The Impact of Exogenous Asymmetry on Trade and Agglomeration in Core-Periphery Model

Sidorov, Alexander

Sobolev Institute of Mathematics SB RAS

2011

The Impact of Exogenous Asymmetry on Trade and Agglomeration in Core-Periphery Model

Alexander V. Sidorov*

Sobolev Institute of Mathematics SB RAS, Novosibirsk, Russia

Abstract

The paper studies the Krugman's CP model in the weakly explored case of asymmetric regions in two settings: international trade and agglomeration processes. First setting implies that the industrial labor is immobile, while second one consider mobile industrial labor and long-run equilibria. Analytical study of both settings requires application of advanced mathematical analysis, e.g. implicit function theory. For international trade we find how equilibrium prices, production, consumption, wages and welfare for all population groups respond to shifts in all exogenous parameters: characteristics of utility function, transportation costs and degree of asymmetry in initial labor endowment. As for agglomeration process, it was found that the asymmetry in the population distribution simplifies pattern of agglomeration, making the direction of migration more definite, so the well-known ambiguity of final destination may disappear under sufficiently large extent of asymmetry. From political point of view, it means that under some conditions, openness of international trade may be harmful to immobile population of the smaller country.

Keywords and Phrases: Agglomeration; international trade; migration dynamics

JEL Classification Numbers: C62, D51, F12, R12, R23

*Sobolev Institute of Mathematics, 4 Acad. Koptyug ave., 630090, Novosibirsk, Russia. Phone:+7(383)363-46-06. Fax: +7(383)333-25-98. E-mail: sidorov@math.nsc.ru.

Contents

1 Introduction

In the course of many years a concept of Perfect Competitive Market was a lodestar for liberal economics. Related widely accepted model of perfect competitive market — Arrow–Debreu model — (see Debreu 1959, p. 30) distinguishes goods at various locations as different goods. However, it is insufficient to explain features of international trade and agglomeration. As pointed out by Mundell (1957), if every activity could be carried out on an arbitrarily small scale in every possible place, without any loss of efficiency, there would be no transportation. This idea was formally supported by Starrett's (1978) *impossibility theorem.* The breakthrough in modifying the main model of the market to the needs of spatial economics, was made by Dixit and Stiglitz (1977) who implemented the Chamberlainian idea of inter-firm increasing returns and market power. Their monopolistic-competition model became a cornerstone of spatial economics, and the seminal step in this direction was implemented by Krugman's papers (1980), (1991). These papers initiated the New Economic Geography, developing rapidly and addressing many interesting questions including agglomeration (see Baldwin et al., 2003; Combes et al., 2008).

This paper belongs to this field. Namely, we study Krugman's (1991) Core-Periphery model (hereafter CP), that considers a two-region economy of the Dixit–Stiglitz type and two types of labor: industrial and agricultural (one can interpret these as skilled and unskilled labor, or any other two industry-specific factors). It is assumed that the industrial workers are mobile between the regions while the agricultural are not. Wherever a worker lives, he/she combines residence, consumption and work at the same place. Krugman (1991) has found the full-agglomeration effect. Namely, under sufficiently small transportation cost, whole industry and industrial labor should concentrate in one country. Moreover, when initially the countries were identical, which of them becomes the core and which the periphery, is ambiguous. This means history-dependence of development. Subsequently, many papers showed simulations of such effects and Fujita et al. (1999) provided a synthesis of the economic geography based on the Dixit–Stiglitz–Krugman approach. However, main *analytical* results for CP model were obtained later, being summarized mainly in Baldwin et al. (2003) (which contains also a wide range of applications), also in Fujita and Thisse (2002), and Combes et al. (2008). The existence of long-run equilibria, their number and stability were found, the existence of switching point between dispersed and agglomerated types of equilibria (see next section for details). With few exceptions mentioned later on, all papers consider *symmetric* regions that means total identity of all exogenous parameters. This assumption of symmetry aimed to answer an important question or intellectual challenge. In contrast to traditional economic geography that derived differences in development from differences in exogeneous physical circumstances of countries, the new geography distinguishes between effects of *first nature* (exogenous heterogeneity) and *second nature*, which is the result of human actions (see Cronon, 1991). The achievement of symmetric model was to explain the endogeneous heterogeneity emerging from homogeneous first nature. The next step in theory would be to explain *how initial exogeneous heterogeneity interferes with economic forces generating endogeneous heterogeneity*.

Agglomeration and dispersion forces: known results

Three driving forces of agglomeration or dispersion in CP model are traditionally discussed. The first is the '*market access effect*' that describes the tendency of monopolistic firms to locate their production in the big market but export to small markets. The second is the '*cost of living effect*' that concerns the impact of firms' location on the local cost of living. Goods tend to be cheaper in the region with numerous industrial firms since consumers in this region will import a narrower range of products and thus avoid part of the trade costs. These two centripetal forces are countervailed by the centrifugal '*market crowding effect*', which reflects the fact that imperfectly competitive firms have a tendency to reside in regions with relatively few competitors. The first two effects encourage spatial concentration while the third one discourages it.

What determines the relative strength of these three forces? The strength of the dispersion force diminishes as trade gets freer. For example, if trade is almost completely free, competition from foreign firms is approximately as important as competition from locally based firms. Then shifting firms from south to north will have very little impact on firm's revenues and thus on the wages they can pay to industrial workers and '*cost of living effect*' becomes negligible. Other two effects also become weaker, but generally we shall see that agglomeration prevails. At the other extreme, near-prohibitive levels of trade cost mean that a change in the number of locally based firms has a very large impact on competition for customers and thus a very big effect on wages and dispersion prevails. The strength of agglomeration forces increases continuously as trade gets freer. It means that at some level ('break point') of trade costs the agglomeration forces overpower the dispersion force and self-reinforcing migration ends up shifting all industry to one region.

The existence of the break point underpins what is perhaps the most striking feature of the CP model — a symmetric reduction in trade costs between initially symmetric regions eventually produces asymmetric regions. This mechanism becomes self-reinforcing since as firms move, for example, northwards, the number of industrial jobs in the south shrinks and the number in the north expands, so the production shifting tends to encourage further expenditure shifting. The key point here is that if there were no change in industrial wages, the increase in northern industrial production would have to be more than proportional to the original expenditure shift in order to re-establish zero profits. (This, of course, is just the famous 'home market effect' of Krugman, 1980). Since the shifting in industrial jobs is more than proportional (holding wages constant), we see that production shift tends to encourage further migration to the north.

In the course of many years the conclusions of this theory were supported by the numerical simulations only. For this moment quite complete analytic study was carried out only for symmetric CP models. In this case the general picture describing stability of long-run equilibria is the following "tomahawk diagram", see Figure 1, left-hand side. Here the measure of trade freeness φ is, in some extent, inverse value to transportation costs, the bold lines contain all stable long-run equilibria, dashed "fork" is a set of all unstable equilibria. Along with "break point" φ^B mentioned above, the diagram contains so-called "sustain point" φ^S — the minimum level of freeness making total agglomeration equilibria stable. Robert-Nicoud (2005) provides the first analytic proof of the CP model's main features, namely that the break point comes before the sustain point (in terms of freeness φ) and that it has at most three locally stable equilibria for any given level of openness. Mossay (2006) proved an existence and uniqueness of short-run equilibrium in symmetric CP model for all admissible values of parameters.

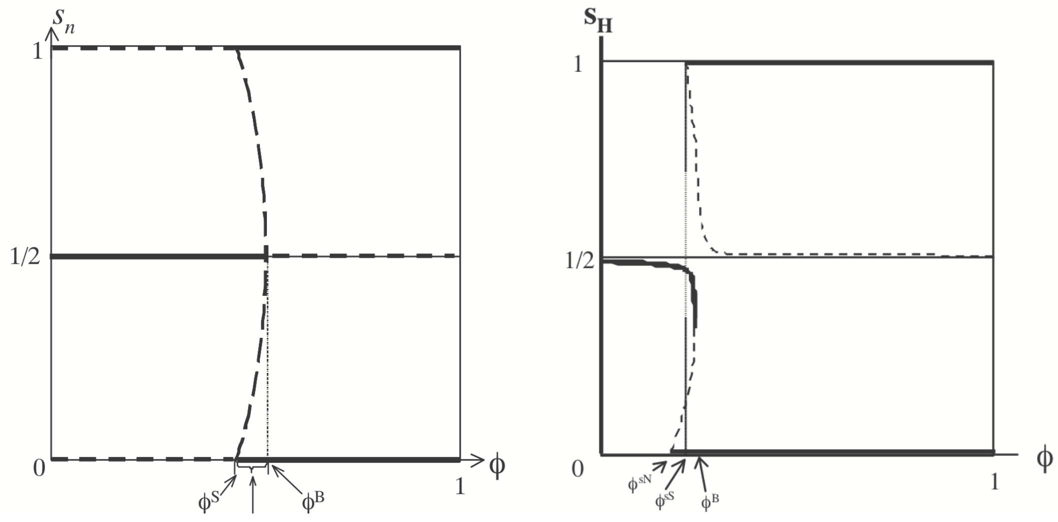


Figure 1: Tomahawk and "broken" tomahawk, taken from Baldwin et al. (2003). Here S_H and S_n denote industrial labor share.

Studying of symmetric CP models allows to obtain very important conclusion: the resulting asymmetry (i.e. agglomeration) is a well-formed result even under condition of initial symmetry. In other words, agglomeration process may be interpreted in frameworks of "second nature", i.e. as result of patterns human activity. However, a theorist can be interested: Are the trade and agglomeration pictures obtained robust against asymmetry or specific to symmetry?

To answer the appeal of the empirics, some numerical simulations show that certain qualitative conclusions can not survive in asymmetric case, e.g. relative positions of break points φ^B and sustain points φ^S (see for example, Appendix C.1 to Chapter 2 of Baldwin et al. (2003), so called "broken tomahawk" on Figure 1, right-hand side). Moreover, Berliant and Kung (2009) show that symmetric case is singular in some sense.

Generally asymmetry remains weakly studied. The reason for lack of comprehensive analytic study of asymmetric CP model is substantial complexity of appearing problems, addressed in Baldwin et al. (2003, p. 53) pessimistically:

> "Unfortunately, the intense intractability of the CP model means that numerical simulation of the model for specific values .... is the only way forward."

Our contribution

The present paper aims to overcome this seeming "intractability" and give a complete study of Krugman's CP model with and without initial heterogeneity, achieving the complete comparative statics of equilibria with respect to main exogenous parameters.

