Decision-form games

Carfì, David

University of Messina, University of California at Riverside

2009

Communications to SIMAI Congress, ISSN 1827-9015, Vol. 3 (2009) 307 (12pp)
DOI: 10.1685/CSC09307

Decision-form games

David Carfi[1]

[1] *Dipartimento DESMAS, Facoltà di Economia*
Università degli Studi di Messina, Italy
davidcarfi71@yahoo.it

Abstract

In this paper we formalize a new form of two-player game, that we call decision-form. A two-player decision-form game consists in a pair of decision rules, representing the rationality of each player. We develop the basic facts of this type of games, showing that this form of game generalizes the normal-form . Indeed, we show that with a normal form game it is possible, in a natural way, to associate a decision form game. In the paper we give examples of decision-form games not-deriving, a priori, by normal-form games. We observe that it is possible to associate with a normal-form game several decision-form games, each representing a possible decisional behavior of the pair of players. The classic best-response behavior is only one of these possible behaviors.

Keywords: `2-person games, Applications of game theory`

1. Introduction.

The concept of decision rule was used by J. P. Aubin in [1] and in [2], there he notes that the interaction of two players can be represented by a pair of decision rules and he gives the definition of equilibrium in this situation. Anyway, he applies the definition only for the canonical decision rules of normal form games (see, for instance, the two books of Aubin). We continue on the path marked by Aubin, we concentrate our attention on the pairs of decision rules of two players, that we call decision-form games, and we begin to build a theory upon them (see also [3] and [4]). The motivation of our study is not the consideration that the decision-form games are generalization of the normal ones, rather the principal motivations are: i) there are aspects of normal-form games (that we desire to consider decision-aspects), for example Nash equilibria and some their properties, which do not depend upon the entire payoff functions of the two players but only upon some features of these ones; ii) following the preceding point, it becomes of great interest to distinguish the solution-concepts of decision-type and those which are not; iii) there are response multifunctions that, even

Received 15/02/2009, in final form 29/06/2009
Published 31/07/2009

though can be viewed as best-reply correspondences, have no a natural associated payoff function (for example, the linear response functions of a physical system), so that the construction of payoff functions is a forcing of the model.

2. Strategy spaces and strategy base of game

The context. We deal with *two-player games*. We shall consider two non-void sets E and F, viewed as the respective sets of strategies at disposal of two players. The aim is to form ordered pairs of strategies $(x, y) \in E \times F$, called *strategy profiles* or *bistrategies*, via the (individual or collective) selection of their components x and y, done by the two players in the sets E and F, respectively, in order that the strategy x of the first player is a good reaction to the strategic behavior y of the second player and vice versa.

Let us formalize our starting point.

Definition 2.1. (strategy base and bistrategy space). *Let (E, F) be a pair of non-empty sets, we call it* ***strategy base of a two-player game****. The first set E is said* **the first player's** ***strategy set****; the second set F is said* **the second player's** ***strategy set****. Any element x of E is said a* ***first player's strategy*** *and any element y in F is said a* ***second player's strategy****. Every pair of strategies $(x, y) \in E \times F$ is said* ***a bistrategy of the strategy base*** *(E, F) and the cartesian product $E \times F$ is said* ***the bistrategy space of the base*** *(E, F).*

Interpretation and terminology. We call the two players of a game Emil and Frances: Emil, simply, stands for "first player"; Frances stands for "second player". Emil's aim is to choose a strategy x in the set E , Frances' aim is to choose a strategy y in F, in order to form a bistrategy (x, y) such that the strategy x is an Emil's good response to the Frances' strategy y and vice versa.

We can make a first distinction between bases of game.

Definition 2.2. (finite and infinite bases). *A strategy base is said* ***finite*** *if it has finitely many bistrategies,* ***infinite*** *on the contrary.*

Example 2.1. (of infinite strategy bases). Two producers offer the identical good on a same market. They can interact *a la Cournot* or *a la Bertrand*. In the first case they choose the quantities to produce; in the second one, they choose the unitary prices of the good. The strategy spaces E and F of the two players coincide with the interval $[0, +\infty[$, or they are infinite subsets of this semi-line. In both cases, the strategy base is the pair of strategy spaces (E, F), and it is infinite.

DOI: 10.1685/CSC09307

3. Decision rules

A standard way for Emil and Frances to choose their reactions to the strategies of the other player is the adoption of *decision rules*. Let us formalize this basic concept (see also [1] and [2]).

