Reactivity in decision-form games

Carfì, David

University of California at Riverside, University of Messina

2009

Reactivity in decision-form games

David Carfì

Abstract

In this paper we introduce the reactivity in decision-form games. The concept of reactivity allows us to give a natural concept of rationalizable solution for decision-form games: the solubility by elimination of sub-reactive strategies. This concept of solubility is less demanding than the concept of solubility by elimination of non-reactive strategies (introduced by the author and already studied and applied to economic games). In the work we define the concept of super-reactivity, the preorder of reactivity and, after a characterization of super-reactivity, we are induced to give the concepts of maximal-reactivity and sub-reactivity; the latter definition permits to introduce the iterated elimination of sub-reactive strategies and the solubility of a decision-form game by iterated elimination of sub-reactive strategies. In the paper several examples are developed. Moreover, in the case of normal-form games, the relation between reactivity and dominance is completely revealed.

1 Introduction

In this paper we introduce the concept of *reactivity for two-player decision-form games* and concentrate upon it. For the concept of decision form game, formally introduced and developed by the author himself, the reader can see [4] and [5]; for the origin of the concept and its motivation the reader can see [1] and [2]. Let $G = (e, f)$ be a decision-form game and let us christen our two player Emil and Frances, it is quite natural that if an Emil's strategy x can react to all the Frances' strategies to which an other strategy x' can react, then we must consider the strategy x *reactive* at least as the strategy x'; moreover, if the strategy x is reactive at least as x' and x can react to a Frances' strategy to which x' cannot react to, then Emil should consider the strategy x strictly more reactive than x'. The previous simple considerations allow us to introduce the *capacity of reaction*, or reactivity, of any Emil's strategy and to compare it with the capacity of reaction of the other Emil's strategies. In this direction, we introduce the *super-reactive strategies* of a player i, i.e. strategies of player i capable to reply to any opponent's actions to which the player i can reply: obviously these strategies (whenever they there exist) are the best ones to use, in the sense explained before. In a second time, we introduce the *reactivity comparison* between strategies and we observe that this relation is a preorder. Then, we define the concept of reactivity and explain the nature of the super-reactivity, this permits to define the concepts of *maximally reactive strategy*, *minimally reactive strategy,* and of *sub-reactive strategy.* The concept of sub-reactivity will give us the opportunity to introduce the principal operative concepts of the paper, i.e. *the elimination of sub-reactive strategies*, the concept of *reducing sequence of a game by elimination of sub-reactive strategies* and, at last, the *solvability of a game by elimination of sub-reactive strategies* with the meaning of solution in the case of solvability.

2 Super-reactive strategies

Definition (of super-reactive strategy). *Let (e, f) be a two player decision-form game. An Emil's strategy x_0 is called* ***super-reactive with respect to the decision rule*** *e if it is a possible reaction to all the Frances' strategies to which Emil can react. In other terms, an Emil's strategy x_0 is called super-reactive if it belongs to the reaction set $e(y)$, for each Frances' strategy y belonging to the domain of the decision rule e. Analogously, a Frances' strategy y_0 is called* ***super-reactive with respect to the decision rule*** *f if it is a possible reaction to all the Emil's strategies to which Frances can react. In other terms, a Frances' strategy y_0 is called super-reactive if it belongs to the reaction set $f(x)$, for each Emil's strategy x in the domain of the decision rule f.*

Remark. Let E' be the domain of the decision rule f and F' be the domain of the decision rule e. The sets of all the Frances' and Emil's super-reactive strategies are the two intersections

$$\cap^{\neq}(e) := \cap_{y \in F'} e(y)$$

and

$$\cap^{\neq}(f) := \cap_{x \in E'} f(x),$$

respectively. If Frances has no disarming strategies toward Emil we have

$$\cap^{\neq}(e) = \cap e := \cap_{y \in F} e(y).$$

Analogously, if Emil has no disarming strategies toward Frances

$$\cap^{\neq}(f) = \cap f := \cap_{x \in E} f(x).$$

Obviously these two intersections can be empty.

We note here an elementary and obvious result.

Proposition. *Let (e, f) be a decision form game and let x_0 and y_0 be two non-disarming and super-reactive strategies of the first and second player respectively. Then the bistrategy (x_0, y_0) is an equilibrium of the game.*

It is straightforward that a game can have equilibria and lack in super-reactive strategy, as the following example shows.

Example (of game without super-reactive strategies). Let (e, f) be the decision form game with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$ and decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} -1 & \text{if } y < 0 \\ E & \text{if } y = 0 \\ 2 & \text{if } y > 0 \end{cases},$$

$$f(x) = \begin{cases} -1 & \text{if } x < 1 \\ F & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases}.$$

Emil has not super-reactive strategies, in fact

$$\cap e = \{-1\} \cap E \cap \{2\} = \varnothing.$$

Also Frances has no super-reactive strategies, in fact

$$\cap f = \{-1\} \cap F \cap \{1\} = \varnothing.$$

Note that this game has three equilibria.

We say that an equilibrium of a game is a *super-reactive equilibrium* when it is a super-reactive cross, i.e. when it is a pair of super-reactive strategies.