Note that there are two partial cases when result of struggle between agglomeration and dispersion forces is obvious. On perfectly competitive market dispersion forces prevail. Starrett's impossibility theorem is one of impressive evidences of this idea. On the other hand, purely monopolistic competitive market is domain of agglomeration forces. More exactly, excluding of perfectly competitive agricultural sector from this model tends to existing of agglomerated equilibria only, i.e. *resulting force* for all market forces listed above in the case of monopolistic competition is agglomeration force. Thus Krugman's CP model may be considered as *mixed economy* model joining features of both perfectly competitive and monopolistic competitive markets. As a natural measure of relative weight of monopolistic-competitive sector an expenditure share of industral varieties may be taken. Moreover, intensity of agglomeration effects depends on elasticity of substitution, because its inverse is interpreted traditionally as "love for variety". Note that this point of view is not brand new. For example, in Combes et al. (2008), pp.151-152, we read "Product differentiation plays a key role in the model... In other words, when varieties are homogeneous, dispersion is the only stable equilibrium. Conversely, when varieties are more differentiated, the likelihood of agglomeration is higher because the competition effect is weakened" and "Regarding the share of the manufactured good, it is readily verified that the values of τ^S and τ^B increase with μ, and so does the probability that agglomeration occurs. This is because a larger share of the manufacturing sector, on which the snowball effect is built, makes the agglomeration force stronger." Nevertheless, this idea was not supported by intensive *analytical studies* and all conclusions concern symmetric case only. In our studies, prevailing or countervailing of agglomeration and dispersion forces is considered in terms of relative weights of perfectly competitive and monopolistic competitive sectors, which is measured mainly by expenditure shares.

We start in section 3 with analyzing the short-run equilibria. These describe international trade without any migration. Our theorem of equlibrium existence generalizes that of Mossay (2006) by allowing for heterogeneity and simplifies the proof. Our technical achievement enabling this and other results is essential simplification of equilibrium conditions into one equation with relative wage as a variable. Based on this technique, we analyze comparative statics of each variable of interest with respect to two parameters: share of agricultural labor θ and

share of industrial labor λ in home country. First we study relative industrial wage which is the fraction of industrial wage in home country to foreign wage. Propositions 1 and 3 show that:
(1) relative wage *increases* monotonically with respect to home share of agricultural population;
(2) under sufficiently small transportation cost (very free trade) relative wage increases monotonically with respect to share of industrial labor;
(3) under sufficiently big transportation cost relative wage *decreases* with respect to industrial labor share.

Further we study industrial price indices in both countries and relative price index, which is home industrial price index divided by foreign one. Proposition 4 says that home industrial price index grows in home share θ of agricultural population, whereas foreign price index decreases, so relative price index grows. Influence of industrial labor share growth is opposite — relative price index decreases with respect to industral labor share.

More important for further study of agglomeration is the study of relative welfare, i.e. utility value obtained by representative consumer in equilibrium. Proposition 5 says that relative welfare always increases in the agricultural share. Proposition 6 say that behavior of relative welfare in industrial labor share is more complicated and generally non-monotone. Under given parameters, including transportation cost and agricultural labor share, there can be zero, one, two or three interior values of industrial share yielding welfare equalization between the two countries (such point is a candidate for interior equilibrium when migration is allowed). No more than one of them is a candidate for a stable equilibrium. Under sufficiently small transportation costs for any fixed agricultural population our industral workers are always relatively better off when having more industrial compatriots. But under sufficiently big transportation costs for any fixed agricultural population there exist an optimal (for them) share.

Sections 4.1 and 4.2 study long-run equilibria describing migration: agglomerated and interior respectively. Obtained analytical results (as well as results of simulations) show that the general picture of agglomeration processes differs depending on degree of region's asymmetry. Under small asymmetry the general picture of agglomeration patterns with respect to trade freeness is qualitatively similar to well-known one under symmetry "tomahawk" picture, it becomes just asymmetric. We provide the formal proof for this robustness of symmetric theory in the case of moderate asymmetry: any country can become the "core". In contrast, under big asymmetry the pattern is qualitatively different: only agriculturally big country can become the core in agglomeration process, but the smaller one is doomed to be a periphery. We find the threshold or lower bound on asymmetry (in terns of other parameters) when only one core is predetermined by "first nature." Below this bound ambiguity remains, but the higher is asymmetry, the more "probable" is agglomeration in the direction of the bigger country. Something like this was previously shown in simulations, see Baldwin et. al (2003). Now we combine known symmetric theory and known asymmetric simulations into general theoretical picture.

As to interior long-run equilibria, in section 4.2 we obtain comprehensive analytical results on their existence, number and stability under given parameters, generalizing results from Robert-Nicoud (2005) and giving more direct and intuitive proof.

To further motivate our interest in asymmetry of immobile (or, "agricultural") populations, note that it may be considered (*ceteris paribus*) as equivalent to the *market potential* differences. There is a series of empirical evidences showing that this asymmetry causes unavoidable differences across regions in wages (nominal and real), cost-of-living's levels (or price indices) etc, see, for example, Cecchetti et al (2002), Hanson (2006), Roos (2006), Beenstock and Felsenstein (2008), Klaesson and Larsson (2009).

Our study can be extended to other types of asymmetry, including consumer's preferences dissimilarity studied in Berliant and Kung (2009). Concerning the robustness of general pitchfork bifurcation patterns to asymmetry, they expressed the following opinion: "... generically in all parameter paths this class of bifurcations does not appear. In other words, conclusions drawn from the use of this bifurcation to generate a core-periphery pattern are not robust. Generically, this class of bifurcations is a myth, an urban legend." We agree that bifurcation itself goes away, but more important is that qualitative features and patterns of agglomeration still remain.

2 Core-Periphery Model

This paper studies the classical Krugman's CP model, only with asymmetry of the two trading regions or countries: "Home" H and "Foreign" F. The manufacturing sector in both countries is a standard Dixit-Stiglitz monopolistic competition industry with specialized labor that produces varieties or brands of "industrial good". Each manufacturing firm employs the labor of industrial workers and produces a single variety subject to increasing returns. Namely, production of variety i requires a variable input involving m units of industrial-worker labor per unit of output produced and a fixed input of f units of this labor. In symbols, the cost function is $C(i) = (f + m \cdot x(i)) \cdot w$, where x is a firm's output and w is the industrial workers' wage. Note that wage w may differ across regions, while fixed f and marginal variable m costs are the same for both H and F. In contrast, the agricultural sector produces a homogeneous good under perfect competition and constant returns using only the labor of agricultural workers. We normalize the model assuming that agricultural production takes $m_a = 1$ unit of agricultural labor to make one unit of the homogeneous good and agricultural fixed costs $f_a = 0$.

The total amount of industrial labor denoted by $L > 0$ and $L_a > 0$ denotes total amount of agricultural labor. The *share* of industral labor resided in home region H is a number $\lambda \in [0,1]$, analogously, $\theta \in (0,1)$ the relative share of agricultural population in region H. It implies that supplies of industral and agricultural labor in region H are λL and θL_a respectively. Analogously, labor supplies in region F are $(1-\lambda)L$ and $(1-\theta)L_a$. In the short-run equilibrium concept is supposed that the values of shares λ and θ are fixed. Without loss of generality we shall assume that $\theta \geqslant 1/2$, i.e. agricultural population of home region is greater or equal to foreign agricultural population. Perfectly competitive agricultural sector technology is uniform for both regions, the only difference concerns the numbers of laborers. Transportation of agricultural good assumed costless, thus prices for both countries are equal, which implies equalization of wage rates w_a across regions. Agricultural marginal cost $m_a = 1$, consequently $w_a = 1$.

The goods of both sectors are traded between countries, and trade in industrial sector trade incur iceberg-type trade costs, whereas trade in agricultural sector goods is frictionless. Specifically, it is costless to ship industrial goods to local consumers but to sell one unit in another region, an industrial firm must produce and ship $\tau \geqslant 1$ units. The idea is that $\tau - 1$ units of the good "melt" in transit like an iceberg melts driving across an ocean. As usual, τ captures all the costs of selling to distant markets, including transport costs and tariffs, $\tau - 1$ being the tariff-equivalent of these costs.

The typical consumer in each region has a two-tier utility function. The upper tier determines the consumer's allocation of expenditure between the homogeneous agricultural good, and the differentiated sector. The second tier dictates the consumer's preferences over the various differentiated industrial varieties and the choice within this sector. The specific functional form of the upper tier is Cobb-Douglas that makes the sectoral expenditure shares constant (under CES at the lower tier). In symbols, preferences of a typical consumers are described by utility function:

$$U = M^{\mu} \cdot Q^{1-\mu}, \; 0 < \mu < 1 \tag{1}$$

where Q is the consumption of the homogeneous good and M is the composite consumption utility of all differentiated varieties of industrial goods. Traditionally the functional form of the lower tier is represented by CES-function (constant elasticity of substitution) function $M = \left(\sum\limits_{i=1}^{N} q_i^{\rho}\right)^{\frac{1}{\rho}}$ for discrete varieties or $M = \left(\int\limits_{0}^{N} q(i)^{\rho} \mathrm{d}i\right)^{\frac{1}{\rho}}$ in continuous case[1]. Here $0 < \rho < 1$ is a parameter related to the elasticity of substitution $1 < \sigma < \infty$ as follows $\rho = \frac{\sigma-1}{\sigma}$. Number N characterizes assortment of varieties or number of firms producing these varieties.

Each consumer (being of industrial or agricultural type) owns a unit of labor supplied inelastically to the market in exchange of both types of goods produced in both countries, so the industrial consumer's problem is

$$U(M^m, Q^m) \to \max \text{ s.t. } \int_0^{N_H} p_{HH}(i) q_{HH}^m(i) \mathrm{d}i + \int_0^{N_F} p_{FH}(j) q_{FH}^m(j) \mathrm{d}j + 1 \cdot Q^m = w_H$$

and the agricultural worker finds

$$U(M^a, Q^a) \to \max \text{ s.t. } \int_0^{N_H} p_{HH}(i) q_{HH}^a(i) \mathrm{d}i + \int_0^{N_F} p_{FH}(i) q_{FH}^a(i) \mathrm{d}i + 1 \cdot Q^a = 1,$$

where $q_{HH}^m(i)$ denotes the industrial worker's consumption of i-th variety produced and consumed in home region, whereas $q_{FH}^m(j)$ denotes her consumption of j-th variety imported from foreign country. Values $p_{HH}(i)$ and $p_{FH}(j)$ are the corresponding prices. Recall that price of homogeneous agricultural good is normalized to 1. Values N_H and N_F denote the numbers of firms in regions H and F respectively. Similar notations are used for consumer's problem of agricultural workers, and for consumers in another country. Now the home aggregate demand for agriculture good is $Q = \lambda L \cdot Q^m + \theta L_a \cdot Q^a$, aggregate demands for each variety $q_{HH}(i) = \lambda L q_{HH}^m(i) + \theta L_a q_{HH}^a(i)$, $q_{HF}(i) = (1-\lambda) L q_{HF}^m(i) + (1-\theta) L_a q_{HF}^a(i)$, etc.