Definition 3.1. (decision rule). *Let (E, F) be a strategy base of a two-player game. An* ***Emil's decision rule on the base*** *(E, F) is a correspondence from F to E, say $e : F \to E$. Symmetrically, a* ***Frances' decision rule on the base*** *(E, F) is a correspondence from E to F, say $f : E \to F$.*

4. Decision-form games

Let us formalize the basic concept of our discourse.

Definition 4.1. (decision-form game). *Let (E, F) be a strategy base of a two-player game. A* ***two-player decision-form game on the base*** *(E, F) is a pair (e, f) of decision rules of the players Emil and Frances, respectively, on the strategy base (E, F).*

Example 4.1. (of a game). Let $E = [-1, 2]$ and $F = [-1, 1]$ be the strategy sets of two players. The multifunctions $e : F \to E$ and $f : E \to F$, defined by $e(y) = -1$ if $y < 0$, E if $y = 0$ and 2 if $y > 0$, $f(x) = -1$ if $x < 1$, F if $x = 1$ and 1 if $x > 1$, for every strategy x in E and y in F, are decision rules, of Emil and Frances respectively, on the base (E, F). The pair (e, f) is a two-player decision-form game on the base (E, F).

Definition 4.2. (finite and infinite games). *A game is said* ***finite*** *if it has a finite number of bistrategies,* ***infinite*** *on the contrary.*

Definition 4.3. (symmetric games). *A decision-form game is said* ***symmetric*** *if the decision rules of the two players coincide (consequently, the two players have the same strategy space).*

Definition 4.4. (of univocal game). *A decision-form game is said, with abuse of language,* ***univocal*** *if its decision rules are everywhere defined and univocal, that is if its decision rules are functions.*

5. Possible reactions

Definition 5.1. (of possible reaction and of capability of reaction). *Let (e, f) be a decision-form game. Let y be a Frances' strategy, the elements of the image of y by the correspondence e (that is, the elements of the set $e(y)$), i.e., the direct corresponding strategies of y by the rule e,* *are called **Emil's possible responses**, or* **Emil's** ***possible reactions, to the Frances' strategy*** *y. Analogously, let x be an Emil's strategy, the elements of the image of x by the decision rule f (that is, the elements of the set $f(x)$), i.e. the direct corresponding strategies of x by the rule f, are said* **Frances'** ***possible responses,*** *or* **Frances'** ***possible reactions, to the Emil's strategy*** *x. The set of Emil's possible reactions (responses) to the Frances' strategy y is said the **Emil's reaction set to the Frances' strategy** y. Finally, we say that **Emil can react to the Frances' strategy** y if the corresponding reaction set $e(y)$ is non-void.*

Interpretation. In the conditions of the above definition, the decision rule e associates, with each strategy $y \in F$ (of Frances), all those strategies x of E among which Emil can choose his response, when Frances is playing y. Analogously, the decision rule f associates, with every strategy $x \in E$, played by Emil, all those strategies y in F among which Frances can choose her own response, to react to the Emil's action x.

Example 5.1. (of reaction). Let (e, f) be the game of the example 4.1. The only possible Emil's response, to a Frances' strategy $y < 0$ is the strategy -1. Emil can choose an arbitrary strategy in E, if Frances plays 0; Emil has only the reaction strategy 2, if Frances plays a strategy $y > 0$. The only possible Frances' response to an Emil's strategy $x < 1$ is the strategy -1; Frances can choose an arbitrary strategy in F if Emil plays 1; Frances has only the reaction strategy 1 if Emil uses a strategy $x > 1$.

Definition 5.2. (of equilibrium). *We call **equilibrium** of a decision form game (e, f) each bistrategy (x, y) of the game such that the strategy x is a possible reaction to the strategy y, with respect to the decision rule e, and y is a possible reaction to x, with respect to f. In other terms, an equilibrium of (e, f) is any bistrategy of the game belonging to the intersection of the graph of f with the inverse (symmetric) graph of e.*

Example 5.2. (Matching pennies). To win a prize, two players 1 and 2 must write a number, chosen among -1 and 1, hiding the choice to the other player. After this, the choices are revealed simultaneously. If the numbers coincide, player 1 wins, if they are different player 2 wins. The preceding scenario can be formalized as a decision-form game $G = (e_1, e_2)$, with both strategy spaces coincident with the finite set $E = \{-1, 1\}$ and decision rules $e_1, e_2 : E \to E$, defined by $e_1(s) = s$ and $e_2(s) = -s$, for every strategy s in E. It is a univocal non-symmetric game.

