beta \cdot (A+1-\delta)$, i.e., both private and public consumption grow at rate γ, i.e., the same growth rate that we have referred to previously.

In this section, we consider that private consumption effectively grows at the specified rate, but the government disturbs the public expenditures growth rate optimal outcome with a counter-cyclical fiscal policy: there is an opposite sign relation between last period's output gap and the growth rate of g_t; a rise in the output gap triggers a fall in the growth rate of public expenditures and vice-versa. In this case, we might say that the government pursues a stabilization policy.

The rule we adopt to translate the described fiscal policy is close in its form to the one we have considered to describe the consumer sentiment. The following functional form is adopted,

$$\hat{g}_{t+1} = \left[1 - \frac{2\cdot(\gamma-\gamma_0)}{\pi\cdot(1+\gamma)} \cdot arctg(x_{t-1})\right] \cdot \hat{g}_t \quad (6)$$

In (6), we take $\hat{g}_t \equiv \dfrac{g_t}{1+\gamma}$. Figure 4 reveals a function symmetric to the one in figure 1.

*** Figure 4 here ***

3.2 Dynamic Results

The procedure for the analysis of this model is very similar to the one in the previous section. We have a three dimensional system, composed by equations $\tilde{k}_{t+1} = \dfrac{1}{\beta}\cdot(\tilde{k}_t + \hat{k}^*) - \dfrac{\hat{c}^* + \hat{g}_t}{\beta\cdot(A+1-\delta)} - \hat{k}^*$, $\tilde{z}_{t+1} = \tilde{k}_t$ and (6), with $x_{t-1} = \ln\left(\dfrac{\tilde{z}_t + \hat{k}^*}{\hat{k}^*}\right)$; note that the level of consumption is constant over time ($\hat{c}^*$) and that we continue to denote the steady state physical capital value by $\hat{k}^*$. The steady state level of public consumption corresponds to $\hat{g}^* = (1-\beta)\cdot(A+1-\delta)\cdot\hat{k}^* - \hat{c}^*$; because this must be a positive value, the following constraint has to be taken into account: $\hat{c}^*/\hat{k}^* < (1-\beta)\cdot(A+1-\delta)$.

The linearization of the system in the steady state vicinity leads to the matricial presentation in (7),

$$\begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{z}_{t+1} \\ \hat{g}_{t+1} - \hat{g}^* \end{bmatrix} = \begin{bmatrix} 1/\beta & 0 & -1/(1+\gamma) \\ 1 & 0 & 0 \\ 0 & -\dfrac{2}{\pi}\cdot\dfrac{\gamma-\gamma_0}{1+\gamma}\cdot\dfrac{\hat{g}^*}{\hat{k}^*} & 1 \end{bmatrix} \cdot \begin{bmatrix} \tilde{k}_t \\ \tilde{z}_t \\ \hat{g}_t - \hat{g}^* \end{bmatrix} \tag{7}$$

Trace, sum of principal minors and determinant of (7) are straightforward to obtain: $Tr(J) = \dfrac{1+\beta}{\beta}$, $M(J) = 1/\beta = Tr(J) - 1$, $Det(J) = \dfrac{2}{\pi}\cdot\dfrac{\gamma-\gamma_0}{(1+\gamma)^2}\cdot\dfrac{\hat{g}^*}{\hat{k}^*}$. The stability result is given by proposition 3.

Proposition 3. *The AK growth model with counter-cyclical fiscal policy is saddle-path stable (this result requires the assumption of a reasonable future utility discount rate). A two dimensional stable trajectory exists in the three dimensional space that defines the underlying system.*

Proof: The trace and the determinant of the Jacobian matrix in (7) are both positive values. Thus, two possibilities relating to the signs of the eigenvalues are admissible,

- Alternative 1: $\lambda_1, \lambda_2, \lambda_3 > 0$;
- Alternative 2: $\lambda_1, \lambda_2 < 0, \lambda_3 > 0$.

Let us start by analyzing the second alternative. The higher than 1 value of the trace immediately imposes that λ_3 lies outside the unit circle. This information is important, when looking at expression $(1+\lambda_1)\cdot(1+\lambda_2)\cdot(1+\lambda_3) > 0$, which is, in the present case, a satisfied stability condition [because, once again, $M(J) = Tr(J) - 1$, then the presented inequality is equivalent to $2Tr(J) + Det(J) > 0$, as before]; from this condition, we understand that $\lambda_1 < -1 \wedge \lambda_2 < -1$ or, alternatively, $\lambda_1 > -1 \wedge \lambda_2 > -1$. The condition $(1-\lambda_1\lambda_2)\cdot(1-\lambda_1\lambda_3)\cdot(1-\lambda_2\lambda_3) > 0$ [or $2 - Tr(J) + Det(J)\cdot[Tr(J) - Det(J)] > 0$] is, in this case, violated, meaning that $\lambda_1\lambda_2 < 1$ must hold. Combining these last two results, we must have $\lambda_1 > -1 \wedge \lambda_2 > -1$.