It is well known under CES-function the analytical form of aggregate demand in country H for home produced i-th variety is equal to

$$q_{HH}(i) = \frac{(w_H \cdot \lambda L + \theta L_a) \cdot \mu \cdot p_{HH}^{-\sigma}(i)}{\int_0^{N_H} p_{HH}^{1-\sigma}(j)\,\mathrm{d}j + \int_0^{N_F} p_{FH}^{1-\sigma}(j)\,\mathrm{d}j}. \tag{2}$$

In turn, aggregate demand in foreign region for i-th variety produced in H is equal to

$$q_{HF}(i) = \frac{(w_F \cdot (1-\lambda) L + (1-\theta) L_a) \cdot \mu \cdot p_{HF}^{-\sigma}(i)}{\int_0^{N_H} p_{HF}^{1-\sigma}(j)\,\mathrm{d}j + \int_0^{N_F} p_{FF}^{1-\sigma}(j)\,\mathrm{d}j}. \tag{3}$$

Home demand for foreign produced good $q_{FH}(j)$ and foreign demand for foreign produced good $q_{FF}(j)$, $j \in [0, N_F]$ may be obtained analogously.

Each home producer maximizes her profit

$$\pi_H(i) = q_{HH}(i) \cdot (p_{HH}(i) - w_H m) + q_{HF}(i) \cdot (p_{HF}(i) - w_H \cdot \tau \cdot m) - w_H \cdot f,$$

where $q_{HH}(i)$, $q_{HF}(i)$ are defined in (2) and (3), correctly anticipating the demand for his/her variety and perceiving the competitors' variables fixed, because everyone is "small enough". Note that production costs m are also uniform for both regions. Maximizing this profit with respect to $p_{HH}(i)$ and $p_{HF}(i)$ we obtain the following values of prices

$$p_{HH}(i) \equiv p_{HH} = \frac{w_H \cdot m \cdot \sigma}{\sigma - 1}, \; p_{HF}(i) \equiv p_{HF} = \frac{w_H \cdot \tau \cdot m \cdot \sigma}{\sigma - 1}, \tag{4}$$

which are uniform for all product varieties $i \in [0, N]$. Analogously,

$$p_{FH}(j) \equiv p_{FH} = \frac{w_F \cdot m \cdot \tau \cdot \sigma}{\sigma - 1}, \; p_{FF}(j) \equiv p_{FF} = \frac{w_F \cdot m \cdot \sigma}{\sigma - 1} \tag{5}$$

[1]The limitations of Cobb-Douglas+CES modeling is that it is too specific to obtain more interesting comparative statics: it shows too much stability of prices against shocks. Besides, use of Cobb–Douglas function makes CP model *singular*. Still, in the rest part of this paper two-tier utility function with underlying CES tier will be used.

The concept of *short-run equilibrium* implies that share λ is fixed (as well as θ), whereas the free-entry condition states that positiveness of profit in industry causes increasing of number of firms until profit becomes zero. For example, for home producers this Zero-Profit Condition $\pi_H(i) = 0$, taking into account formulas (2)-(5), is of the following form

$$\frac{(w_H\lambda L+\theta L_a)\mu p_{HH}^{-\sigma}\cdot(p_{HH}-w_H\cdot m)}{N_H\cdot p_{HH}^{1-\sigma}+N_F\cdot p_{FH}^{1-\sigma}}+\frac{(w_F(1-\lambda)L+(1-\theta)L_a)\mu p_{HF}^{-\sigma}\cdot(p_{HF}-w_H\cdot\tau\cdot m)}{N_H\cdot p_{HF}^{1-\sigma}+N_F\cdot p_{FF}^{1-\sigma}}=w_Hf.$$

Let's substitute for this $p_{HH}-w_Hm=\dfrac{w_H\cdot m\cdot\sigma}{\sigma-1}-w_H\cdot m=\dfrac{w_H\cdot m}{\sigma-1}$ and $p_{HF}-w_H\cdot\tau\cdot m=\dfrac{w_H\cdot\tau\cdot m}{\sigma-1}$ into previous equation. We obtain

$$\frac{(w_H\cdot\lambda L+\theta L_a)\cdot\mu\cdot p_{HH}^{-\sigma}}{N_H\cdot p_{HH}^{1-\sigma}+N_F\cdot p_{FH}^{1-\sigma}}+\frac{\tau\cdot(w_F\cdot(1-\lambda)L+(1-\theta)L_a)\cdot\mu\cdot p_{HF}^{-\sigma}}{N_H\cdot p_{HF}^{1-\sigma}+N_F\cdot p_{FF}^{1-\sigma}}=\frac{(\sigma-1)f}{m}.$$

The left-hand expression is equal to total output of home-produced variety covering the total demand. This production requires the following amount of industrial labor

$$f+m\frac{(\sigma-1)f}{m}=f\cdot\sigma.$$

Consequently the total industrial labor demand is equal to $N_H\cdot f\cdot\sigma$ and from labor market equilibrium condition we obtain that the mass of home firms

$$N_H=\frac{\lambda L}{f\cdot\sigma},$$

analogously

$$N_F=\frac{(1-\lambda)L}{f\cdot\sigma}.$$

Substituting expressions for prices (4), (5) and just obtained N_H, N_F into Zero-Profit equation we obtain, after simplifying, the following equation:

$$\mu\left(\frac{\lambda w_H+\theta\cdot\frac{L_a}{L}}{\lambda w_H^{1-\sigma}+(1-\lambda)\varphi\cdot {w_F}^{1-\sigma}}+\frac{\varphi\left((1-\lambda)w_F+(1-\theta)\frac{L_a}{L}\right)}{\lambda\varphi {w_H}^{1-\sigma}+(1-\lambda)w_F^{1-\sigma}}\right)=w_H^{\sigma}, \tag{6}$$

where $\varphi=\tau^{1-\sigma}\in(0,1)$ may be interpreted as a measure of trade "freeness" or "openness". Taking into account normalization of agricultural parameters (i.e. marginal costs wage rate are supposed to be equal to 1), a commodity balance of agricultural product may be written as

$$(1-\mu)\left(w_H\cdot\lambda L+w_F\cdot(1-\theta)L+1\cdot L_a\right)=L_a$$

or

$$\lambda w_H+(1-\lambda)w_F=\frac{\mu}{1-\mu}\cdot\frac{L_a}{L} \tag{7}$$

where L_a, L are the total amounts of agricultural and industrial labor in the world, w_H and w_F are industrial wages, home and foreign respectively, μ characterizes expenditure share of industrial production and λ is a share of industrial labor in home region.

It will be shown further that the equation system (6), (7) defines short run equilibrium well. This system is *not* analytically solvable, which leads to the pessimistic notion about the intractability of the CP model mentioned in Introduction. One of the aims of this article is to show that this insolubility is not a hindrance for analytical study of CP model.

In contrast, the concept of *long-run equilibrium* implies that there can be a location choice. Agricultural laborers still assumed to be immobile, but share of industrial workers λ can change due to migration between the regions. Let's fix an arbitrary $\lambda \in (0,1)$ and calculate all *short-run* equilibrium values of prices $p_{HH}, p_{HF}, p_{FH}, p_{FF}$, demands $q^m_{HH}, q^m_{HF} q^m_{FH} q^m_{FF}$, and numbers of firms N_H, N_F (see (4), (5)). To obtain the demand of *industrial* workers only, it is sufficient to put $L_a = 0$ in (2) and (3). Then substitute the obtained equilibrium values of industrial labor demand (for both countries) into utility function (1), as result we get the *industrial welfares* $V_H(\lambda)$ and $V_F(\lambda)$ for Home and Foreign countries at the value of share $\lambda \in (0,1)$ (for rather explicit formula of welfare see subsection 3.4). An inequality $V_H(\lambda) > V_F(\lambda)$ is considered *in long run* as incentive to migration from Foreign to Home country and vice versa. One of the popular migration dynamics is so-called *ad hoc* migration equation:

$$\dot{\lambda} = \lambda\left(V_H(\lambda) - (\lambda V_H(\lambda) + (1-\lambda)V_F(\lambda))\right) = \lambda\,(1-\lambda)\,(V_H(\lambda) - V_F(\lambda))$$

In the *long-run equilibrium* the market and agglomeration determine the equilibrium value λ^0 of share of industrial labor in our country, such that no migration occurs: $\dot{\lambda} = 0$. The equilibria can be "agglomerated" that means either $\lambda^0 = 0$ or $\lambda^0 = 1$, i.e. one country becomes pure agricultural one. Alternatively, a "interior equilibrium" implies both countries producing manufacturing goods and the equal real wages for both regions $V_H(\lambda^0) = V_F(\lambda^0)$, $0 < \lambda^0 < 1$.

We are going to analyze the asymmetric CP model in two stages. First we derive solutions and their comparative statics for short-run equilibria, interpreted as equilibrium in international trade, and having independent value itself. Then, using these results we completely describe the long-run equilibria, i.e., the resulting agglomeration/dispersion outcomes.

3 Short-run Equilibria

3.1 Equilibrium equations and existence

We get the following two equations (see equations (6), (7) above) connecting nominal wages with exogenous parameters

$$\mu\left(\frac{\lambda w_H + \theta\cdot\frac{L_a}{L}}{\lambda w_H^{1-\sigma} + (1-\lambda)\varphi\cdot {w_F}^{1-\sigma}} + \frac{\varphi\left((1-\lambda)w_F + (1-\theta)\frac{L_a}{L}\right)}{\lambda\varphi {w_H}^{1-\sigma} + (1-\lambda)w_F^{1-\sigma}}\right) = w_H^{\sigma},$$

$$\frac{L_a}{L} = \frac{1-\mu}{\mu}\cdot(\lambda w_H + (1-\lambda)w_F),$$

where $\varphi = \tau^{1-\sigma}$ is "trade freeness" and w_H and w_F are wages of industrial labor in Home and Foreign countries respectively.