Example (of game with super-reactive strategies). Let (e, f) be the game with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$ and decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} [-1, 1] & \text{if } y < 0 \\ E & \text{if } y = 0 \\ [0, 2] & \text{if } y > 0 \end{cases},$$

$$f(x) = \begin{cases} -1 & \text{if } x < 1 \\ F & \text{if } x = 1 \\ \{-1, 1\} & \text{if } x > 1 \end{cases}.$$

Emil has infinite super-reactive strategies, in fact the intersection of the family (correspondence) e is

$$\cap e = [-1, 1] \cap E \cap [0, 2] = [0, 1],$$

all the strategies x between 0 and 1 are super-reactive for Emil. Frances has only one super-reactive strategy, in fact

$$\cap f = \{-1\} \cap F \cap \{-1, 1\} = \{-1\}.$$

Note that this game has infinitely many equilibria, their set is the graph of the correspondence $f_1 : E \to F$ defined by

$$f_1(x) = \begin{cases} -1 & \text{if } x < 1 \\ F & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases} .$$

On the other hand, only the equilibria belonging to the segment $[0, 1] \times \{-1\}$ are super-reactive equilibria. Thanks to super-reactivity, in this game an equilibrium is non-cooperatively reachable; indeed, it is reasonable for Frances to play his unique super-reactive strategy -1 and for Emil to play one of his super-reactive strategies x in $[0, 1]$, consequently the game finishes in the equilibrium $(x, -1)$.

Remark (independence of the super-reactivity on the rival's rule). The Emil's (Frances's) super-reactive strategies depend only upon the Emil's (Frances's) decision rule, and not on both the decision rules.

Example (game with super-reactive strategies). Let E be the compact interval $[0, 1]$ and let F be the interval $[-1, 1]$, let $e : F \to E$ be the correspondence defined by $e(y) = [0, |y|]$, for each y in F. Frances has no disarming strategies toward Emil. The strategy 0 is the only Emil's super-reactive strategy, because

$$\cap e = \bigcap_{y \in F} [0, |y|] = \{0\} .$$

Let $f : E \to F$ be defined by $f(x) = [-x, x]$. Emil has no disarming strategies toward Frances. The strategy 0 is the only Frances' super-reactive strategy, because

$$\cap f = \bigcap_{x \in E} [-x, x] = \{0\} .$$

In this case we have again infinitely many equilibria, the points of the graph of the correspondence $f_1 : E \to F$ defined by $f_1(x) = \{-x, x\}$, but we have only one super-reactive equilibrium: the strategy profile $(0, 0)$.

3 Comparison of reactivity

The definition of super-reactive strategy can be generalized.

Definition (of comparison among reactivity). *Let (e, f) be a two player decision form game. Let x_0 and x be two Emil's strategies. We say that the strategy x_0 is* ***more reactive (in wide sense), with respect to the decision rule** e**, than the strategy** x, and we write $x_0 \geq_e x$, if x_0 is a possible reaction* *to all the Frances' strategies to which x can react. In other terms, an Emil's strategy x_0 is said more reactive than an other strategy x when x_0 belongs to the reaction set $e(y)$, for each strategy $y \in e^-(x)$. Analogously, let y_0 and y be two Frances' strategies. We say that y_0 is more* ***reactive, with respect to the decision rule*** *f,* ***than the strategy*** *y, and we write $y_0 \geq_f y$, if the strategy y_0 is a possible reaction to all the Emil's strategies to which y is a possible reaction. In other terms, a Frances' strategy y_0 is more reactive than y when y_0 belongs to the reaction set $f(x)$, for each strategy $x \in f^-(y)$.*

Memento (reciprocal correspondence). We remember that the set $e^-(x)$ is the set of those Frances' strategies to which the strategy x can reply, with respect to the decision rule e. In fact, the reciprocal image of the strategy x with respect to the correspondence e is

$$e^-(x) = \{y \in F : x \in e(y)\},$$

therefore it is defined, exactly, as the set of all those Frances' strategies y for which x is a possible response strategy. The reciprocal correspondence of e, i.e. the correspondence $e^- : E \to F$ defined by $x \mapsto e^-(x)$, associates with every Emil's strategy x the set of all those Frances's strategies for which x is a possible reaction. This last circumstance explains the interest in the determination of the reciprocal correspondence e^-.

Example (of comparison of reactivity). Let (e, f) be the decision form game with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$ and decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} \{-1\} & \text{if } y < 0 \\ E & \text{if } y = 0 \\ \{2\} & \text{if } y > 0 \end{cases},$$

$$f(x) = \begin{cases} \{-1\} & \text{if } x < 1 \\ F & \text{if } x = 1 \\ \{1\} & \text{if } x > 1 \end{cases}.$$

we want to determine the reciprocal multifunctions of e and f. We have

$$e^-(x) = \begin{cases} [-1, 0] & \text{if } x = -1 \\ \{0\} & \text{if } x \in \,]-1, 2[\\ [0, 1] & \text{if } x = 2 \end{cases},$$

$$f^-(y) = \begin{cases} [-1, 1] & \text{if } y = -1 \\ \{1\} & \text{if } y \in \,]-1, 1[\\ [1, 2] & \text{if } y = 1 \end{cases}.$$

From these two relations, we can easily observe that Emil's strategies -1 and 2 are more reactive than all the other Emil's strategies of the interval $]-1, 2[$, with respect to the rule e. To this aim, we have to prove that the strategies 1 and 2 belong to the reaction set $e(y)$, for each strategy $y \in e^-(x)$. Let x be an Emil's strategy in the open interval $]-1, 2[$, we have $e^-(x) = \{0\}$, then the relation $y \in e^-(x)$ is equivalent to $y = 0$, but the image $e(0)$ is the whole of E, therefore it includes both -1 and 2. Analogously, we prove that Frances' strategies -1 and 1 are more reactive than every other strategy $y \in]-1, 1[$, with respect to the decision rule f.

The following theorem expresses the reactivity comparison in a conditional form.