Now, note that $M(J) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 1/\beta$. This result is not compatible with the previously found eigenvalues signs, because while $1/\beta>1$, it was remarked that $\lambda_1\lambda_3 < 0$, $\lambda_2\lambda_3 < 0$ and $\lambda_1\lambda_2 < 1$. This contradiction leads to the conclusion that the second alternative that we have suggested for the eigenvalues signs is not feasible, and thus alternative 1 prevails: all eigenvalues have positive signs. To know whether the eigenvalues lie inside or outside the unit circle under alternative 1, follow the reasoning below.

From $(1-\lambda_1)\cdot(1-\lambda_2)\cdot(1-\lambda_3) < 0$ [which is equivalent to $Det(J) > 0$], we will have two possibilities: first, $\lambda_1, \lambda_2 < 1, \lambda_3 > 1$ or, second, $\lambda_1, \lambda_2, \lambda_3 > 1$. This second case is not feasible under $M(J) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 1/\beta$, if one considers a reasonable value for the discount factor (note that, in this case, we would have $\beta<1/3$, which is equivalent to say that the discount rate of future utility would be above 200%). Therefore, we conclude that $0 < \lambda_1, \lambda_2 < 1, \lambda_3 > 1$. Two eigenvalues lie inside the unit circle and, as a consequence, a two dimensional stable path exists■

The exact location of the feasible dynamic results in the trace-determinant diagram is computable in a similar way to the one used to draw figure 3. A relation between trace and determinant is obtained through the discount factor, and corresponds to

$$Det(J) = 2\cdot\frac{Tr(J)-1}{\pi}\cdot\left[\frac{Tr(J)-2}{Tr(J)-1}-\frac{1}{A+1-\delta}\cdot\frac{\hat{c}^*}{\hat{k}^*}\right]\cdot\left[1-\frac{(1+\gamma_0)\cdot(Tr(J)-1)}{A+1-\delta}\right] \qquad (8)$$

Note that (8) is valid only for admissible values of the trace; in the case, $2 < Tr(J) < \dfrac{2\cdot(A+1-\delta)-\hat{c}^*/\hat{k}^*}{(A+1-\delta)-\hat{c}^*/\hat{k}^*}$; the upper bound on the trace is the result of imposing $\hat{c}^*/\hat{k}^* < (1-\beta)\cdot(A+1-\delta)$. As in section 2, computing $\dfrac{dDet(J)}{dTr(J)} = 0$ allows to find an extreme point (in this case, a maximum), which corresponds to the average point between the boundaries taken for the trace.

Figure 5 presents (8) for the specified interval. Note that, once again, no bifurcation exists: saddle-path stability, characterized by the derived eigenvalues' signs, exists independently of the values of parameters. The difference between the two models regarding the number of stable arms relates to the fact that the possible outcomes are located in different places of the diagram in figure 2; basically, the difference in sign of the determinant determines the different dimension of the stable area.

*** Figure 5 here ***

Proposition 4. *In the AK growth model with countercyclical fiscal policy, the convergence to the steady state is achieved through one of the two stable trajectories:*

$$\hat{g}_t = \hat{g}^* + \frac{1-\lambda_1\beta}{\beta^2\cdot(A+1-\delta)}\cdot\tilde{k}_t + \frac{\lambda_1\cdot(1-\lambda_1\beta)}{\beta^2\cdot(A+1-\delta)}\cdot\tilde{k}_{t-1};$$

$$\hat{g}_t = \hat{g}^* + \frac{1-\lambda_2\beta}{\beta^2\cdot(A+1-\delta)}\cdot\tilde{k}_t + \frac{\lambda_2\cdot(1-\lambda_2\beta)}{\beta^2\cdot(A+1-\delta)}\cdot\tilde{k}_{t-1}.$$

Proof: The Jacobian matrix under analysis has two associated eigenvalues inside the unit circle: $0 < \lambda_1, \lambda_2 < 1$; thus, two eigenvectors are computable. These can be presented as follows: $P_1 = \left[1 \quad \frac{1}{\lambda_1} \quad \frac{1-\lambda_1\beta}{\beta^2 \cdot (A+1-\delta)}\right]'$ and $P_2 = \left[1 \quad \frac{1}{\lambda_2} \quad \frac{1-\lambda_2\beta}{\beta^2 \cdot (A+1-\delta)}\right]'$. The stable trajectories correspond to expressions $\hat{g}_t - \hat{g}^* = \frac{p_{31}}{p_{11}} \cdot \tilde{k}_t + \frac{p_{31}}{p_{21}} \cdot \tilde{z}_t$ and $\hat{g}_t - \hat{g}^* = \frac{p_{32}}{p_{12}} \cdot \tilde{k}_t + \frac{p_{32}}{p_{22}} \cdot \tilde{z}_t$ with p_{ij} the i^{th} element of vector j. From the above expressions, it is straightforward to obtain the conditions in the proposition■