Substituting second equation into the first one we derive the following simpler one with respect to *relative wage* $\frac{w_H}{w_F}$:

$$\begin{aligned}(1-\lambda)&\left[1-((1-\theta)+\mu\theta)(1-\varphi^2)-\varphi\cdot\left(\tfrac{w_H}{w_F}\right)^{\sigma}\right]-\\ -\lambda&\left[\left(1-((1-\mu)\theta+\mu)\left(1-\varphi^2\right)\right)\tfrac{w_H}{w_F}-\varphi\cdot\left(\tfrac{w_H}{w_F}\right)^{1-\sigma}\right]=0\end{aligned} \tag{8}$$

Solution of this equation allows to obtain *nominal* wages from the following equation, which is equivalent to (7)

$$w_H = \frac{\mu}{1-\mu}\cdot\frac{L_a\cdot\frac{w_H}{w_F}}{L\left(\lambda\frac{w_H}{w_F} + (1-\lambda)\right)}. \tag{9}$$

Based on the simplification obtained, the following result generalizes existence and uniqueness of short-run equilibrium (see Mossay, 2006) to the asymmetric case.

Proposition 1. *i) There exists a unique positive value of Relative Nominal Wage* $\frac{w_H}{w_F}$ *defined by equation* (8).

ii) This value satisfies the following inequalities $\varphi^{\frac{1}{\sigma}} < \frac{w_H}{w_F} < \varphi^{-\frac{1}{\sigma}}$. *In the case of "costless trade",* $\varphi = 1$, *an identity* $w_H \equiv w_F$ *holds ("integrated equilibrium").*

For analytical proof see Lemma 1 in Appendix A.

From equations (8) one can see that *relative wage* depends on shares θ, λ only but not on absolute values of industrial and agricultural labors (L and L_a). Similarly, based on (9), the *wage* itself depends on fraction of industrial labor to agricultural one $\frac{L_a}{L}$ but not on the absolute size of global labor. Note that this is a natural consequence of the linear homogeneity of preferences. Costs do not affect wages.

3.2 Changes in wages

From equation (8), the relative wage is an implicit function of labor shares θ and λ, as well as trade freeness φ and utility function parameters μ and σ. We start deriving the comparative statics of equilibria with respect to labor shares.

Denote $x = \frac{w_H}{w_F}$ for short, then equation (8) takes the following form

$$(1-\lambda)(1-((1-\theta)+\mu\theta)(1-\varphi^2)-\varphi\cdot x^{\sigma})-\lambda(1-((1-\mu)\theta+\mu)(1-\varphi^2))x-\varphi x^{1-\sigma})=0.$$

An implicit function $x(\lambda, \theta, \mu, \varphi, \sigma)$ is solution to this equation. Immediately one can derive two special cases.

Under $\varphi = 0$ transportation costs are infinitely large, trade is impossible and this *autarky* yields $\frac{w_H}{w_F} = \frac{(1-\lambda)\theta}{(1-\theta)\lambda}$ which increases with respect to θ, decreases with respect to λ and remains unaffected by other parameters. Under $\varphi = 1$ trade is costless and so-called *integrated equilibrium* yields wage equalization $\frac{w_H}{w_F} \equiv 1$ independently of any parameters. The equalization was known under symmetry of labor, but now we see that it is a general fact.

Now consider the most interesting case $0 < \varphi < 1$ of non-trivial trade cost. Applying the implicit function theorem to previous equation we obtain the following

Proposition 2. ***(Comparative statics with respect to agricultural labor share)***

i) Nominal wage in Home country w_H *increases while foreign wage* w_F *decreases with respect to Home agricultural population share* θ*, thereby relative industrial wage* $\frac{w_H}{w_F}$ *also increases.*

ii) Home nominal wage is greater than foreign wage ($w_H > w_F$) *iff the share of home agricultural labor is sufficiently large, namely* $\theta > \lambda + \frac{(1-2\lambda)\varphi}{(1+\varphi)(1-\mu)}$. *The opposite strict inequality implies reverse wage relation.*

iii) Under sufficiently big asymmetry of industrial labor, namely $\lambda + \frac{(1-2\lambda)\varphi}{(1+\varphi)(1-\mu)} > 1$ *or* $\lambda + \frac{(1-2\lambda)\varphi}{(1+\varphi)(1-\mu)} < 0$ *there is no wage equalization for any shares* $\theta \in (0,1)$.

For analytical proofs see Lemmas 1, 2 and 3 in Appendix A.

Figure 2 illustrates all these facts by simulations (using Mathematica 6.3). Taking three values of industrial labor share: (small, medium and big) we observe monotonic increase of wages predicted by claim (i) of Proposition and three qualitatively different outcomes predicted by claims (ii), (iii). Namely, too small industrial share $\lambda = 0.05$ yield small industrial wage $w_H < w_F$ everywhere. Similarly, big share $\lambda = 0.85$ yield big industrial wage $w_H > w_F$, for all values of θ . Only under medium industrial share wages w_H, w_F may be equalized for some critical value of

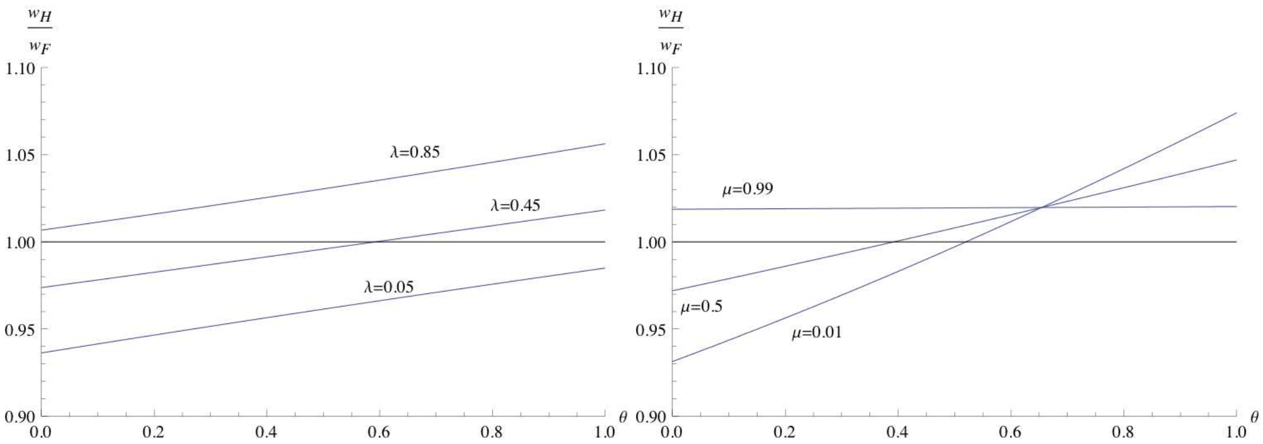


Figure 2: Left-hand plot: Relative wages with respect to θ for $\mu = 0.7$, $\rho = 0.75$, $\varphi = 0.75$ and three values of λ. Right-hand plot: Relative wages with respect to θ for $\lambda = 0.65$, $\rho = 0.75$, $\varphi = 0.75$ and three values of μ.

$\theta^0 \approx 0.6$. Also we plot relative wage three values of expenditure share – $\mu = 0.01$ (close to perfect competition), intermediate $\mu = 0.5$ and $\mu = 0.99$ (close to monopolistic competition).

To interpret the monotonic increase, take into account that agricultural share θ indirectly reflects the size of the market for industrial goods. Bigger market requires more industrial labor whose supply is fixed in the short-run. Note also there is no difference in tendencies for perfectly competitive and monopolistic competitive sectors of economy. It should be mentioned only that for "almost monopolistic competitive" economies ($\mu \approx 1$) relative wage is (almost) neutral to agricultural labor allocation θ because of its negligibility. On the other hand, in the case when economy close to perfect competition ($\mu \approx 0$) its monopolistic competitive sector is more sensitive to home market size. As to realism of monotonicity conclusion, there is empirical evidence that market size really positively affects wages and that the industrial structure matters, see, for example, discussion in Klaesson and Larsson (2009).

Considering dependence of wages on λ we find that in this case answer is ambiguous and heavily depends on *agricultural heterogeneity index* defined as follows $\alpha = 2\theta - 1 = \theta - (1-\theta)$ (we assume that $\theta \geqslant 1/2$). It measures degree of *asymmetry* in the agricultural labor distribution, $\alpha = 0$ for symmetric model and $\alpha = 1$ in the case of total asymmetry for $\theta = 1$. Define the following value

$$\varphi^w(\mu,\alpha) = \sqrt{\frac{1-\alpha^2}{\left(\frac{1+\mu}{1-\mu}\right)^2 - \alpha^2}} \in (0,1)$$

which increases with respect to asymmetry α and decreases w.r.t. expenditure share μ.

Proposition 3. ***(Comparative statics with respect to industrial labor share)***
When "freeness" exceeds the threshold: $\varphi > \varphi^w(\mu,\alpha)$, then relative industrial wage $\frac{w_H}{w_F}$ increases with respect to industrial labor share λ but decreases under smaller "freeness" $\varphi < \varphi^w(\mu,\alpha)$. In the case $\varphi = \varphi^w(\mu,\alpha)$ relative wage $\frac{w_H}{w_F}$ does not depend on λ.

For analytical proof see Lemma 4 in Appendix A.

We can also reformulate Proposition 3 in equivalent form as follows

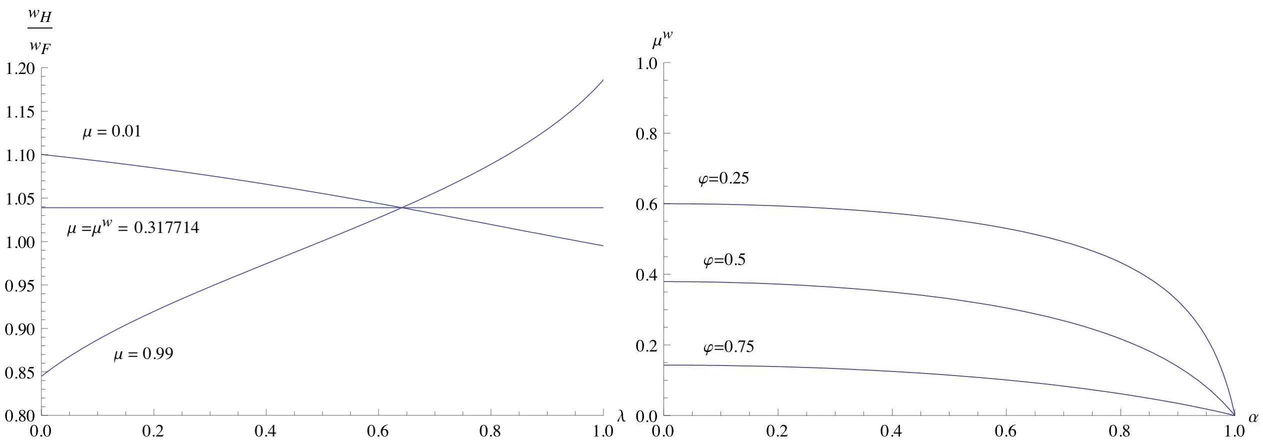


Figure 3: Left-hand plot: Relative wages with respect to λ for $\theta = 0.65$, $\rho = 0.75$, $\varphi = 0.5$ and three values of μ. Right-hand plot: Threshold values of expenditure share μ^w with respect to asymmetry measure $\alpha = 2\theta - 1$ for three values of φ.