Theorem. *In the conditions of the above definition. An Emil's strategy x_0 is more reactive than another Emil's strategy x, with respect to the decision rule e, if, for each Frances' strategy y, from the relation $x \in e(y)$ it follows $x_0 \in e(y)$. In symbols, the relation $x_0 \geq_e x$ holds if and only if*

$$(\forall y \in F)(x \in e(y) \Rightarrow x_0 \in e(y)).$$

Analogously, a Frances' strategy y_0 is more reactive than another Frances' strategy y, with respect to the decision rule f, if for each Emil's strategy x, from the relation $y \in f(x)$ we deduce $y_0 \in f(x)$. In symbols, the relation $y_0 \geq_f y$ holds if and only if

$$(\forall x \in E)(y \in f(x) \Rightarrow y_0 \in f(x)).$$

4 The reactivity preorder

It is immediate to verify that the relation of reactivity comparison determined by the decision rule f upon the strategy space F - defined, for each pair of strategies (y, y'), by $y \geq_f y'$, and that we denote by $\geq_f$ - is a preorder. This justifies the following definition.

Definition (of reactivity preorder). *Let (e, f) be a decision form game upon the underlying strategy pair (E, F). The binary relation $\geq_f$ on the strategy set F is called* ***preorder of reactivity induced by the*** **decision** ***rule*** *f* ***on Frances' strategy space****. Symmetrically, the binary relation $\geq_e$ on the strategy space E is called* ***preorder of reactivity induced by the*** **decision** ***rule*** *e* ***on Emil's*** **strategy** ***space*** *.*

Remark (strict preorder of reactivity). Since the reactivity comparison $\geq_f$ is a preorder, it has an associated strict preorder: the preorder $>_f$ defined, as usual in Preorder Theory, for each pair of strategies (y_0, y), by $y_0 >_f y$ if and only if $y_0 \geq y$ and $y \not\geq y_0$. Analogous consideration holds for Emil.

Now we see an example of strict comparison of reactivity among strategies.

Example (of strict reactivity comparison). Let (e, f) be the decision form game with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$ and with decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} -1 & \text{if } y < 0 \\ E & \text{if } y = 0 \\ 2 & \text{if } y > 0 \end{cases},$$

$$f(x) = \begin{cases} -1 & \text{if } x < 1 \\ F & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases}.$$

We apply the conditional characterization to prove (again) that Emil's strategies -1 and 2 are more reactive than all the strategies of the open interval $]-1, 2[$. In fact, for each Frances' strategy y , if an Emil's strategy $x \in]-1, 2[$ belongs to the reaction set $e(y)$ of y, the Frances' strategy y must necessarily be 0 (because the unique reaction set $e(y)$ containing strategies different from -1 and 2 is just $e(0)$), but, in this case, we have also that -1 and 2 belong to $e(y)$ (inasmuch, the reaction set $e(0)$ is the whole of E). We have so proved that the inequality $-1, 2 \geq_e x$ holds for each strategy x in E. Now we want to prove that the strict inequality $-1, 2 >_e x$ holds true, for each $x \in]-1, 2[$ (i.e. that the strategies -1 and 2 are strictly more reactive than any other Emil's strategy). It is sufficient to prove that, for instance, the relation $2 \leq_e x$ is false, for any x in $]-1, 2[$; for, fixed such an x, we have to show that there exists a strategy y in F such that $2 \in e(y)$ and $x \notin e(y)$ (i.e. a strategy y in F to which 2 reacts and x does not). Let $y = 1$, we have $e(y) = \{2\}$, then 2 is in $e(y)$ and any $x \in]-1, 2[$ does not.

5 The reactivity of a strategy

Terminology (reciprocal decision rule). Let $f : E \to F$ be a Frances' decision rule. We can associate, in a natural way, with the correspondence f the Emil's decision rule

$$f^{-} : F \to E : y \mapsto f^{-}(y),$$

that we call *Emil's decision rule reciprocal of the Frances's decision rule f*. This reciprocal decision rule is canonically associated with the application of F into the set of subsets of E associating with every Frances's strategy y the set of all Emil's strategies for which y is a possible reaction: the function

$$f^{\leftarrow} : F \to \mathcal{P}(E) : y \mapsto f^{-}(y).$$

With abuse of language, we will name this application *reciprocal function of the correspondence f*.

Theorem (characterization of the preorder of reactivity). *The (opposite) reactivity preorder $\leq_f$ is the preorder induced (in the usual sense) by the reciprocal function of the decision rule f, that is by the function*

$$f^{\leftarrow} : F \to \mathcal{P}(E) : y \mapsto f^{-}(y),$$

endowing the set of parts of E (denoted by $\mathcal{P}(E)$) with the set inclusion order $\subseteq$. In other terms, the (opposite) preorder of reactivity $\leq_f$ is the reciprocal image of the set inclusion order with respect to the reciprocal function $f^{\leftarrow}$ of the decision rule f.

Proof. Let $x \in E$ and $y \in F$ be strategies. The relation $y \in f(x)$ is equivalent to the relation $x \in f^{-}(y)$, therefore a Frances's strategy y_0 is more reactive than y if and only if $f^{-}(y) \subseteq f^{-}(y_0)$. ■

The above characterization allows to give the following definition.

Definition (of reactivity). *Let (e, f) be a decision form game. For each strategy x in E, the reciprocal image of the strategy x by the correspondence e , that is the set $e^{-}(x)$, is called the* ***reactivity of*** *x with respect to the decision rule e. Analogously, for each France' strategy y in F, the reciprocal image of y by the decision rule f is called the* ***reactivity of*** *y with respect to f.*

6 Super-reactive strategies as maxima

The following obvious result characterizes super-reactive strategies of a player as maxima (upper optima) of the strategy space of the player with respect to the reactivity preorder induced by his decision rule. Therefore it allows to reduce the concept of super-reactive strategy to the concept of optimum.