Observe that there are two important differences between the derived stable paths for public expenditures and the stable path found for private consumption under the consumers' sentiment rule. First, as highlighted before, there are two trajectories through which one can achieve the long run steady state; second, although in both cases consumption rises with present period physical capital increases, now public expenditures also rise with the previous period amount of capital, given that any of the eigenvalues corresponding to stable manifolds is, in this case, positive.

4. Pro-cyclical Investment

4.1 Firms' Response to Output Fluctuations

In the previous sections, we have analyzed the implications of households and government reactions to departures of the effective income relatively to the potential / steady state benchmark income level. In this section, we use a similar procedure to discuss non optimal decisions taken by firms. The assumption is that investment is pro-cyclical, i.e., firms invest more in periods of expansion (effective output above potential output) than in periods of recession. The form of the investment function is similar to the consumption and public expenditures functions that one as used before.

Consider once again the simple AK model, where $\tilde{k}_{t+1} = \frac{1}{\beta} \cdot \left(\tilde{k}_t + \hat{k}^*\right) - \frac{1}{1+\gamma} \cdot \hat{c}^* - \hat{k}^*$. Investment corresponds to the flow of accumulated capital, i.e., $i_t \equiv k_{t+1} - (1-\delta) \cdot k_t$, and thus we may write the following capital accumulation expression,

$$\tilde{k}_{t+1} = \frac{1}{1+\gamma} \cdot \hat{i}_t + \frac{1-\delta}{1+\gamma} \cdot \left(\tilde{k}_t + \hat{k}_t\right) - \hat{k}^* \qquad (9)$$

with $\hat{i}_t \equiv \frac{i_t}{1+\gamma}$. Observe that investment is just the difference between output and consumption, that is, $i_t = y_t - c_t$, or $\hat{i}_t = A \cdot \left(\tilde{k}_t + \hat{k}^*\right) - \hat{c}^*$.

We can present the dynamics of investment by taking ratio $\frac{\hat{i}_{t+1}}{\hat{i}_t} = \frac{A \cdot \left(\tilde{k}_{t+1} + \hat{k}^*\right) - \hat{c}^*}{A \cdot \left(\tilde{k}_t + \hat{k}^*\right) - \hat{c}^*}$, which is equivalent to $\frac{\hat{i}_{t+1}}{\hat{i}_t} = \theta_t$, with $\theta_t \equiv \frac{\frac{A}{\beta} \cdot \left(\tilde{k}_{t+1} + \hat{k}^*\right) - \left(1 + \frac{A}{1+\gamma}\right) \cdot \hat{c}^*}{A \cdot \left(\tilde{k}_t + \hat{k}^*\right) - \hat{c}^*}$. The growth rate of investment is $\gamma_t^i \equiv \frac{i_{t+1}}{i_t} - 1 = (1+\gamma) \cdot \theta_t - 1$; given that, in the steady state, θ_t=1, investment grows in the long run at the same rate as consumption, output and capital.

Investment growth will not remain always on the optimal level; it varies pro-cyclically with the output gap in the precedent time period. The rule is similar to the ones used before, that is,

$$\hat{i}_{t+1} = \left[\theta_t + \frac{2 \cdot (\gamma_t^i - \gamma_0^i)}{\pi \cdot (1+\gamma_t^i)} \cdot arctg(x_{t-1})\right] \cdot \hat{i}_t \qquad (10)$$

Recalling that $x_{t-1} = \ln \frac{\tilde{z}_t + \hat{k}^*}{\hat{k}^*}$ and assuming $\gamma_0^i = (1+\gamma_0) \cdot \theta_t - 1$, equation (10) is presentable as

$$\hat{i}_{t+1} = \left[\theta_t + \frac{2 \cdot (\gamma - \gamma_0)}{\pi \cdot (1+\gamma)} \cdot arctg\left(\ln \frac{\tilde{z}_t + \hat{k}^*}{\hat{k}^*}\right)\right] \cdot \hat{i}_t \qquad (11)$$

Figure 6 represents graphically the investment dynamic equation in (11).