Corollary 1. *Relative wage $\frac{w_H}{w_F}$ increases with respect to industrial labor share λ in the case $\mu > \mu^w(\varphi, \alpha)$, decreases when $\mu < \mu^w(\varphi, \alpha)$ and does not depend on λ for $\mu = \mu^w(\varphi, \alpha)$ where*

$$\mu^w(\varphi, \alpha) = \frac{\sqrt{1-(1-\varphi^2)\alpha^2} - \varphi}{\sqrt{1-(1-\varphi^2)\alpha^2} + \varphi}.$$

This corollary allows more clear interpretation of struggle in the outcomes between agglomeration and dispersion forces. On a near perfectly-competitive market ($\mu \approx 0$) dispersion forces prevail, and increasing the home industrial labor share λ (or, equivalently number of firms $N_H = \dfrac{\lambda L}{f \cdot \sigma}$) negatively affects the relative wage $\dfrac{w_H}{w_F}$. On the other hand, on a monopolistic-competitive market ($\mu \approx 1$) domination of agglomeration forces tends to increase the relative wage with respect to λ. Increasing the expenditure share μ of the monopolistic competitive sector weakens dispersion forces, while agglomeration forces are enhanced. At threshold value μ^w, agglomeration and dispersion forces compensate each other. On left-hand side of Figure 2 three outcomes of force's struggle are represented: when expenditure share $\mu = 0.01$ is small dispersion wins, when it is large $\mu = 0.99$ then agglomeration prevail and only for $\mu = \mu^w \approx 0.317714$ tie happens. The right-hand side illustrates dependence of threshold value on asymmetry measure of agricultural labor allocation $\alpha = 2\theta - 1 \in (0, 1)$ and trade "freeness" φ for three values. Due to Corollary 1, an area *above* the threshold curve $\mu^w(\varphi, \alpha)$ covers the cases of increasing of relative wage. We see that increasing of trade "freeness" makes this area wider, i.e. helps generate agglomeration. The same occurs under increasing of asymmetry in the agricultural population allocation $\alpha = 2\theta - 1$. The large concentration of agricultural population in home country attracts more industrial (i.e. agglomeration) forces and amplifies their impact.

3.3 Changes in Price Indices (or Agricultural Welfare)

Prices p_{ij} defined in equations (4)-(5) allow to define the index of cost of living, somewhat vaguely called "price index", P_H for Home-Country and P_F for foreign one. This index is the expenditure function at utility 1, i.e. the amount of money needed to maintain a unit level of gross utility under these prices. It is standard in CP model to express the index in both as

$$P_H = \left(\lambda w_H^{1-\sigma} + (1-\lambda) w_F^{1-\sigma} \varphi\right)^{\frac{1}{1-\sigma}}, \; P_F = \left(\lambda \varphi w_H^{1-\sigma} + (1-\lambda) {w_F}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

(see for details **Comparative Statics of Price Indices** in Appendix I).

Since agricultural workers always have unit income, the inverse price index directly measures their utility as $V^a = \dfrac{w_a}{P^\mu} = \dfrac{1}{P^\mu}$. The changes in both with respect to relative agricultural population are described as follows.

Proposition 4. *i) Nominal price index for any region increases with respect to its own agricultural population share, while foreign nominal price decreases, i.e., P_H increases in θ whereas P_F decreases.*

ii) Relative price indices P_H/P_F increases with respect to agricultural population share θ, thereby relative welfare of agricultural workers $V_H^a/V_F^a = \dfrac{1}{(P_H/P_F)^\mu}$ becomes more favorable for the country with decreasing θ.

iii) Relative price indices P_H/P_F decreases with respect to industrial population share λ, thereby relative welfare of agricultural workers $V_H^a/V_F^a = \dfrac{1}{(P_H/P_F)^\mu}$ becomes more favorable for the country with increasing λ.

iv) Price indices P_H and P_F (as well as agricultural welfares V_H^a and V_F^a) are equalized under condition

$$\frac{1-((1-\theta)+\mu\theta)(1-\varphi)}{1-((1-\mu)\theta+\mu)(1-\varphi)} = \left(\frac{\lambda}{1-\lambda}\right)^{\frac{\sigma}{\sigma-1}}.$$

For all given $\theta,\varphi,\mu,\sigma$ there exists $\lambda \in (0,1)$ satisfying this equation. On the other hand, for some values of $\lambda,\varphi,\mu,\sigma$ price indices of two countries may be non-equalizible for all $\theta \in (0,1)$.

v) Price index inequality $P_H > P_F$ holds if and only if

$$\theta > \frac{(1-\mu)+\mu\varphi}{(1-\mu)(1-\varphi)} - \frac{(1-\mu)+(1+\mu)\varphi}{(1-\mu)(1-\varphi)\left(1+\left(\frac{\lambda}{1-\lambda}\right)^{\frac{\sigma}{\sigma-1}}\right)}$$

or, equivalently,

$$\frac{\lambda}{1-\lambda} < \left(\frac{1-((1-\theta)+\mu\theta)(1-\varphi)}{1-((1-\mu)\theta+\mu)(1-\varphi)}\right)^{\rho}.$$

For analytical proof see Lemmas 5, 6 and 7 in Appendix A.

Note that left-hand side of equalization condition $\frac{1-((1-\theta)+\mu\theta)(1-\varphi)}{1-((1-\mu)\theta+\mu)(1-\varphi)}$ under variation of $\theta \in (0,1)$ takes the values in limited interval between $\frac{\varphi}{(1-\mu)+\mu\varphi}$ and $\frac{(1-\mu)+\mu\varphi}{\varphi}$, whereas the right-hand side changes in unlimited interval, i.e, $\left(\frac{\lambda}{1-\lambda}\right)^{\frac{\sigma}{\sigma-1}} \in (0,\infty)$ for various $\lambda \in (0,1)$. Thereby, there always exist some shares of industrial labor λ that equalize price indices and welfares of agricultural workers, but changes in agricultural labor may be insufficient for this task under given $\theta \in (0,1)$ and other parameters. *In other words, migration of industrial labor producing varieties matters more for agricultural welfares equalization than changes in agricultural labor, even if concede possibility of the agricultural migration.*

At Figure 4 the proposition and all three (equalizible and non-equalizible) situations are represented. We see that under sufficient asymmetry in industrial labor, price indices need not equalize between countries, it means Law of one price fails. We can compare our two propositions with traditional international trade and factor-price equalization in Hecksher-Ohlin model with specific factors used in two industries. Like there, factor endowments of the countries matter in our model and differences in wages or prices depend on parameters in natural direction: a factor in shorter supply becomes more expensive.

To mention realism of these conclusion, an empirical study Roos (2003) claims that "the most important factors driving price level differentials are population size and the average wage level" (see also Cecchetti et al (2002)).

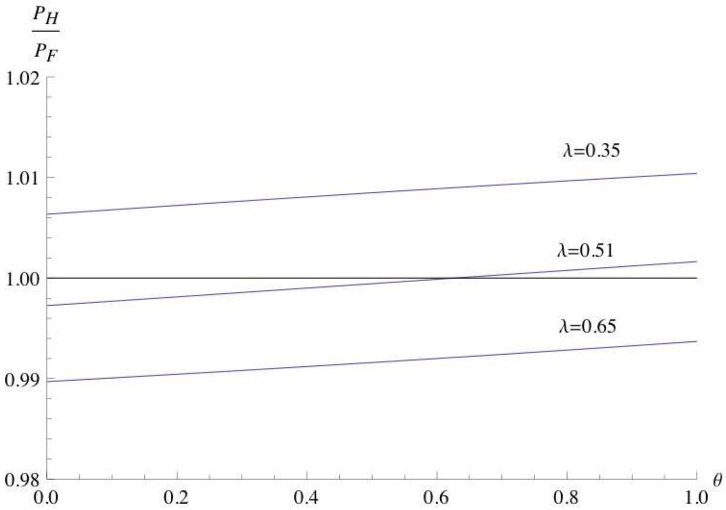


Figure 4: Changes in relative price index with respect to asymmetry in θ for $\mu = 0.7$, $\rho = 0.75$, $\varphi = 0.75$ and three values of λ.

3.4 Changes in industrial welfares

The industrial welfare in a country is defined by dividing industrial nominal wage w by the price index P. For Home country it equals to

$$V_H = \frac{(Lf\sigma)^{\frac{\mu}{1-\sigma}}}{\left(\frac{m\sigma}{\sigma-1}\right)^{\mu}} \cdot \frac{w_H}{\left(\lambda w_H^{1-\sigma} + (1-\lambda) w_F^{1-\sigma}\varphi\right)^{\frac{\mu}{1-\sigma}}},$$

and for Foreign it is

$$V_F = \frac{(Lf\sigma)^{\frac{\mu}{1-\sigma}}}{\left(\frac{m\sigma}{\sigma-1}\right)^{\mu}} \cdot \frac{w_F}{\left(\lambda\varphi w_H^{1-\sigma} + (1-\lambda) w_F^{1-\sigma}\right)^{\frac{\mu}{1-\sigma}}}.$$

Thus relative welfare may be expressed in terms of relative wage as follows

$$\frac{V_H}{V_F} = \frac{w_H}{w_F} \left(\frac{\lambda + (1-\lambda)\varphi \left(\frac{w_H}{w_F}\right)^{\sigma-1}}{\lambda\varphi + (1-\lambda)\left(\frac{w_H}{w_F}\right)^{\sigma-1}} \right)^{\frac{\mu}{\sigma-1}}. \tag{10}$$

Next proposition extends the comparative statics of previous two propositions onto real wages (utilities) of industrial labor in two countries.