Theorem (characterization of super-reactivity). *Let (e, f) be a decision form game. Any Frances' super-reactive strategy is a maximum of the preorder space $(F, \geq_f)$ and vice versa.*

Remark (on the nature of super-reactive strategies). After the realization of the true nature of super-reactive strategies, we can observe some of the previous examples in another way. We have, in fact, seen that there are situations in which Frances has no super-reactive strategies, this simply means that the preordered space $(F, \geq_f)$ has no maxima; this does not surprise, in fact a preordered space has maxima only in very particular cases. Obviously, when a space has no maxima (as observed in preordered space theory) we have to look for other solutions of the corresponding decision problem (Pareto boundaries, cofinal and coinitial parts, suprema and infima, and so on) but we shall analyze these aspects in the following paragraphs.

We should notice that, in general, the space $(F, \geq_f)$ is not an ordered space, and therefore several maxima can exist (they must necessarily be indifferent between themselves by the theorem of indifference of optima in preordered spaces), as the following example shows.

Example (of distinct super-reactive strategies). Let (e, f) be the decision form game with strategy spaces the two intervals of the real line $E = [a, b]$ and $F = [c, d]$ and with decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} \{a, b\} & \text{if } y < 0 \\ E & \text{if } y = 0 \\ \{a, b\} & \text{if } y > 0 \end{cases},$$

$$f(x) = \begin{cases} \{c, d\} & \text{if } x < 1 \\ F & \text{if } x = 1 \\ \{c, d\} & \text{if } x > 1 \end{cases},$$

for each bistrategy (x, y) of the game. It is easy to realize that the two strategies a and b are super-reactive for Emil, and, because they are maxima of the set E with respect to the preorder $\geq_e$, they are indifferent. Let us see this directly. The set of Frances' strategies for which a is a possible reaction is $e^-(a) = F$, from which immediately follows that a is a maximum of the space $(E, \geq_e)$ (no Emil's strategy can be more reactive than a inasmuch the strategy a reacts to all the Frances' strategies). Analogously, we can proceed for b (for which the situation is exactly the same).

Remark (on the indifference in reactivity of strategies). We note that the reactivity indifference of two Emil's strategies x and x' is equivalent to the relation $e^-(x) = e^-(x')$. In fact, the preorder $\leq_e$ is induced by the function $e^{\leftarrow}$ of E in $\mathcal{P}(F)$ defined by $x \mapsto e^-(x)$ with respect of the set inclusion, and therefore x and x' are equivalent in reactivity if and only if they have the same value in $e^{\leftarrow}$.

7 Maximally reactive strategies

For the concept of maximal element in preordered spaces and its developments we follow [3].

Definition (of maximally reactive strategy). *Let (e, f) be a decision form game upon the underlying strategy pair (E, F). A Frances' strategy $y \in$*

F is called ***maximally reactive*** *if does not exist another Frances' strategy strictly more reactive than y (i.e., as we shall see later, if the strategy y is not a sub-reactive strategy). In other terms, a Frances' strategy is called maximally reactive if it is (Pareto) maximal in the preordered space $(F, \geq_f)$. Analogously, an Emil's strategy is called maximally reactive if it is (Pareto) maximal in the preordered space $(E, \geq_e)$.*

Example (of maximally reactive strategy). Let (e, f) be the decision form game with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$ and decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} \{-1\} & \text{if } y < 0 \\ E & \text{if } y = 0 \\ \{2\} & \text{if } y > 0 \end{cases},$$

$$f(x) = \begin{cases} \{-1\} & \text{if } x < 1 \\ F & \text{if } x = 1 \\ \{1\} & \text{if } x > 1 \end{cases}.$$

The reciprocal correspondences of e and f are defined by

$$e^-(x) = \begin{cases} [-1, 0] & \text{if } x = -1 \\ \{0\} & \text{if } x \in \,]-1, 2[\\ [0, 1] & \text{if } x = 2 \end{cases},$$

$$f^-(y) = \begin{cases} [-1, 1] & \text{if } y = -1 \\ \{1\} & \text{if } y \in \,]-1, 1[\\ [1, 2] & \text{if } y = 1 \end{cases}.$$

Hence we can easily note that the Emil's strategies -1 and 2 are maximally reactive. For instance, we shall study the strategy 2. It is sufficient to show that the subset $e^-(2)$ is not strictly included in any other image $e^-(x)$, and this is evident. We have seen before that these two maximal strategies are more reactive than all other Emil's strategies $x \in \,]-1, 2[$, with respect to the rule e: therefore all the Emil's strategies, with the exception of the two maximal ones, are strictly less reactive than the maximal ones; moreover, all Emil's strategies in $]-1, 2[$ are indifferent between them (they have the same image through e^-), we see, so, that the interval $]-1, 2[$ is even the set of all the minima of the preordered space $(E, \geq_e)$. Analogously, we can prove that the strategies -1 and 1 form the maximal boundary of the preordered space $(F, \geq_f)$.

8 Sub-reactive strategies

Definition (of sub-reactive strategy). *A strategy s of a player in a decision form game is said* ***sub-reactive*** *if there exists a strategy s' of the same player strictly more reactive than the strategy s. In other terms, a Frances' strategy is* *said sub-reactive if it is not (Pareto) maximal in the preordered space* $(F, \geq_f)$. *Analogously, an Emil's strategy is said sub-reactive if it is not Pareto maximal in the preordered space* $(E, \geq_e)$.