*** Figure 6 here ***

4.2 The Dynamics of Capital and Investment

The dynamic system that is now subject to evaluation includes equation (9), $\tilde{z}_{t+1} = \tilde{k}_t$ and equation (11). The linearization in the steady state vicinity yields,

$$\begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{z}_{t+1} \\ \hat{i}_{t+1} - \hat{i}^* \end{bmatrix} = \begin{bmatrix} \frac{1-\delta}{1+\gamma} & 0 & \frac{1}{1+\gamma} \\ 1 & 0 & 0 \\ \frac{1-\beta}{\beta} \cdot A & \frac{2}{\pi} \cdot \frac{\gamma-\gamma_0}{1+\gamma} \cdot \frac{\hat{i}^*}{\hat{k}^*} & 1 \end{bmatrix} \cdot \begin{bmatrix} \tilde{k}_t \\ \tilde{z}_t \\ \hat{i}_t - \hat{i}^* \end{bmatrix} \qquad (12)$$

In the steady state, the following relation holds: $\frac{\hat{i}^*}{\hat{k}^*} = \delta + \gamma$. Trace, sum of principal minors and determinant are easily computed: $Tr(J) = 1 + \frac{1-\delta}{1+\gamma}$, $M(2) = \frac{1-\delta}{1+\gamma} - \frac{1-\beta}{\beta} \cdot \frac{A}{1+\gamma}$ and $Det(J) = \frac{2}{\pi} \cdot \frac{(\gamma-\gamma_0)\cdot(\delta+\gamma)}{1+\gamma}$. With these values, it is possible to present the stability result.

Proposition 5. *Under conditions* $A < 1+\gamma$ *and* $Det(J)<1$, *the endogenous growth / non optimal investment model is saddle-path stable. The stable manifold is two-dimensional.*

Proof: Trace and determinant of the Jacobian matrix in (12) are clearly positive values. Condition $A < 1+\gamma$ implies that the sum of principal minors is also a positive value. This inequality is compatible with a low depreciation rate and a low discount rate, and thus it is empirically plausible.

Also empirically realistic is the condition $Det(J)<1$, that holds for plausible depreciation and growth rates.

As in the fiscal policy case, we encounter two alternatives regarding the sign of the eigenvalues,

- Alternative 1: $\lambda_1, \lambda_2, \lambda_3 > 0$;
- Alternative 2: $\lambda_1, \lambda_2 < 0, \lambda_3 > 0$.

The higher than 1 trace implies immediately that λ_3>1, in alternative 2. We look at this second alternative first. Observe that $(1+\lambda_1)\cdot(1+\lambda_2)\cdot(1+\lambda_3)>0$ is a true condition because it is equivalent to $1+Tr(J)+M(J)+Det(J)>0$; thus, we reduce the possibilities concerning eigenvalues signs to $\lambda_1<-1\wedge\lambda_2<-1$ or, alternatively, $\lambda_1>-1\wedge\lambda_2>-1$. A determinant lower than 1 excludes the first possibility, and thus we are able to guarantee that $\lambda_1>-1\wedge\lambda_2>-1$.

Relatively to alternative 1, we observe that $(1-\lambda_1)\cdot(1-\lambda_2)\cdot(1-\lambda_3)<0$, since this is equivalent to $1-Tr(J)+M(J)-Det(J)=-\frac{1-\beta}{\beta}\cdot\frac{A}{1+\gamma}-\frac{2}{\pi}\cdot\frac{(\gamma-\gamma_0)\cdot(\gamma+\delta)}{1+\gamma}$. Hence, it is true that $\lambda_1,\lambda_2<1,\lambda_3>1$ or, alternatively, $\lambda_1,\lambda_2,\lambda_3>1$. This second case is not possible under $M(J)=Tr(J)-1-\frac{1-\beta}{\beta}\cdot\frac{A}{1+\gamma}$.

Putting together the obtained results, we will have $-1<\lambda_1,\lambda_2<1,\lambda_3>1$, therefore concluding that two and only two eigenvalues of J lie inside the unit circle, and consequently we have a saddle-path stable equilibrium■

As in the previous cases, the stable trajectories are straightforward to obtain, by computing the eigenvectors associated with the eigenvalues lower than 1 in modulus.