DOI: 10.1685/CSC09307

6. Disarming strategies

Our definition of game does not exclude the existence of Emil's strategies x such that the Frances' reaction set to x, that is the image $f(x)$, is empty. In other words, it may happen that Frances could not be able to react to a certain Emil's strategy x, as she does not consider any own strategy appropriate to face up to the Emil's action x. It makes harder and harder the comprehension of what we can define as a solvable game or the solution of a game. This consideration prompts us to give the following definition.

Definition 6.1. (of a disarming strategy). *Let (e, f) be a game. The Emil's strategies x to which Frances cannot react, i.e. such that the image $f(x)$ is empty, are called **Emil's disarming strategies (for Frances)**. The Frances' strategies y to which Emil cannot react, namely such that the reaction set $e(y)$ is empty, are called **Frances' disarming strategies (for Emil)**.*

Example 6.1. (of disarming strategies). Let $E = [-1, 2]$ and $F = [-1, 1]$ be two strategy spaces and let $e : F \to E$ and $f : E \to F$ be two decision rules defined by

$$e(y) = \begin{cases} \{-1\} & if \;\; y < 0 \\ E & if \;\; y = 0 \\ \varnothing & if \;\; y > 0 \end{cases}, \quad f(x) = \begin{cases} \{-1\} & if \;\; x < 1 \\ \varnothing & if \;\; x = 1 \\ \{1\} & if \;\; x > 1 \end{cases},$$

for every x in E and y in F. Emil has no reaction strategies if Frances chooses a strategy $y > 0$: then, any positive Frances' strategy is disarming for Emil. Instead, Frances has no reaction strategy if Emil plays 1: the Emil's strategy 1 is disarming for Frances.

Remark 6.1. For the previous example, consider the graphs of the two correspondences e and f in the cartesian products $F \times E$ and $E \times F$, respectively, and the graph of the reciprocal correspondence of e and that of the correspondence f in the same space $E \times F$. It is easily seen (geometrically and algebraically) that the intersection of the graph of the reciprocal of e with the graph of f contains just the point $(-1, 1)$.

Remark 6.2. (about the domain of a decision rule). From previous definitions we can gather that the set of Emil's strategies to which Frances can oppose a reaction is the domain of the correspondence f, dom$f$. Similarly, the set of Frances' strategies to which Emil can oppose a reaction is the domain of the correspondence e, dom$e$. Consequently, the set of Emil's disarming strategies is the complement of dom$f$ with respect to E and the set of Frances' disarming strategies is the complement of dom$e$ with respect to F.

A game with decision rules everywhere defined is said *a game without disarming strategies.*

The instance that a decision rule is univocal at any point can be interpreted in the context of game theory, as in the following definition.

Definition 6.2. (obliged strategies). *Let (e, f) be a decision-form game. If, with respect to the decision rule f, there is only one Frances' reaction y to a certain Emil's strategy x, that is if $f(x)$ is the singleton $\{y\}$, such strategy y is called **Frances's obliged strategy by the Emil's strategy** x. Analogous definition can be given for Emil's strategies.*

7. Subgames

We now introduce another fundamental notion, that of subgame.

Definition 7.1. (of subgame). *Let $G = (e, f)$ be a decision-form game with strategy base (E, F) and let (E', F') be a **subbase** of (E, F), namely a pair of subsets of E and F, respectively. We call **subgame of G with strategy base** (E', F') the pair (e', f') of the restrictions of the decision rules e and f to the pairs of sets (F', E') and (E', F'), respectively. It is important to remember that e' is the correspondence from F' to E' which associates with every strategy y' in F' the part $e(y') \cap E'$. In other words, it sends every strategy y' of F' into the corresponding Emil's reaction strategies to y' which belong to E'. We also call the subgame (e', f') the **restriction of the game G to the strategy pair** (E', F').*

Example 7.1. (of subgame). Let $(\mathbb{R}, \mathbb{R})$ be the strategy base of the game $G = (e, f)$, defined by $e(y) = y^2$ and $f(x) = x^2$, for every couple of real numbers x and y. The subgame $G' = (e', f')$, with base $([-2, 2], [0, 1])$ is defined by $e'(y) = y^2$, if $y \in [0, 1]$ and $f'(x) = x^2$ if $x \in [-1, 1]$ and $\varnothing$ if $x \notin [-1, 1]$, for each x in $[-2, 2]$. Even though in the game G there were no disarming strategies, its restriction to the subbase $([-2, 2], [0, 1])$ detects disarming strategies.