Example (of sub-reactive strategy). Let (e, f) be the game with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$ and decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} \{-1\} & \text{if } y < 0 \\ E & \text{if } y = 0 \\ \{2\} & \text{if } y > 0 \end{cases}, \quad f(x) = \begin{cases} \{-1\} & \text{if } x < 1 \\ F & \text{if } x = 1 \\ \{1\} & \text{if } x > 1 \end{cases}.$$

We have seen before that the two Emil's maximal strategies -1 and 2 are more reactive than any other Emil's strategy $x \in]-1, 2[$, with respect to the rule e: therefore all Emil's strategies, except the maximal, are sub-reactive.

9 Elimination of sub-reactive strategies

Definition (of reduced game by elimination of sub-reactive strategies). *A game* (e, f) *is said* ***reduced by elimination of*** **sub-reactive** ***strategies*** *if the maximal (Pareto) boundaries of the preordered spaces* $(E, \geq_e)$ *and* $(F, \geq_f)$ *coincide with the strategy sets* E *and* F*, respectively.*

Example (of not reduced game). Let (e, f) be the decision form game with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$ and decision rules $e : F \to E$ and $f : E \to F$ defined by

$$\begin{aligned} e(y) &= \begin{cases} \{-1\} & \text{if } y < 0 \\ E & \text{if } y = 0 \\ \{2\} & \text{if } y > 0 \end{cases}, \\ f(x) &= \begin{cases} \{-1\} & \text{if } x < 1 \\ F & \text{if } x = 1 \\ \{1\} & \text{if } x > 1 \end{cases}. \end{aligned}$$

The maximal boundaries of the preordered spaces $(E, \geq_e)$ and $(F, \geq_f)$ are the sets $\{-1, 2\}$ and $\{-1, 1\}$, therefore the game is not reduced by elimination of sub-reactive strategies.

Before to proceed with the following definition, we recall the notion of subgame of a decision form game.

Definition (of subgame). *Let* (e, f) *be a decision form game upon the strategy pair* (E, F) *and let* (E', F') *be a sub-strategy pair of* (E, F)*, i.e. a pair* *of subsets of E and F, respectively. We call* ***subgame of*** *(e, f)* ***with underlying pair*** *(E', F') the pair of correspondence (e', f') having as components the restrictions of the rules e and f to the pairs of sets (F', E') and (E', F'), respectively. We remember that, for example, e' is the correspondence from F' into E' which sends a strategy y' of F' into the intersection $e(y') \cap E'$. In other terms, e' sends every strategy y' of F' into all Emil's reaction strategies to y' which are in E'.*

Definition (reduction of a game by elimination of sub-reactive strategies). *Let $G = (e, f)$ be a decision form game with underlying pair (E, F). We call* ***reduction of the game*** *(e, f)* ***by elimination of*** **sub-reactive** ***strategies*** *the subgame (e', f') of G with underlying strategy pair the pair of the maximal Pareto boundaries $\overline{\partial}_e E$ and $\overline{\partial}_f F$ of the preorder spaces $(E, \geq_e)$ and $(F, \geq_f)$. In other terms, the* ***reduction of the game*** *(e, f)* ***by elimination of the*** **sub-reactive** ***strategies*** *is the game with decision rules the restrictions $e_{|(F',E')}$ and $f_{|(E',F')}$, where E' and F' are the maximal Pareto boundaries $\overline{\partial}_e E$ and $\overline{\partial}_f F$ of the preordered spaces $(E, \geq_e)$ and $(F, \geq_f)$.*

Example (of reduction). Let (e, f) be the game with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$ and decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} \{-1\} & \text{if } y < 0 \\ E & \text{if } y = 0 \\ \{2\} & \text{if } y > 0 \end{cases},$$

$$f(x) = \begin{cases} \{-1\} & \text{if } x < 1 \\ F & \text{if } x = 1 \\ \{1\} & \text{if } x > 1 \end{cases}.$$

The maximal boundaries of the preordered spaces $(E, \geq_e)$ and $(F, \geq_f)$ are the sets $E_1 = \{-1, 2\}$ and $F_1 = \{-1, 1\}$, therefore the game is not reduced, because they don't coincide with the respective spaces. The reduction of the game (e, f) by elimination of sub-reactive strategies is the game with decision rules $e_1 : F_1 \to E_1$ and $f_1 : E_1 \to F_1$ defined by

$$e_1(y) = \begin{cases} -1 & \text{if } y = -1 \\ 2 & \text{if } y = 1 \end{cases},$$

$$f_1(x) = \begin{cases} -1 & \text{if } x = -1 \\ 1 & \text{if } x = 2 \end{cases}.$$

Example (of reduced game). We note that the game (e_1, f_1) of previous example is reduced. In fact, the reciprocals correspondences of the rules e_1 and f_1 are defined by

$$e_1^-(x) = \begin{cases} \{-1\} & \text{if } x = -1 \\ \{1\} & \text{if } x = 2 \end{cases},$$

$$f_1^-(y) = \begin{cases} \{-1\} & \text{if } y = -1 \\ \{2\} & \text{if } y = 1 \end{cases}.$$

The maximal boundaries of the preordered spaces $(E_1, \geq_{e_1})$ and $(F_1, \geq_{f_1})$ are the sets $E_2 = \{-1, 2\}$ and $F_2 = \{-1, 1\}$, respectively, therefore the game is reduced because E_2 and F_2 coincide with the respective spaces. For an easy determination of the two boundaries, we note that, for example, the preordered space $(E_1, \geq_{e_1})$ is isomorphic to the preordered space with two elements $(\{\{1\}, \{-1\}\}, \subseteq)$.