Proposition 6. *In the AK model with firms' response to business cycles, the convergence to the steady state is guaranteed when one of the presented trajectories is followed,*

$$\hat{i}_t=\hat{i}^*+[\lambda_1\cdot(1+\gamma)-(1-\delta)]\cdot\tilde{k}_t+\left[\lambda_1^2\cdot(1+\gamma)-\lambda_1\cdot(1-\delta)\right]\cdot\tilde{k}_{t-1};$$
$$\hat{i}_t=\hat{i}^*+[\lambda_2\cdot(1+\gamma)-(1-\delta)]\cdot\tilde{k}_t+\left[\lambda_2^2\cdot(1+\gamma)-\lambda_2\cdot(1-\delta)\right]\cdot\tilde{k}_{t-1}.$$

Proof: The procedure to reach the stable trajectories in the proposition is the same as usual. We first compute two eigenvectors, associated with each one of the eigenvalues lying inside the unit circle: $P_1=[1\quad 1/\lambda_1\quad \lambda_1\cdot(1+\gamma)-(1-\delta)]'$ and $P_2=[1\quad 1/\lambda_2\quad \lambda_2\cdot(1+\gamma)-(1-\delta)]'$. The stable trajectories are

$\hat{i}_t - \hat{i}^* = \frac{p_{31}}{p_{11}} \cdot \tilde{k}_t + \frac{p_{31}}{p_{21}} \cdot \tilde{z}_t$ and $\hat{i}_t - \hat{i}^* = \frac{p_{32}}{p_{12}} \cdot \tilde{k}_t + \frac{p_{32}}{p_{22}} \cdot \tilde{z}_t$, with p_{ij} the i^{th} element of vector j. These are equivalent to the equations in the proposition■

In the present case, we have not produced a clear statement about the signs of the stable eigenvalues; these can be positive or negative. In this sense, it is not feasible to point out the direction that the variables follow as the path to the steady state is undertaken. We leave this analysis to the numerical treatment of the model in the next section.

5. Numerical Evaluation

In this section, we re-consider the three models previously discussed, through a numerical example. For all the analyzed models, only four parameters were considered: β, A, δ and γ_0. For the discount rate and the depreciation rate we take values that are considered in the literature as representing well real economic conditions; in particular, we take the values by Guo and Lansing (2002): β=0.962 (this guarantees a discount rate around 4%) and δ=0.067. Relatively to the technology index, we choose a value that is compatible with a reasonable economic growth rate of 3% when effective output equals potential output, i.e., $1+\gamma = 1.03 = \beta \cdot (A + 1 - \delta) \Rightarrow A = 0.138$. Finally, we consider three different cases for the value of γ_0; this allows to explore how the extent of the reaction of agents to output fluctuations impacts on the problems' results. The chosen values are γ_0=(0.025; 0.02; 0.015), that is, we allow the growth rate to deviate from the reference level γ, 0.5, 1 and 1.5 percentage points.

We begin by recovering the consumers' sentiment model. For this, the Jacobian matrix is now,

$$J = \begin{bmatrix} 1.03950 & 0 & -0.97087 \\ 1 & 0 & 0 \\ 0 & 0.00075 - 0.02515 \cdot \gamma_0 & 1 \end{bmatrix}$$

For matrix J above, we compute the eigenvalues and present them in table 1.

γ_0	λ_1	λ_2	λ_3
0.025	-0.00011	1.0032	1.0364
0.02	-0.00023	1.0074	1.0323
0.015	-0.00035	1.0140	1.0259

Table 1 – Eigenvalues in the Consumers' sentiment model.

Table 1 allows to confirm the signs of the eigenvalues one has found in section 2. A unique eigenvalue inside the unit circle is computed, independently of the value of γ_0. Moreover, one regards that the stable eigenvalue rises with a decrease in γ_0; this also occurs for one of the eigenvalues above unity.

The stable trajectory is, in this case, $\hat{c}_t = \hat{c}^* + 1.0093 \cdot \tilde{k}_t - 0.00011 \cdot \tilde{k}_{t-1}$ (for γ_0=0.025), $\hat{c}_t = \hat{c}^* + 1.0094 \cdot \tilde{k}_t - 0.00023 \cdot \tilde{k}_{t-1}$ (for γ_0=0.02) and $\hat{c}_t = \hat{c}^* + 1.0096 \cdot \tilde{k}_t - 0.00035 \cdot \tilde{k}_{t-1}$ (for γ_0=0.025). Thus, in the convergence to the steady state, a change in last period capital stock provokes an almost insignificant variation in today's consumption, while a unit change in contemporaneous capital stock occurs with an almost identical variation in the level of consumption. No significant changes are observed as we vary parameter γ_0.

Consider now the fiscal policy problem. One has concluded that the countercyclical government policy transforms the AK model by introducing a two dimensional stable trajectory.

One additional element is necessary to analyze the stability properties of the model under the numerical example; we have to establish a relation between the steady state values of private and public consumption. Let $\hat{c}^* = 2\hat{g}^*$; this relation and the defined array of parameter values, allows to present the Jacobian matrix,

$$J = \begin{bmatrix} 1.03950 & 0 & -0.97087 \\ 1 & 0 & 0 \\ 0 & -0.00025 + 0.00839 \cdot \gamma_0 & 1 \end{bmatrix}$$

Table 2 presents the eigenvalues of the above matrix, for different values of γ_0,

γ_0	λ_1	λ_2	λ_3
0.025	0.00004	0.99903	1.0404
0.02	0.00008	0.99807	1.0414
0.015	0.00016	0.99714	1.0422

Table 2 – Eigenvalues in the fiscal policy model.