8. Rules induced by utility functions

In this section we introduce a standard method to define a decision rule when a player has a preference (preorder) on the bistrategy space induced by an utility function.

Definition 8.1. (decision rule induced by a utility function). *Let $(u_1, \geq)$ be an Emil's utility function on the bistrategy space $E \times F$, that is a function $u_1 : E \times F \to \mathbb{R}$ endowed with the usual upper order of the real line.*

DOI: 10.1685/CSC09307

*We call **Emil's best reply decision rule induced by the utility function** $(u_1, \geq)$, the rule $B_1 : F \to E$ defined by $B_1(y) = \max_{u_1(.,y)}(E)$, for every Frances' strategy y. In other words, Emil's reaction set to a Frances' strategy $y \in F$, with respect to the rule B_1, is the set of every Emil's strategy maximizing the section $u_1(., y)$. Symmetrically, let $(u_2, \geq)$ be a Frances' utility function on the bistrategy space $E \times F$, that is a real function u_2 defined upon the bistrategy space $E \times F$ together with the canonical upper order of the real line. We call **Frances' best reply decision rule induced by the utility function** $(u_2, \geq)$, the rule $B_2 : E \to F$ defined by $B_2(x) = \max_{u_2(x,.)}(F)$, for each Emil's strategy x. In other words, Frances' reaction set to the Emil's strategy $x \in E$, with respect to the rule B_2, is the set of every Frances' strategy maximizing the section $u_2(x, .)$.*

Memento. We write $\max_{u_1(.,y)}(E)$ to denote the set of maxima of the preordered space $(E, \leq_{u_1(.,y)})$, where by $\leq_{u_1(.,y)}$ we denote the preorder induced by the section $u_1(., y)$ on the set E. Such set of maxima is the set of maximum points (on E) of the function $u_1(., y)$, it is also denoted by $\mathrm{argmax}_E\ u_1(., y)$. There are symmetric notations for Frances.

Example 8.1. (of induced rule). Let $E = [-1, 2]$ and $F = [-1, 1]$ be two strategy spaces and let $f : E \to F$ be the decision rule defined by $f(x) = -1$ if $x < 0$, F if $x = 0$ and 1 if $x > 0$, for every Emil's strategy x in E. The rule f is induced by the utility function $u_2 : E \times F \to \mathbb{R}$ defined by $u_2(x, y) = xy$, for each bistrategy (x, y) of the game. Indeed, fix an Emil's strategy x, the section of partial derivative $\partial_2 u_2(x, .)$ coincide with the derivative $u_2(x, .)'$, therefore the function $u_2(x, .)$ is strictly increasing if $x > 0$, strictly decreasing if $x < 0$ and constant if $x = 0$, in particular: 1) if $x < 0$, the only Frances' strategy maximizing the function $u_2(x, .)$, on the compact interval $[-1, 1]$, is the strategy -1; 2) if $x > 0$, the only Frances' strategy maximizing the function $u_2(x, .)$, on the interval $[-1, 1]$, is the strategy 1; 3) if $x = 0$, each Frances' strategy maximizes the function $u_2(x, .)$, on the interval $[-1, 1]$, (since the value of the section $f_2(0, .)$ is zero in the whole domain).

Remark 8.1. (about the never-best reply strategies). In the conditions of the above definition, an Emil's strategy x is called *never-best reply strategy with respect to the utility function* u_1 if and only if there is no $y \in F$ such that $x \in B_1(y)$. Moreover, a strategy x in E is said *non-reactive with respect to an Emil's decision rule* e if there is no y in F such that x lies in $e(y)$. The u_1 -never-best reply strategies are, so, the non-reactive strategies with respect to the decision rule B_1.

9. Rules induced by preorders

In this section we point out a generalization of the standard method to define decision rules of the previous section.

Note that, if $\geq_1$ is an Emil's preference on the bistrategy space $E \times F$ and if y is a Frances' strategy, the preorder $\geq_1$ induces, through y, a *section preorder* $\geq_1^y$ on E, that defined by $x_0 \geq_1^y x$ iff $(x_0, y) \geq_1 (x, y)$, for each pair (x_0, x) of Emil's strategies.