Proposition 5. *i) Relative industrial real wage* $\frac{V_H}{V_F}$ *as well as home welfare* V_H *of industrial labor increase with respect to home agricultural population share* θ.

ii) There exist values of $\lambda, \varphi, \mu, \sigma$ *when industrial welfares of two countries are non-equalizible, i.e.,* $V_H \neq V_F$ *for all* $\theta \in (0,1)$.

For analytical proof see Lemma 5 in Appendix A.

Figure 5 illustrates monotonicity claim (i) and proves claim (ii) by example. In contrast to previous propositions, it is not easy here to obtain explicit condition of non-equability. Our conjecture induced by this and other simulations is that non-eligibility happens when industrial labor asymmetry is sufficiently large.

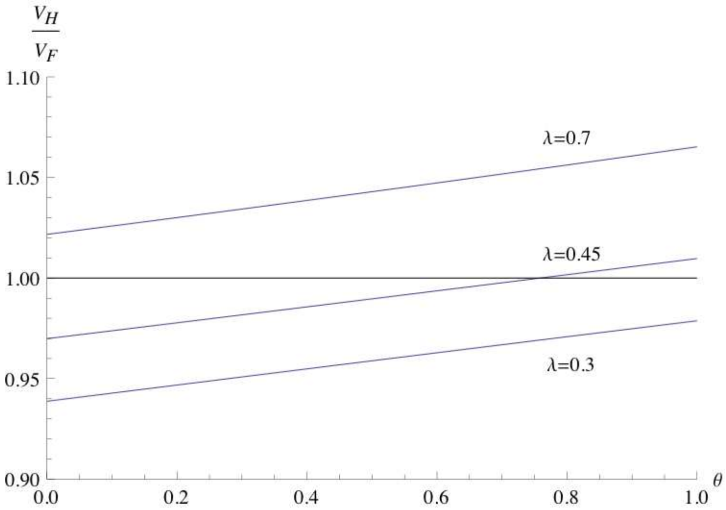


Figure 5: Comparative statics of relative welfare with respect to θ for $\mu = 0.7$, $\rho = 0.75$, $\varphi = 0.75$ and three values of λ.

Interpreting the monotonicity observed, we can say that in this case there is no conflicts between perfect competitive and monopolistic competitive sectors. Indeed, in the case $\mu = 0$ we obtain from (10) that relative welfare is similar to relative wage $\frac{w_H}{w_F}$ that is increasing function with respect to agricultural labor share θ. To approximate monopolistic competitive case $\mu = 1$ we use equivalent representation of relative welfare (see for details **Comparative Statics of Industrial Welfares** in Appendix A)

$$\frac{V_H}{V_F} = \left(x^{\frac{\rho}{\mu}} \frac{(1-\mu)\theta - ((1-\mu)\theta + \mu)\varphi x}{(1-\mu)(1-\theta)x - ((1-\theta) + \mu\theta)\varphi} \right)^{\frac{\mu}{\sigma-1}},$$

where $x = \left(\frac{w_H}{w_F}\right)^{\sigma}$. Then for $\mu = 1$ we obtain that relative welfare is similar to positive power of relative wage $\left(\frac{w_H}{w_F}\right)^{\frac{2\sigma-1}{\sigma-1}}$ that is increasing function with respect to agricultural labor share θ.

Comparative statics with respect to industrial labor share λ is more sophisticated. Traditionally general case is divided into two sub-cases. The first one is so called **Black Hole,** characterized by inequality $\rho = \frac{\sigma-1}{\sigma} \leqslant \mu$, or equivalently, $\frac{1}{\sigma} = 1 - \rho \geqslant 1 - \mu$ with the following interpretation — *love for industrial variety exceeds the expenditure share for agricultural good.* The opposite case characterized by **No-Black-Hole Condition** $\rho > \mu$ will be considered as main one. In this case one can define the following critical point

$$\varphi^B(\rho,\mu,\alpha) = \frac{\rho-\mu}{\rho+\mu} \sqrt{\frac{1-\alpha^2}{\left(\frac{1+\mu}{1-\mu}\right)^2 - \alpha^2}}$$

where $\alpha = 2\theta - 1$, coinciding in symmetric case $\alpha = 0$ with break-point φ^B mentioned in Introduction.

Proposition 6. *i) Let Black-Hole Condition $\rho \leqslant \mu$ holds. Then relative welfare $\frac{V_H}{V_F}$ increases with respect to industrial labor share λ. There exists the unique value $\lambda^0 \in (0,1)$ yielding industrial welfare equalization $V_H(\lambda^0) = V_F(\lambda^0)$.*

ii) Assume No-Black-Hole Condition $\rho > \mu$ holds and $\varphi \geqslant \varphi^B(\rho,\mu,\alpha)$. Then relative welfare $\frac{V_H}{V_F}$ increases with respect to industrial labor share λ and there is at most one value $\lambda^0 \in (0,1)$ yielding welfare equalization $V_H(\lambda^0) = V_F(\lambda^0)$.

iii) Assume No-Black-Hole Condition $\rho > \mu$ holds and $\varphi < \varphi^B(\rho,\mu,\alpha)$. Then relative welfare $\dfrac{V_H}{V_F}$ increases with respect to industrial labor share λ for sufficiently small $\lambda \geqslant 0$ and derivative $\dfrac{\partial}{\partial\lambda}\dfrac{V_H}{V_F}$ changes its sign not more than twice for all $\lambda \in (0,1)$. Therefore there is at most one three values $\lambda \in (0,1)$ yielding welfare equalization $V_H(\lambda) = V_F(\lambda)$.

For analytical proof see Lemmas 9, 10 in Appendix A.

We obtain this threshold effect in terms of trade freeness φ. Proposition 6 may be equivalently reformulated in terms of threshold value of expenditure share μ, though there is no the corresponding *explicit* form of this threshold.

Corollary 2. *For all $(\rho,\alpha,\varphi) \in (0,1)^3$ there is defined a continuous function $\mu^B(\rho,\alpha,\varphi)$ satisfying the following conditions:*

i) function values $0 < \mu^B(\rho,\alpha,\varphi) < \rho$, moreover $\mu^B(\rho,\alpha,\varphi)$ decreases with respect to φ and α

ii) For all $\mu \geqslant \mu^B(\rho,\alpha,\varphi)$ relative welfare $\dfrac{V_H}{V_F}$ increases with respect to industrial labor share λ and there exists not more than one value $\lambda^0 \in (0,1)$ yielding welfare equalization $V_H(\lambda^0) = V_F(\lambda^0)$

iii) For $\mu < \mu^B(\rho,\alpha,\varphi)$ relative welfare $\dfrac{V_H}{V_F}$ increases with respect to industrial labor share λ for sufficiently small $\lambda \geqslant 0$ and derivative $\dfrac{\partial}{\partial\lambda}\dfrac{V_H}{V_F}$ changes its sign not more than twice for all $\lambda \in (0,1)$. Therefore there exists not more than three values $\lambda \in (0,1)$ yielding industrial welfare equalization $V_H(\lambda) = V_F(\lambda)$.

Indeed, the threshold function $\mu^B(\rho,\alpha,\varphi)$ may be obtained as implicit function defined by equation

$$F(\varphi,\rho,\mu,\alpha) = \varphi - \frac{\rho-\mu}{\rho+\mu}\sqrt{\frac{1-\alpha^2}{\left(\frac{1+\mu}{1-\mu}\right)^2 - \alpha^2}} = 0.$$

It is obvious that $F(\varphi,\rho,\mu,\alpha)$ strictly increases with respect to μ, i.e. $\dfrac{\partial F}{\partial\mu} > 0$ for all admissible arguments, thus by the Implicit Function Theorem there exists differentiable function $\mu^B(\rho,\alpha,\varphi)$ and

$$\frac{\partial\mu}{\partial\alpha} = -\frac{\left(\frac{\partial F}{\partial\alpha}\right)}{\left(\frac{\partial F}{\partial\mu}\right)} < 0, \; \frac{\partial\mu}{\partial\varphi} = -\frac{\left(\frac{\partial F}{\partial\varphi}\right)}{\left(\frac{\partial F}{\partial\mu}\right)} < 0$$

because $F(\varphi,\rho,\mu,\alpha)$ obviously increases with respect to α and φ. We need not consider Black-Hole case separately, since $\mu \geqslant \rho$ implies $\mu \geqslant \mu^B(\rho,\alpha,\varphi)$. In other words, *Black Hole is simply a partial case of monopolistic competition's prevailing*.

Note that equalization values $V_H(\lambda^0) = V_F(\lambda^0)$ are exactly interior long run equilibrium values of industrial labor shares. More detailed consideration of interior long run equilibria is postponed until section 4.2. Claims (ii), (iii) with No-Black-Hole Condition are illustrated in Figure 6. When "freeness" φ increases gradually from 0.003, to 0.1, the sinus-shape curve of relative welfare $\dfrac{V_H}{V_F}(\lambda)$ becomes less and less steep in the middle, arriving finally at monotone increasing shape. The curve changes as a piece of wire, whose left end is pulled down but right end up (seemingly, all curves turn around one point, but it is an illusion). In the beginning, under 0.003, the curve intersects the horizontal line $\dfrac{V_H}{V_F} = 1$ only once, as predicted by claim (iv) with *one* utility-equalization point. Then, at some stage before $\varphi = 0.01$ the right end hits 1 and at the next moment *two* interior equilibria emerge, the right one being unstable because the curve intersects 1 *from below*. These two equilibria, stable and unstable, are