We confirm that the stable manifold is two-dimensional, with all eigenvalues above zero. We observe, as well, that one of the stable eigenvalues rises with γ_0, while the other one falls.

The stable trajectories are the following:

γ_0	λ_1	λ_2
0.025	$\hat{g}_t = \hat{g}^* + 1.0092 \cdot \tilde{k}_t + 0.00004 \cdot \tilde{k}_{t-1}$	$\hat{g}_t = \hat{g}^* + 0.03929 \cdot \tilde{k}_t + 0.03925 \cdot \tilde{k}_{t-1}$
0.02	$\hat{g}_t = \hat{g}^* + 1.0091 \cdot \tilde{k}_t + 0.00008 \cdot \tilde{k}_{t-1}$	$\hat{g}_t = \hat{g}^* + 0.04022 \cdot \tilde{k}_t + 0.04015 \cdot \tilde{k}_{t-1}$
0.015	$\hat{g}_t = \hat{g}^* + 1.0091 \cdot \tilde{k}_t + 0.00016 \cdot \tilde{k}_{t-1}$	$\hat{g}_t = \hat{g}^* + 0.04112 \cdot \tilde{k}_t + 0.04101 \cdot \tilde{k}_{t-1}$

Table 3 – Stable trajectories in the fiscal policy model.

Table 2 confirms that convergence towards equilibrium implies that public expenditures increase as capital in periods *t* and *t*-1 increase. The trajectories for each case are distinguishable in terms of the weight of each period's capital change on the variation of g_t; in concrete, we can have a more significant impact of contemporaneous capital accumulation over the change in public expenditures, or both period's capital stock may have a similar impact over the evolution towards equilibrium of the policy variable.

Finally, consider the investment case. As before, we present the Jacobian matrix,

$$J = \begin{bmatrix} 0.90583 & 0 & 0.97087 \\ 1 & 0 & 0 \\ 0.00545 & 0.0018 - 0.05995 \cdot \gamma_0 & 1 \end{bmatrix}$$

For this particular model, we have found through a generic analysis that two eigenvalues are inside the unit circle, but we were unable to state without any doubt if these were positive or negative values. For the example in appreciation, table 4 presents the eigenvalues.

γ_0	λ_1	λ_2	λ_3
0.025	0.00032	0.86434	1.0412
0.02	0.00065	0.86245	1.0427
0.015	0.00097	0.86059	1.0443

Table 4 – Eigenvalues in the investment model.

In table 4, all eigenvalues are positive, and we confirm the result of a dimension 2 stable arm. The stable trajectories are shown in table 5.

γ_0	λ_1	λ_2
0.025	$\hat{i}_t = \hat{i}^* - 0.93267 \cdot \tilde{k}_t - 0.0003 \cdot \tilde{k}_{t-1}$	$\hat{i}_t = \hat{i}^* - 0.04273 \cdot \tilde{k}_t - 0.03693 \cdot \tilde{k}_{t-1}$
0.02	$\hat{i}_t = \hat{i}^* - 0.93233 \cdot \tilde{k}_t - 0.00061 \cdot \tilde{k}_{t-1}$	$\hat{i}_t = \hat{i}^* - 0.04468 \cdot \tilde{k}_t - 0.03853 \cdot \tilde{k}_{t-1}$
0.015	$\hat{i}_t = \hat{i}^* - 0.932 \cdot \tilde{k}_t - 0.0009 \cdot \tilde{k}_{t-1}$	$\hat{i}_t = \hat{i}^* - 0.04659 \cdot \tilde{k}_t - 0.0401 \cdot \tilde{k}_{t-1}$

Table 5 – Stable trajectories in the investment model.

The computed stable trajectories all point to a relation of opposite sign between capital accumulation (present and past) and today's investment, as the system converges to the steady state. Intuitively, we may explain this relation under the idea that the economy invests more as a response to a decline in the accumulated levels of physical capital. The weight of the variations on past accumulated capital over investment is higher when the second trajectory is followed; the opposite occurs for the contemporaneous stock of capital. Again, no significant qualitative changes arise from considering different values of γ_0.

6. Conclusions

The simple AK endogenous growth model was re-evaluated at the light of a non optimal behaviour assumed by different economic agents.

The optimal control growth problem with a constant returns to scale production function does not allow for a transitional dynamics analysis; only in the circumstance in which the representative agent is able to choose an initial level of consumption that is already in the steady state, the steady state will indeed represent the long run locus of the economy.