Definition 9.1. (decision rule induced by a preorder on the bistrategy space). *Let $\geq_1$ be an Emil's preference on the bistrategy space $E \times F$. We call* ***Emil's best reply decision rule induced by the preorder*** *$\geq_1$, the correspondence $B_1 : F \to E$ defined by $B_1(y) = \max_{\geq_1^y}(E)$, for each Frances' strategy y. In other words, the Emil's reaction set to the Frances' strategy $y \in F$ is the set of all those Emil's strategies maximizing the section preorder $\geq_1^y$. Similarly, let $\geq_2$ be a Frances' (utility) preorder on the bistrategy space $E \times F$. We call* ***Frances' best reply*** **decision rule** ***induced by the utility preorder*** *$\geq_2$, the correspondence $B_2 : E \to F$ defined by $B_2(x) = \max_{\geq_2^x}(F)$, for each Emil's strategy x. In other words, the Frances' reaction set to the Emil's strategy $x \in E$ is the set of all those Frances' strategies maximizing the section preorder $\geq_2^x$.*

Memento. We denote by $\max_{\geq_1^y}(E)$ the set of maxima in the preordered space $(E, \geq_1^y)$. Such set of maxima is as well the set of maximum points of the preorder $\geq_1^y$ and it may also be denoted by $\mathrm{argmax}_E \;\; \geq_1^y$. There are similar notations for Frances.

10. A first price auction

In this section we study a first price auction as a decision-form game. **The context.** Two players 1 and 2 take part to an auction to obtain an item in return for a payment. **Rules of the game.** The auction has the following rules: a) each player i makes a public evaluation v_i of the item; b) if the two evaluations are equal and if no one of the two participants changes his own evaluation (or withdraws), the item will be drawed lots and the winner will pay an amount equal to his evaluation ; c) if the evaluations are different, the two players will make simultaneously an offer for the item; d) the bid b_i of the player i cannot exceed the evaluation v_i; e) the item is assigned to the player that offers the greatest bid, or, in case of two same offers, to the player with the biggest evaluation; f) the winner i^* pays his own bid b_{i^*} and receives the item. *Our aim is to describe the previous situation as a decision-form game, in case the auction actually takes place, that is when an evaluation is strictly greater than the other one.* Let us suppose that

DOI: 10.1685/CSC09307

the first player evaluated the item more than the second one. The strategy spaces E and F of the two players are the spaces of the possible offers of the same players. The utility of the player i is zero, if he does not win; it is $v_i - b_i$, if he carries off the item paying b_i. *Strategy spaces and utility functions.* Emil's and Frances' strategy spaces are the compact intervals $[0, v_1]$ and $[0, v_2]$, respectively. The utility functions of the two players are defined by $u_1(x, y) = v_1 - x$ if $x \geq y$ and 0 otherwise, $u_2(x, y) = v_2 - y$ if $x < y$ and 0 otherwise. **Decision rules.** The best reply rules induced by the two utility functions are defined, respectively, by $B_1(y) = y$, for each y in $[0, v_2]$ and $B_2(x) = \varnothing$ if $x < v_2$ and F otherwise, for each x in E. As a matter of fact, if Emil offers a price x strictly smaller than v_2, Frances could carry off the prize, but she should maximize her own utility function on F, fixed the choice x of Emil, that is she has to maximize the section $u_2(x, .)$, which, when the Frances' offer is strictly greater than x (those that would assure her the item) is defined by $u_2(x, .)(y) = v_2 - y$, for every $y \in]x, v_2]$. Unfortunately, the supremum of $u_2(x, .)$ is the difference $x - y$, and such utility value is a shadow maximum (!), it is unreachable on F: therefore Frances has no best reply to the Emil's offer x. If, instead, $x \geq v_2$, the section $u_2(x, .)$ is constantly null, hence it assumes its maximum 0 on the whole F. **Best reply graphs.** Emil's (inverse) best reply graph is the compact segment with end points $(0, 0)$ and (v_2, v_2). Frances' best reply graph is the compact interval $[v_2, v_1] \times F$. **Equilibrium.** The two graphs intersect in the point (v_2, v_2) alone. An equilibrium solution, therefore, is that Emil awards the item and pays Frances' evaluation.

11. ε-best reply induced by a utility function

In this section we shall give a generalization of the concept of best reply.