10 Iterated elimination of sub-reactivity

Definition (of reducing sequence of a game). *Let $G_0 = (e_0, f_0)$ be a game on a strategy base (E_0, F_0). We call* ***reducing sequence by elimination of*** **sub-reactive** ***strategies of*** *G_0 the sequence of subgames $G = (G_k)_{k=0}^{\infty}$, with 0-term the game G_0 itself and with k-th term the game $G_k = (e_k, f_k)$, such that the strategy base (E_k, F_k) of the game G_k be the pair of maximal boundaries of the preordered spaces $(E_{k-1}, \geq_{e_{k-1}})$ and $(F_{k-1}, \geq_{f_{k-1}})$, of the $(k-1)$-th subgame, for each positive integer k. So, the decision rules e_k and f_k are the restrictions to the pairs (F_k, E_k) and (E_k, F_k) of the decision rules e_{k-1} and f_{k-1}, respectively.*

Definition (of solubility by iterated elimination of sub-reactive strategies). *Let $G_0 = (e_0, f_0)$ be a decision form game, and let G be its reducing sequence by elimination of sub-reactive strategies. The game G_0 is called* ***solvable by iterated elimination of*** **sub-reactive strategies** *if there exists only one bistrategy common to all subgames of the sequence G. In that case, that bistrategy is called the* ***solution*** **by iterated elimination of** ***sub-reactive strategies of the game*** *G_0.*

Remark. The definition of solubility by iterated elimination of sub-reactive strategies is so equivalent to contain the intersection $\bigcap_{k=1}^{\infty} E_k \times F_k$ only one element.

Remark. If the game G_0 is finite, it is solvable by iterated elimination of sub-reactive strategies if and only if there exists a subgame of the sequence G with only one bistrategy; in that case, that bistrategy is the solution by iterated elimination of sub-reactive strategies of the game G_0.

11 Relative super-reactivity

Definition (of relatively super-reactive strategy). *Let (e, f) be a two player decision form game. Let E' be a set of Emil's strategies to which Frances can react and let y_0 be a Frances' strategy. The strategy y_0 is called* ***relatively***

***super-reactive for** E' **(with respect to the decision rule** f**)** if it is a possible reaction to all the Emil's strategies in E'. In other terms, a Frances' strategy y_0 is called relatively super-reactive for E' if it belongs to the set $f(x)$, for each Emil's strategy x in E'. Analogously, let F' be a set of Frances' strategies to which Emil can react and x_0 an Emil's strategy. The strategy x_0 is called **relatively super-reactive for** F' **(with respect to the decision rule** e**)** if it is a possible reaction to all the Frances' strategies in F'. In other terms, an Emil's strategy x_0 is called relatively super-reactive for F' if it belongs to the set $e(y)$, for each Frances' strategy y in F'.*

Remark. So the sets of Emil and Frances' relatively super-reactive strategies for F' and for E' are the two intersections $\cap_{F'} e = \bigcap_{y \in F'} e(y)$ and $\cap_{E'} f = \bigcap_{x \in E'} f(x)$. Evidently these intersections can be empty.

Example (of relatively super-reactive strategies). Let (e, f) be the decision-form game with strategy spaces $E = [-1, 2]$ and $F = [-1, 1]$ and decision rules $e : F \to E$ and $f : E \to F$ defined by

$$e(y) = \begin{cases} -1 & \text{if } y < 0 \\ E & \text{if } y = 0 \\ 2 & \text{if } y > 0 \end{cases} ,$$

$$f(x) = \begin{cases} -1 & \text{if } x < 1 \\ F & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases} .$$

Emil has only a relatively super-reactive strategy for the Frances' nonnegative strategies and only one relatively super-reactive strategy for the Frances' nonpositive strategies. Indeed, we have

$$\bigcap_{y \in [0,1]} e(y) = E \cap \{2\} = \{2\}$$

and

$$\bigcap_{y \in [-1,0]} e(y) = \{-1\} \cap E = \{-1\} .$$

Frances is in a similar situation for the Emil's strategies greater or equal to 1 and for the Emil's strategies less or equal to 1, in fact, we have

$$\bigcap_{x \in [1,2]} f(x) = F \cap \{1\} = \{1\}$$

and

$$\bigcap_{x \in [-1,1]} f(x) = F \cap \{-1\} = \{-1\} .$$

The following theorem has an obvious proof.

Theorem (on reactivity). *Let x be an Emil's strategy. Then, the greatest among the sets F' of Frances's strategies such that the strategy x is relatively super-reactive for is the reactivity of x.*

Example (of reactivity). Let (e, f) be the game of above example. The reactivity of the Emil's strategy 2 is the interval $[0, 1]$, the reactivity of the Emil's strategy -1 is the interval $[-1, 0]$. Indeed, these intervals are the biggest sets to which the above strategies can react, respectively.

12 Dominant strategies

For the definition of normal form game used in this paper see [6], for the theory of normal form games we follow [1], [2], [8], [7] and [9].

Definition (of dominant strategy). *Let $(u_1, \geq)$ be an Emil's utility function on the bistrategy space $E \times F$ of a strategy pair (E, F). An Emil's strategy x_0 in E is said **dominant with respect to the utility function** u_1 if, for each strategy x in E, the inequality*

$$u_1(x_0, y) \geq u_1(x, y),$$

holds, for each strategy y in F. In other terms, an Emil's strategy x_0 in E is said dominant if, for each other strategy x in E, the function inequality

$$u_1(x_0, .) \geq u_1(x, .)$$

holds true. Analogously, let $(u_2, \geq)$ be a Frances' utility function on the bistrategy space $E \times F$ of a strategy pair (E, F). A strategy y_0 in F is said **dominant with respect to the *utility* function** *u_2 if, for each y in F, the inequality*

$$u_2(x, y_0) \geq u_2(x, y),$$

holds, for each strategy x in E. In other terms, a Frances' strategy y_0 in F is said u_2-dominant if, for each other strategy y in F, the function inequality

$$u_2(., y_0) \geq u_2(., y)$$

holds true.