Departing from the idea of full rationality, we have identified and explored three cases in which transitional dynamics arise, and where a saddle-path stable equilibrium is present. The three cases are analyzed under a same notion that agents look, in each time moment, to the performance of the economy relatively to its potential to grow. Agents take in consideration the business cycle, and take their decisions accordingly. Instead of fully rational agents, we have agents that change the growth rate of the economic aggregates they control as the economy performs better or worse than expected.

The previous reasoning can be applied to households, government and firms, concerning the variables each agent controls. We have explored the three scenarios under simple logical arguments regarding the reaction to the business cycle. First, one has assumed that consumers react pro-cyclically to the economic activity; this seems logical and has empirical support. Second, a counter-cyclical fiscal policy was taken into consideration; although in practice governments not always adopt such kind of policy, this seems in theory the type of stabilization policy that should be pursued. Third, we assume that firms invest pro-cyclically; this seems logical as well, in the sense that forecasting a larger demand as a result of an expansionary phase, firms will be stimulated to invest more.

In all the three cases, we have identified the presence of a saddle-path equilibrium. In the consumers' sentiment case, the stable path is one dimensional. In the other two cases, a two dimensional stable trajectory was computed. In each case, it was possible to represent the stable paths analytically and therefore to characterize the process of convergence to the steady state.

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Figures

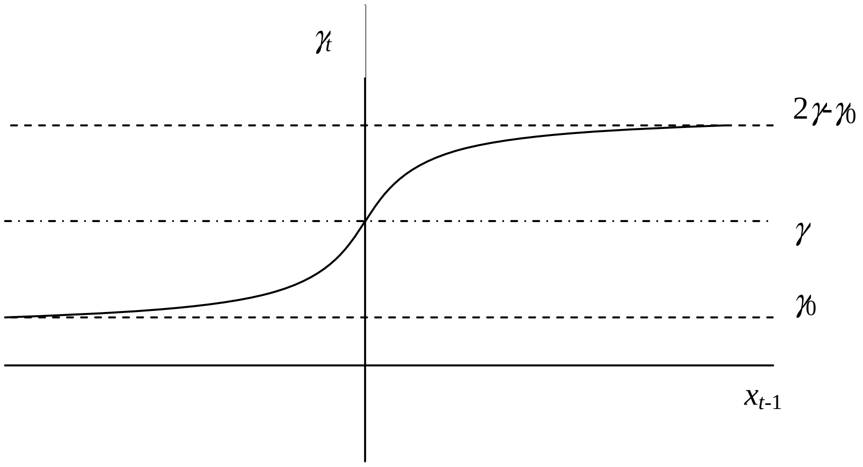

Figure 1 – Consumption growth rate as a function of the output gap.

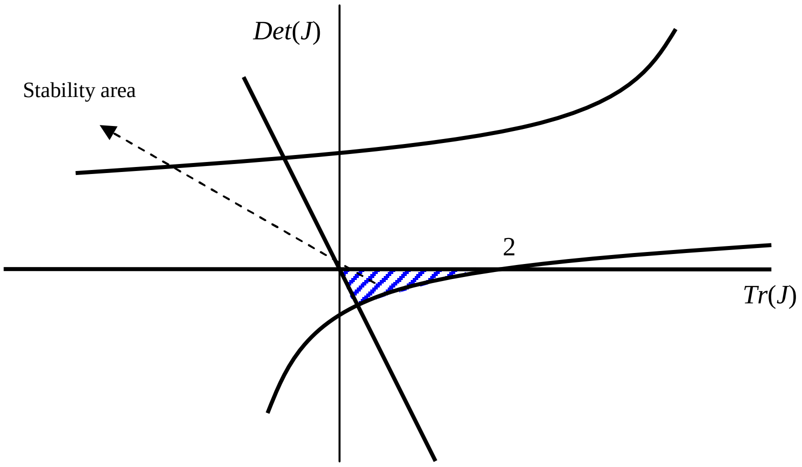

Figure 2 – The trace-determinant relation and the area of stability.

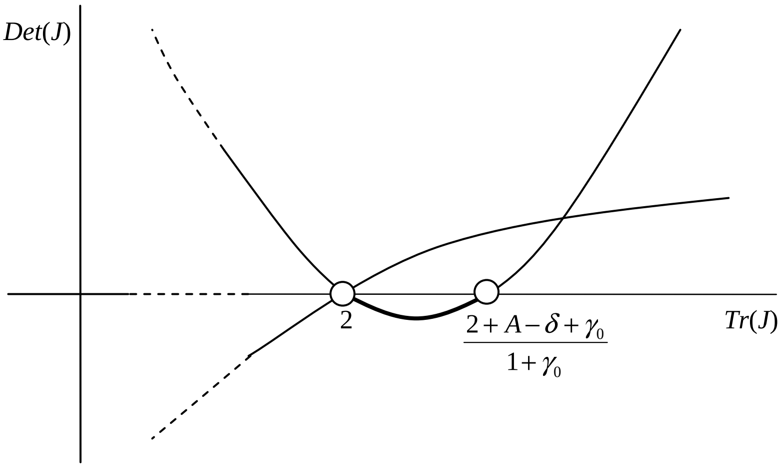

Figure 3 – Saddle-path stability on the consumers' sentiment endogenous growth model.