Definition 11.1. (ε-best reply induced by a utility function). *Let $(u_1, \geq)$ be an Emil's utility function on the bistrategy space $E \times F$, that is a function $u_1 : E \times F \to \mathbb{R}$ endowed with the usual upper order of the real line. For each positive real ε, we call* **Emil's *ε-best reply decision rule induced by the utility function*** *$(u_1, \geq)$, the rule ${}^{\varepsilon}B_1 : F \to E$ defined by ${}^{\varepsilon}B_1(y) = \{x \in E : u_1(x, y) \geq \sup_E u_1(., y) - \varepsilon\}$, for every Frances' strategy y. In other words, Emil's reaction set to a Frances' strategy $y \in F$, with respect to the rule ${}^{\varepsilon}B_1$, is the set of every Emil's strategy whose utility distance from the shadow utility $\sup_E u_1(., y)$ is less than ε. Symmetrically, we can do for Frances.*

Remark 11.1. The ε-best reply reaction set ${}^{\varepsilon}B_i(s)$ is always non-void by definition of supremum. Moreover, it contains the best reply $B_i(s)$.

Example 11.1. In the case of the above auction, we have

$$^{\varepsilon}B_2(x) = \begin{cases}]x, x+\varepsilon] \cap F & if \;\; x < v_2 \\ F & if \;\; x \geq v_2 \end{cases},$$

for each x in E, and hence Emil has no longer disarming strategies. Note, however, that, also in this case, there is only one equilibrium.

12. Examples of different equilibria in a game

Scope of the section. In this section we associate with a normal-form game some decision-form games different from the canonical one (the pair of the best-reply rules). Each decision-form game which we shall consider represents a pair of player *behavioural ways*. In particular we introduce two types of behaviour: the *devote behaviour* and that *offensive behaviour*.

Definition 12.1. (devote response). *We say that an Emil's action x is a **devote response to the Frances' strategy** y in the game $G = (f, \leq)$, if x minimizes the Frances' partial loss function $f_2(., y)$. We define **Emil's devotion decision rule** $L_1 : F \to E$ by $L_1(y) = \min_{f_2(.,y)} E$, for each y in F. In other terms, for any y, the reply-set $L_1(y)$ is the set of all Emil's strategies minimizing the partial loss function $f_2(., y)$. Analogously, we can define the Frances' reaction-set $L_2(x)$, for every Emil's action x.We call the equilibria of the game (L_1, L_2) **devote equilibria of the loss game** G.*

Interpretation. The decision-form game (L_1, L_2) represents the interaction of the two players when they are devoted each other.

Definition 12.2. (offensive response). *We say that an Emil's action x is an **offensive response to the Frances' strategy** y, in the loss game $G = (f, \leq)$ with biloss function f, if x maximizes the Frances' partial loss function $f_2(., y)$. We define **Emil's offensive decision rule** $O_1 : F \to E$ by $O_1(y) = \max_{f_2(.,y)} E$, for each y in F. In other terms, for any y, the responce-set $O_1(y)$ is the set of all Emil's strategies maximizing the partial loss function $f_2(., y)$. Analogously, we can define the Frances' reaction-set $O_2(x)$, for every Emil's action x.We call the equilibria of the game (O_1, O_2) **offensive equilibria of the loss game** G.*

Interpretation. The decision-form game (O_1, O_2) represents the interaction of the two players when they are offensive each other.

Example 12.1. (offensive correspondences and equilibria). We refer to the above example. We already saw that the players's (worst) offensive correspondences are defined by $O_1(y) = 1$, for every strategy $y \in F$, and $O_2(x) = F$ if $x = 0$, 0 if $x > 0$, respectively. The intersection of the graph of O_2 with the reciprocal graph of O_1 is the unique offensive equilibrium B.