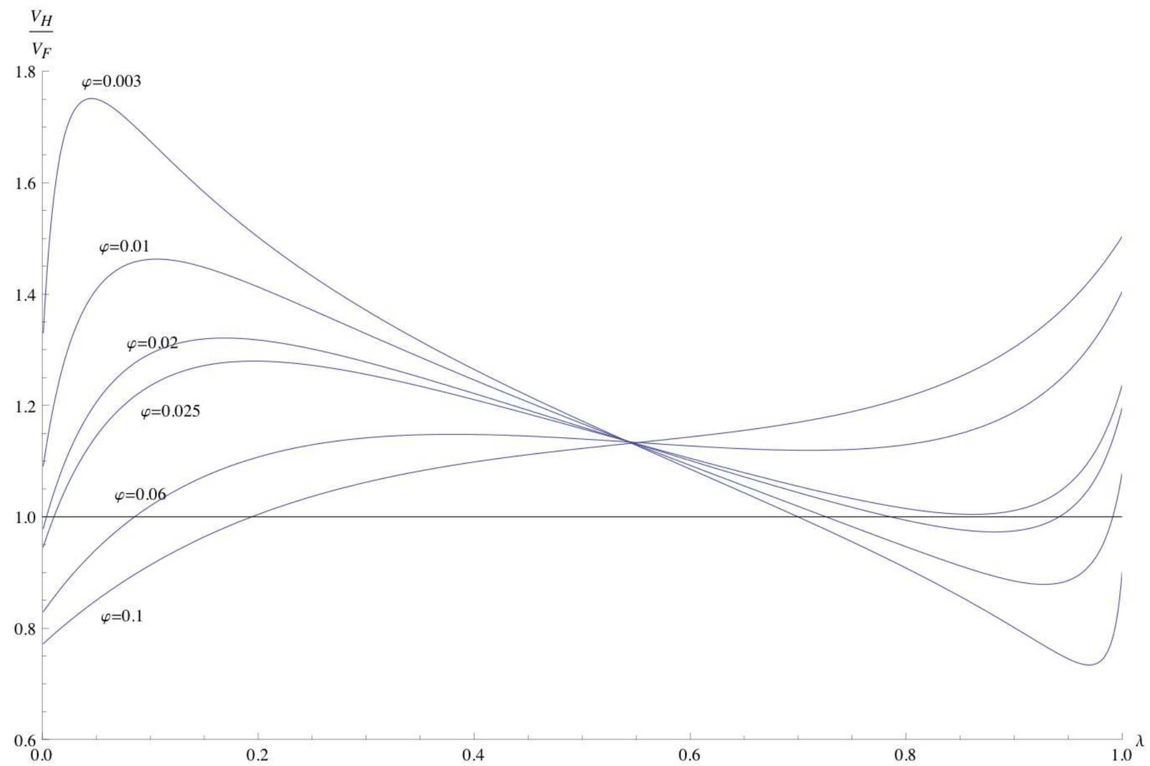


Figure 6: Different patterns of relative welfare for $\theta = 0.55$, $\mu = 0.2$, $\rho = 0.25$ and various values of freeness φ.

supplemented by the *third equilibrium* (unstable one) after the left end of the curve hits 1 (say, under $\varphi = 0.02$), but after bigger $\varphi = 0.01$ the middle and the rightmost equilibria disappear together, only one unstable intersection is possible. Similar picture—one intersection from below—remains true under further increase of $\varphi > 0.01$, and remains also under Black-Hole Condition discussed in claim (ii), for all φ. Note that increasing of relative welfare $\dfrac{V_H}{V_F}$ reflects predominance of agglomeration forces with direction *toward home country*, while decreasing reflects the opposite process. Figure 6 demonstrates also that increasing of trade freeness supports agglomeration forces.

Further, Figure 7 illustrates how strengthening of monopolistic competition (i.e. increasing of expenditure share μ) changes correlation of agglomeration and dispersion forces. For small values of μ dispersion forces prevail and relative welfare decreases forming stable interior long run equilibria (see for details section 4.2). For sufficiently large values of μ outcome is opposite — relative welfare increases with respect to home industrial labor share. As for intermediate values — for example, $\mu = 0.55$ — dynamics of relative welfare may change not more then twice.

Possible absence of interior utility equalization under sufficiently big agricultural asymmetry occurring under $\theta = 0.9$ or $\theta = 0.1$ is illustrated in left-hand side of Figure 8. The industrial workers neighboring very big agricultural population benefit so much from this that this cannot be completely outweighed by any change in their own population. Finally, right-hand side of Figure 8 illustrates how threshold value $\mu^B(\rho, \alpha, \varphi)$ depends on asymmetry in agricultural labor allocation $\alpha = 2\theta - 1$ and trade freeness φ. Recall that area of relative welfare's increasing lies to the right of the curve. The more free is trade, the wider is this area, i.e. increasing of trade freeness intensify agglomeration forces. Moreover, increasing of asymmetry in agricultural population's allocation also amplify agglomeration.

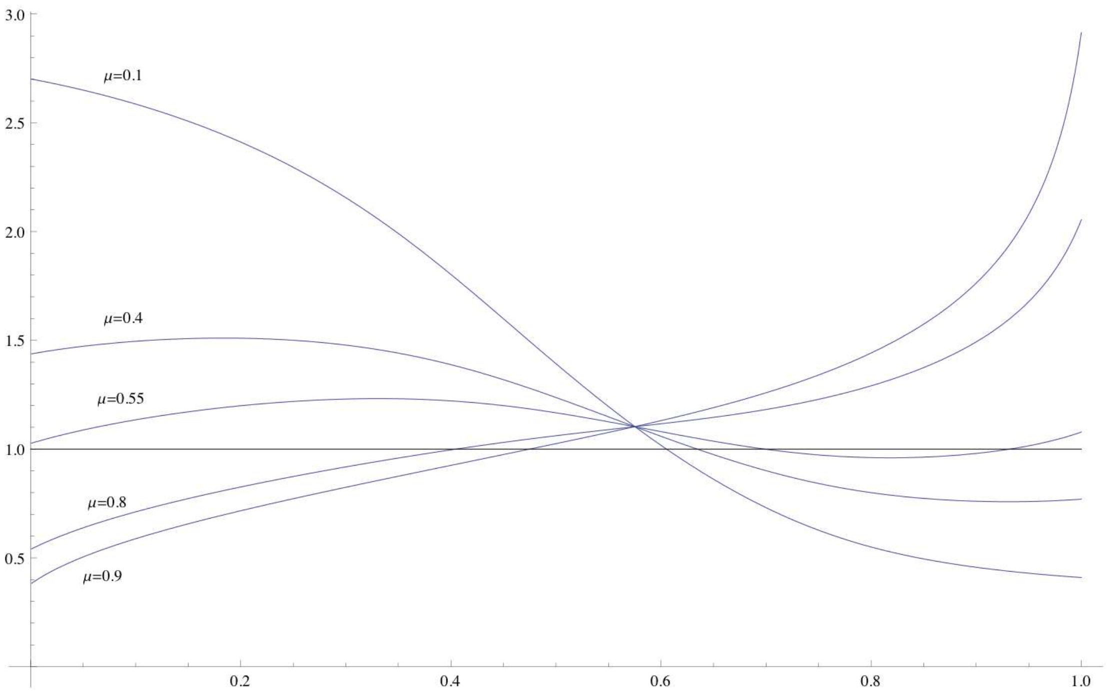


Figure 7: Different patterns of relative welfare for $\theta = 0.6$, $\varphi = 0.005$, $\rho = 0.75$ and various values of expenditure share λ.

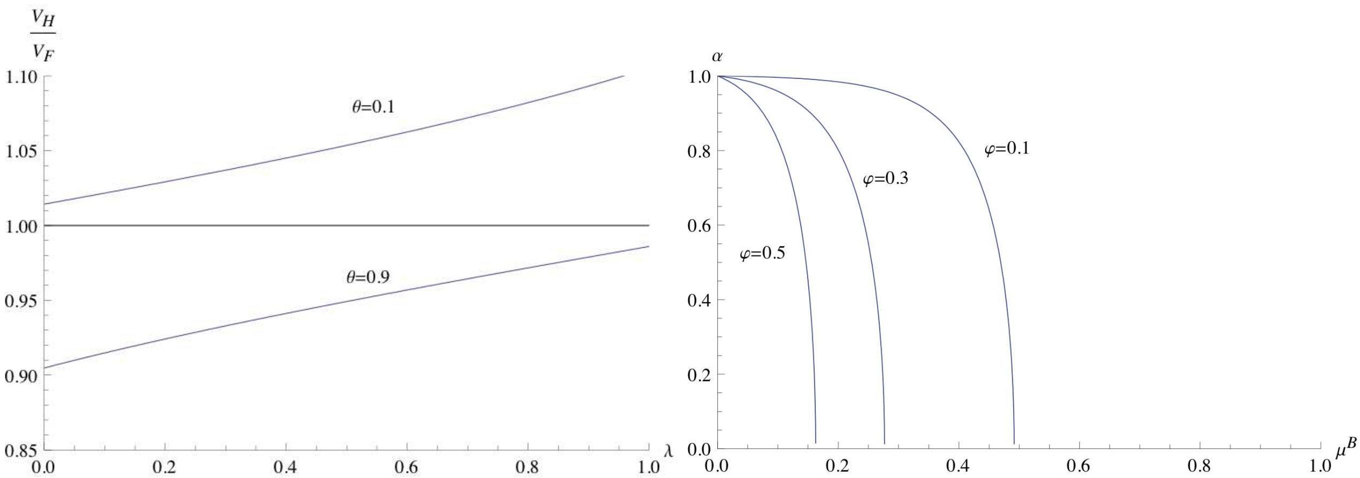


Figure 8: Left-hand plot: Absent utility equalization under big agricultural asymmetry: $\theta = 0.9$, $\mu = 0.25$, $\varphi = 0.7$, $\rho = 0.75$.
Right-hand plot: Threshold values of expenditure share μ^B with respect to asymmetry measure $\alpha = 2\theta - 1$ for three values of φ.

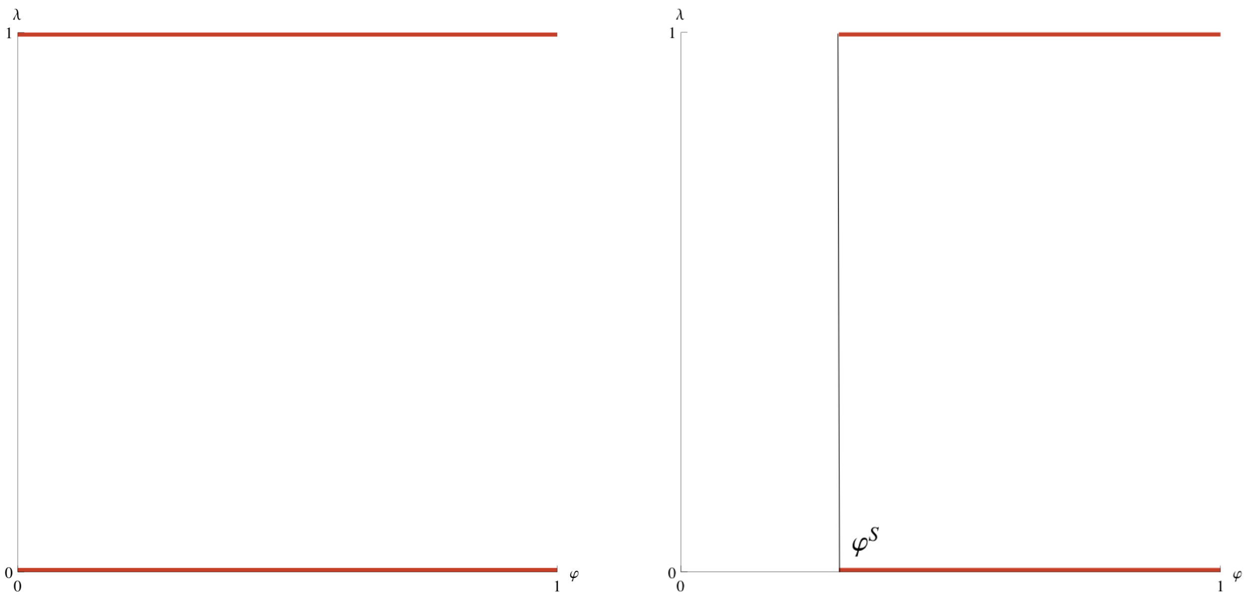
4 Asymmetry in Long Run

4.1 Agglomerated Long-Run Equilibria

The most popular migration dynamics model is so called *ad hoc* dynamics for industrial labor share λ

$$\dot{\lambda} = M(\lambda) = \lambda\,(1-\lambda)\,(V_H(\lambda) - V_F(\lambda))\,.$$