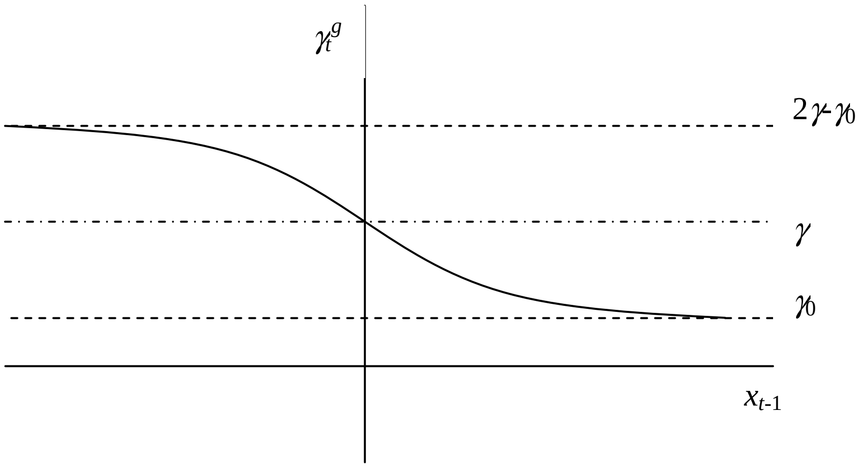

Figure 4 – Public expenditures growth rate as a function of the output gap.

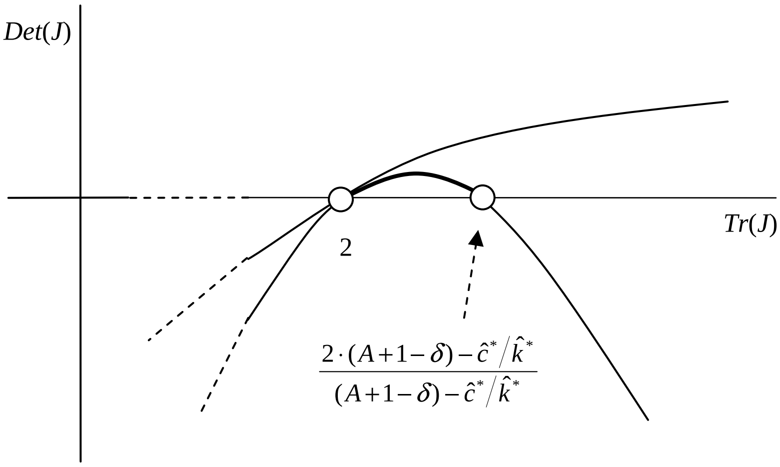

Figure 5 – Saddle-path stability on the endogenous growth model with a counter-cyclical fiscal policy.

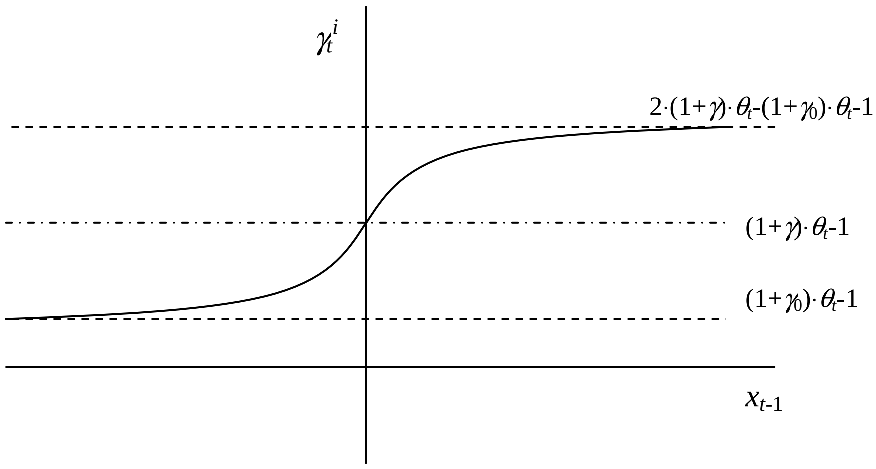

Figure 6 –Investment growth rate as a function of the output gap.