13 Dominant and super-reactive strategies

Let us see the first relationship between dominance and reactivity.

Theorem (characterization of dominant strategies). *Let $(u_1, \geq)$ and $(u_2, \geq)$ be two Emil's and Frances' utility functions, respectively, and let B_1 and B_2 be the respective best reply decision rules induced by the two functions u_1 and u_2. Then, an Emil's strategy x_0 is u_1-dominant if and only if it is B_1-super reactive and, analogously, a Frances' strategy y_0 is u_2-dominant if and only if it is B_2-super reactive.*

Proof. Let x_0 be a super-reactive strategy with respect to the decision rule B_1. Then, the strategy x_0 belongs to the reaction set $B_1(y)$, for each y in F. So, for each y in F, we have the equality

$$u_1(x_0, y) = \max u_1(., y),$$

that means

$$u_1(x_0, y) \geq u_1(x, y),$$

for each x in E and for each y in F, that is the definition of dominance. The vice versa can be proved by following the preceding steps in opposite sense. ∎

14 The preorder of dominance

Definition (of dominance). *Let $(u, \geq)$ be a normal-form game on the bistrategy space $E \times F$ of a strategy base (E, F). We say that* ***an* Emil's *strategy*** *x_0* ***dominates (in wide sense) an other*** **Emil's** ***strategy*** *x* ***with respect to the*** **utility** ***function*** *u_1 if the partial function $u_1(x_0, \cdot)$ is greater (in wide sense) of the partial function $u_1(x, \cdot)$. In this case we write $x_0 \geq_{u_1} x$. We say that* ***an* Emil's *strategy*** *x_0* ***dominates strictly an other*** **Emil's** ***strategy*** *x* ***with respect to the*** **utility** ***function*** *u_1 if the partial function $u_1(x_0, \cdot)$ is strictly greater than the partial function $u_1(x, \cdot)$. In this case we write $x_0 >_{u_1} x$. We say that* ***an* Emil's *strategy*** *x_0* ***dominates strongly an other*** **Emil's** ***strategy*** *x* ***with respect to the*** **utility** ***function*** *u_1 if the partial function $u_1(x_0, \cdot)$ is strongly greater than the partial function $u_1(x, \cdot)$. In that case we will write $x_0 \gg_{u_1} x$.*

Memento (usual order on $\mathcal{F}(X, \mathbb{R})$). Let X be a non-empty set, we remember that a real function $f : X \to \mathbb{R}$ is said greater (in wide sense) than an other function $g : X \to \mathbb{R}$, and we write $f \geq g$, if the wide inequality

$$f(x) \geq g(x),$$

holds for each x in X. The above relation is said strict, and we will write $f > g$, if the function f is greater (in wide sense) than g but different. The f is said strongly greater than g, and we write $f \gg g$, if the strict inequality

$$f(x) > g(x),$$

holds true, for each x in X. The majoring relation $\geq$ on the function's space $\mathcal{F}(X, \mathbb{R})$ is a order and it is called *usual order of the space* $\mathcal{F}(X, \mathbb{R})$. We note that the relation $f \geq g$ is equivalent to the inequality

$$\inf(f - g) \geq 0.$$

Remark. We easily prove that the relation of dominance $\geq_{u_1}$ is a preorder on E. Actually, it is the reciprocal image of the usual order of the space of real functionals on F (the space $\mathcal{F}(F, \mathbb{R})$) with respect to the application $E \to \mathcal{F}(F, \mathbb{R})$ defined by $x \mapsto u_1(x, .)$.

Theorem (Characterization of the strict dominance for Weierstrass' functions). *Let $f_1 : E \times F \to \mathbb{R}$ be a Weierstrass' functional (that is, assume that there are topologies σ and τ on the sets E and F respectively such that the two topological spaces $(E)_\sigma$ and $(F)_\tau$ are compact topological spaces and the function f_1 is continuous with respect to the product of those topologies). Then, if the functional f_1 represents the Emil's disutility, the condition $x_0 \gg_{f_1} x$ is equivalent to the inequality*

$$\sup(f_1(x_0, .) - f_1(x, .)) < 0.$$

Proof. Necessity. Let the strong dominance $x_0 \gg_{f_1} x$ hold. Then the difference function $g = f_1(x_0, .) - f_1(x, .)$ is negative and moreover there exists (by the Weierstrass Theorem) a point y_0 in F such that the real $g(y_0)$ is the supremum of g, hence

$$\sup g = g(y_0) < 0.$$

Sufficiency (the Weierstrass' hypothesis is not necessary). If the supremum of g is negative, every value of g must be negative. ■

15 Dominance and reactivity

The following theorem explains the relationship between dominance and reactivity comparison.

Theorem (on the preorder of reactivity). *Let $(u_1, \geq)$ and $(u_2, \geq)$ be, respectively, two Emil's and Frances' utility functions, and let B_1 and B_2 be the best reply decision rules induced by the two functions u_1 and u_2 respectively. Then, the reactivity preorder $\geq_{B_i}$ is a refinement of the preorder of dominance $\geq_{u_i}$.*

Proof. We shall show before that the preorder of reactivity refines the preorder of dominance. Let $x_0 \geq_{u_1} x$, then $u_1(x_0, .) \geq u_1(x, .)$, from this functional inequality we deduce that, if $y \in F$ and $x \in B_1(y)$ we have $x_0 \in B_1(y)$. In fact, $x \in B_1(y)$ means that

$$u_1(x, y) = \max u_1(., y)$$

but, because $u_1(x_0, y) \geq u_1(x, y)$, we have also

$$u_1(x_0, y) = \max u_1(., y),$$

i.e., $x_0 \in B_1(y)$. ■

The preorder of reactivity, in general, is a proper refinement of the preorder of dominance, as the following example shows.