13. Supply and demand

Supply-demand model is an economic model based on price, consumer-utility and quantity of a certain good produced in a market. Roughly speaking, it affirms that in a competitive market there are prices such that the quantity demanded by consumers equals the quantity supplied by producers, resulting in economic equilibria price-quantity. We can consider this model as a decision-form game. Indeed, for every price p fixed by the producer the consumer decides to buy one of the quantity q belonging to the set $d(p)$ of all (indifferent) quantities that the consumer can and will buy at that price. On the other hand, for every quantity q demanded by the consumer, the producer establishes a unitary price $o(q)$ to sell its own good. We can consider the game $G = (d, o)$, the set of all the economic equilibria is the set $\mathrm{Eq}(G)$ of equilibria of the decision-form game G. We notice that the game G cannot (in a natural way) be considered as a game in normal form. It is true that the demand correspondence $d : P \to Q$, of the price space in the quantity space, is often (but not always) obtained by the maximization of a utility function $u : Q \to \mathbb{R}$ of the consumer on a budget-constraint V_p (subset of Q) depending upon the price p, but there is not a natural way to obtain the price $o(q)$ as a maximum point of a utility function. We emphasize that often an interaction between two subjects can appear (in natural way) as a decision-form game (e, f). If $e : F \to E$ is the reaction correspondence of the first player, as in the case of demand correspondence, the functions that have e as best-reply correspondence are not a priori relevant for the interaction since infinitely many completely different normal form games can have the same corresponding decision-form game of best-reaction. Let us see another example. Let a function $s : X \times \mathbb{N} \to X$ be a discrete dynamical system, that is, assume that, for each state x_0 in X and each time n, is $s(s(x_0, n), m) = s(x_0, n+m)$. This system s can be obtained by the decision form game (e, f), with reaction functions $e, f : X \to X$ defined by $e(x) = s(x, 1)$, and $f = e$, as it follows: $s(x_0, n) = e^n(x_0)$, for $s(x_0, 1) = e(x_0)$, $s(x_0, 2) = f(e(x_0)) = e^2(x_0)$ and so on. A state x_0 is an equilibrium of the dynamical system s if and only if it is an equilibrium of the decision-form game (e, f). Also in this case there is no natural way to consider the equilibria as Nash equilibria (although formally it is possible, as we shall see).

We, however, desire to examine more deeply the question of existence of payoff functions inducing a reaction-correspondence. In the following if f_1 is such a function by B_1 we shall denote its best-reply correspondence $y \mapsto \max_{f_1(.,y)} E$.

Theorem. *Let $(Q, .)$ be a pre-Hilbert space, assume the demand correspondence $d : P \to Q$ be a function. Then, the function $f_1 : Q \times P \to \mathbb{R}$ defined by*

$$f_1(q, p) = -(d(p) - q)^2,$$

for each (q, p) in $Q \times P$, is such that $d = B_1$.

Proof. In fact, for each price p, the value of the partial function $f_1(., p)$ at $d(p)$ is $f_1(d(p), p) = 0$, and moreover, for every q in Q, $f_1(q, p) \leq 0$. ∎

Theorem. *Let (E, F) be a pair of nonempty sets, let $e : F \to E$ be a correspondence and let d_E be the discrete metric of the set E. Then the function $f_1 : E \times F \to \mathbb{R}$ defined by $f_1(x, y) = -d_E(x, e(y))$, for each pair (x, y) in $E \times F$, is such that $e = B_1$.*

Proof. Indeed, fixed y in F, for every x in $e(y)$, $f_1(x, y) = 0$ since the distance $d_E(x, S) = \min_{s \in S} d_E(x, S)$ is 0 if and only if the point x belongs to S. Moreover, it is clear that $\max_E f_1(., y) \leq 0$. ∎

The above theorem is a particular case of the following one.

Theorem. *Let (E, F) be a pair of nonempty sets, let $e : F \to E$ be a correspondence and let d be a metric on the set E such that the correspondence e is with d-closed values. Then the function $f_1 : E \times F \to \mathbb{R}$ defined by $f_1(x, y) = -d(x, e(y))$, for each pair (x, y) in $E \times F$, is such that $e = B_1$.*

Proof. Indeed, fixed y in F, a point $x \in E$ belongs to the image $e(y)$ if and only if $f_1(x, y) = 0$, since the distance $d(x, S) = \inf_{s \in S} d(x, S)$ is 0 if and only if the point x belongs to the d-closure of S. Moreover, it is clear that $\max_E f_1(., y) \leq 0$, and consequently $B_1(y) = e(y)$. ∎

REFERENCES

1. J. P. Aubin, *Mathematical Methods of Game and Economic Theory,* North-Holland.
2. J. P. Aubin, *Optima and Equilibria,* Springer Verlag
3. D. Carfì, *Fondamenti di Teoria delle Decisioni, vol. 3, Teoria dei Giochi in forma decisionale*, Edizioni Il Gabbiano 2008
4. D. Carfì, A. Ricciardello, *Non-reactive strategies in decision-form games*, Atti dell'Accademia Peloritana dei Pericolanti, issue 1 Volume LXXXVII (2009)