Example. Let (B_1, B_2) be the Cournot decision form game with bistrategy space $[0, 1]^2$ and net cost functions f_1 and f_2 defined by

$$f_1(x, y) = x (x + y - 1) ,$$

and, symmetrically,

$$f_2(x, y) = y (x + y - 1) .$$

We easily see that every strategy in $[0, 1/2]$ is strictly more reactive than any strategy $x > 1/2$, in fact the reactivity of any strategy $x > 1/2$ is the empty set (it is a non-reactive strategy). In particular, we have $0 >_{B_1} 3/4$. On the other hand, the function $f_1(0, .)$ is the zero real functional on $[0, 1]$; on the contrary the partial function $f_1(3/4, .)$ is defined by

$$f_1(3/4, .)(y) = (3/4)(y - 1/4),$$

for each y in $[0, 1]$; since this last function has positive and negative values, it is incomparable with the zero function, with respect to usual order of the space of functions $\mathcal{F}(F, \mathbb{R})$. Consequently, the preorder $\geq_{B_1}$ is a *proper refinement* of the preorder $\geq_{f_1}$.

16 Non-reactivity and strong dominance

Another concept used for normal-form games is that of strongly dominated strategy (it is known in the literature also as strictly dominated strategy, but we use this term for a less demanding concept).

Definition (of strongly dominated strategy). *Let $(u, \geq)$ be a multi-utility function on the bistrategy space of a two player game. Let (E, F) be the pair of the strategy sets of the two players (a game base). We say that a strategy*

x in E is ***an*** **Emil's** ***strongly dominated strategy*** *if there exists an other strategy x' in E such that the strict inequality*

$$u_1(x, y) < u_1(x', y),$$

holds for each strategy y in F. In other terms, we say that a strategy $x' \in E$ ***strongly dominates*** *a strategy $x \in E$, and we write $x' \gg_{u_1} x$, if the partial function $u_1(x, \cdot)$ is* ***strongly*** *less then the partial function $u_1(x', \cdot)$.*

The following theorem explains the relationships between the non-reactive strategies and the strongly dominated strategy.

Theorem (strongly dominated strategies as never-best response strategies). *Let $(u_1, \geq)$ and $(u_2, \geq)$ be respectively two Emil's and Frances' utility functions and let B_1 and B_2 be the best reply decision rules induced by the two functions u_1 and u_2 respectively. Then, if a strategy is strongly dominated with respect to the utility function u_i it is non-reactive with respect to the decision rule B_i.*

Proof. Let x_0 be an Emil's u_1-strongly dominated strategy, then there is at least a strategy x in E such that the inequality $u_1(x_0, y) < u_1(x, y)$ holds true, for every y in F. Hence the strategy x_0 cannot be a best response to any strategy y in F, since x is a response to y strictly better than x_0, for every y in F; so the reactivity of x_0, that is the set $B_1^-(x_0)$, is empty. ■

To be a strongly dominated strategy is more restrictive than to be a never best response strategy, as the following example shows.

Example (an undominated and never-best response strategy). Let $E = \{1, 2, 3\}$ and $F = \{1, 2\}$ be the strategy sets of a two player normal-form game $(u, \geq)$, and let u_1 be the Emil's utility function defined by

$$\begin{aligned} u_1(1,1) &= u_1(1,2) = 0 \\ u_1(2,1) &= u_1(3,2) = 1, \\ u_1(2,2) &= u_1(3,1) = -1. \end{aligned}$$

We can summarize the function u_1 in a utility matrix m_1, as it follows

$$m_1 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

It is evident in the matrix m_1 that the Emil's strategy 1 (leading to the first payoff-row) is u_1-incomparable with the other strategies 2 and 3, and then it cannot be strongly dominated (neither strictly dominated). On the other hand, 1 is an Emil's never best response strategy (i.e., it is non-reactive with respect to the best reply rule B_1).

References

[1] J. Aubin, *Mathematical methods of Game and Economic Theory*, North-Holland

[2] J. Aubin, *Optima and Equilibria*, Springer Verlag

[3] D. Carfì, *Optimal boundaries for decisions*, Atti della Accademia Peloritana dei Pericolanti, classe di Scienze Fisiche Matematiche e Naturali, Vol. LXXXVI issue 1, 2008, pp. 1-12 http://antonello.unime.it/atti/

[4] D. Carfì, *Decision-form games*, Proceedings of the IX SIMAI Congress, Rome, 22 - 26 September 2008, Communications to SIMAI congress, vol. 3, (2009) pp. 1-12, ISSN 1827-9015

[5] D. Carfì (with Angela Ricciardello), *Non-reactive strategies in decision-form games*, Atti della Accademia Peloritana dei Pericolanti, Classe di Scienze Fisiche Matematiche e Naturali, Vol. LXXXVII, issue 1, 2009, pp. 1-18. http://antonello.unime.it/atti/

[6] D. Carfì, *Payoff space in C1-games*, in print on Applied Sciences (APPS), vol. 11, 2009, pg. 1 - 16 ISSN 1454-5101. http://www.mathem.pub.ro/apps/v11/a9.htm

[7] M. J. Osborne, A. Rubinstein, *A course in Game theory*, Academic press (2001)

[8] G. Owen, *Game Theory*, Academic press (2001)

[9] R. B. Myerson, *Game Theory*, Harvard University press (1991)