From this point of view there are two *agglomerated* long-run equilibria $\lambda^0 = 0$ and $\lambda^0 = 1$ which are steady states of this differential equation. Stability conditions for this type of dynamics may be standardly expressed as



Black Hole case $\mu \geqslant \rho$ Symmetric No-Black Hole case $\rho > \mu$, $\alpha = 0$

Figure 9: Symmetric patterns of agglomeration stability

$\dfrac{\partial M}{\partial \lambda}(0) < 0$ and $\dfrac{\partial M}{\partial \lambda}(1) < 0$ which is equivalent to inequalities $\dfrac{V_H}{V_F}(0) < 1$ and $\dfrac{V_H}{V_F}(1) > 1$, respectively. Note that these inequalities may be well interpreted without any differential equations, the first inequality means that all industrial workers left Home country and none want to return, the second one means that all gathered in Home country and none want to leave.

In symmetric case $\theta = 1/2$ one can obtain one of two typical pictures of stability depending on transportation costs $\tau \geqslant 1$ or, which is equivalent, on trade freeness $\varphi = \tau^{1-\sigma} \in (0,1]$. The first one is so called "Black Hole" when both agglomerated equilibria are stable for all values of $\varphi \in (0,1]$ and it appears in the case $\rho = \dfrac{\sigma-1}{\sigma} \leqslant \mu$. Under "No Black Hole Condition" $\rho > \mu$ there exists "sustain point" $\varphi^S \in (0,1)$ such that both agglomerated equilibria are unstable for all $\varphi \in (0, \varphi^S)$ and they are both stable for all $\varphi \in (\varphi^S, 1]$.

Here the bold lines are sets of *stable* agglomerated long run equilibria for various values of trade freeness φ. Asymmetry in agricultural population brings two new cases. Without loss of generality we assume that $\theta \geqslant 1/2$.

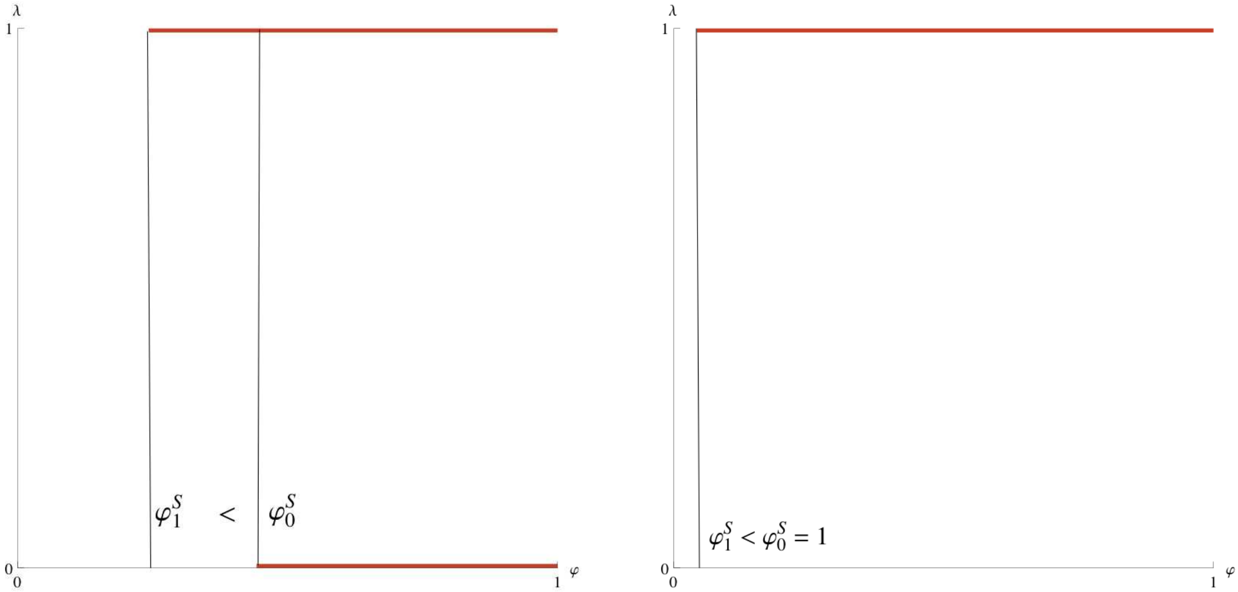
Proposition 7. *i) Let Black-Hole condition $\rho \leqslant \mu$ holds. Then both agglomerated equilibria $\lambda = 0$ and $\lambda = 1$ are stable for all values of φ, θ.*

ii) Let No-Black-Hole condition $\rho > \mu$ holds and $\mu \geqslant \dfrac{\rho}{1+2\rho}$. Then there exist $0 < \varphi_1^S \leqslant \varphi_0^S < 1$ such that agglomerated equilibrium $\lambda = 0$ is stable if and only if $\varphi > \varphi_0^S$ and $\lambda = 1$ is stable if and only if $\varphi > \varphi_1^S$. Moreover, $\varphi_0^S = \varphi_1^S$ if and only if $\theta = 1/2$. Sustain point φ_0^S strictly increases with respect to θ while φ_1^S strictly decreases.

iii) Let No-Black-Hole condition $\rho > \mu$ holds and $\mu < \dfrac{\rho}{1+2\rho}$. Then there exist $0 < \varphi_1^S \leqslant \varphi_0^S \leqslant 1$ such that agglomerated equilibrium $\lambda = 0$ is stable if and only if $\varphi > \varphi_0^S$ and $\lambda = 1$ is stable if and only if $\varphi > \varphi_1^S$. Moreover, $\varphi_0^S = \varphi_1^S$ if and only if $\theta = 1/2$, $\varphi_0^S < 1$ for all $\theta < \dfrac{\rho+\mu}{2(1-\mu)\rho}$ and $\varphi_0^S \equiv 1$ for all $\theta \geqslant \dfrac{\rho+\mu}{2(1-\mu)\rho}$. Sustain point φ_0^S strictly increases with respect to θ for all $\theta \in \left[\dfrac{1}{2}, \dfrac{\rho+\mu}{2(1-\mu)\rho}\right)$ while φ_1^S strictly decreases for all θ.

For technical details see Lemmas 12, 13 in Appendix B.

The asymmetric Black-Hole case i) is identical to symmetric Black Hole. More exactly, for sufficiently large weight of industrial sector, asymmetry degree of agricultural sector does not affect direction of relative welfare change, i.e. it is a case of domination of monopolistic competition suppressing agricultural heterogeneity.



Moderate asymmetry $\alpha < \dfrac{(1+\rho)\mu}{(1-\mu)\rho}$, No-Black-Hole Large asymmetry $\alpha \geqslant \dfrac{(1+\rho)\mu}{(1-\mu)\rho}$, No-Black-Hole

Figure 10: Asymmetric patterns of agglomeration stability

Sub-case ii) is analogous to its symmetric case except that each agglomerated equilibrium has specified sustain point φ_0^S for $\lambda = 0$ and φ_1^S for $\lambda = 1$. Finally, sub-case iii) when one of sustain points disappears is specific to "sufficiently large" asymmetry in agricultural labor allocation. It is obvious that increasing of trade freeness supports agglomeration *in both directions*, while asymmetry in agricultural population is favorable for agglomeration in big country only. For sufficiently large measure of monopolistic competition $\mu \geqslant \dfrac{\rho}{1+2\rho}$ the *general* (i.e. non-directed) agglomeration has some influence and ambiguity in agglomeration directions still remains, though asymmetry makes one of directions "more probable". In opposite case, when monopolistic competitive sector is sufficiently weak, i.e. $\mu < \dfrac{\rho}{1+2\rho}$ directed agglomeration prevails and for sufficiently large asymmetry ambiguity disappears.

Note additionally that for any given $\rho > \mu$ and θ sustain points φ_0^S, φ_1^S may be found as the *smallest* roots of equations

$$G(\varphi) = \theta(1-\mu)\varphi^{\frac{\mu-\rho}{\rho}} + (1-\theta(1-\mu))\varphi^{\frac{\mu+\rho}{\rho}} = 1,$$

$$H(\varphi) = (1-\theta)(1-\mu)\varphi^{\frac{\mu-\rho}{\rho}} + (1-(1-\theta)(1-\mu))\varphi^{\frac{\mu+\rho}{\rho}} = 1,$$

respectively (see proof of Lemma 13 in Appendix B). On the Figure 11 functions $G(\varphi)$, $H(\varphi)$ are plotted for No-Black-Hole (left-hand side) and Black-Hole (right-hand side) cases. Note that $\varphi = 1$ is trivial root of both equations and sometimes (namely, for all $\theta \geqslant \dfrac{\rho+\mu}{2(1-\mu)\rho}$) it is a smallest one. In other cases $\varphi_0^S, \varphi_1^S \in (0,1)$. In No-Black-Hole case $\rho > \mu$ for all $\varphi \in (0,1)$ inequalities $\varphi^{\frac{\mu+\rho}{\rho}} < 1 < \varphi^{\frac{\mu-\rho}{\rho}}$ hold, while left-hand sides of these equations are the weighted sums of terms $\varphi^{\frac{\mu-\rho}{\rho}}$ and $\varphi^{\frac{\mu+\rho}{\rho}}$. It is obvious that increasing in agricultural labor share θ, or equivalently in corresponding weight coefficient, pulls plot of $G(\varphi)$ up, while plot of $H(\varphi)$ goes down. Thus sustain points φ_0^S, φ_1^S shift to the right and left, correspondingly. In Black-Hole case changes in θ do not have significant consequences.

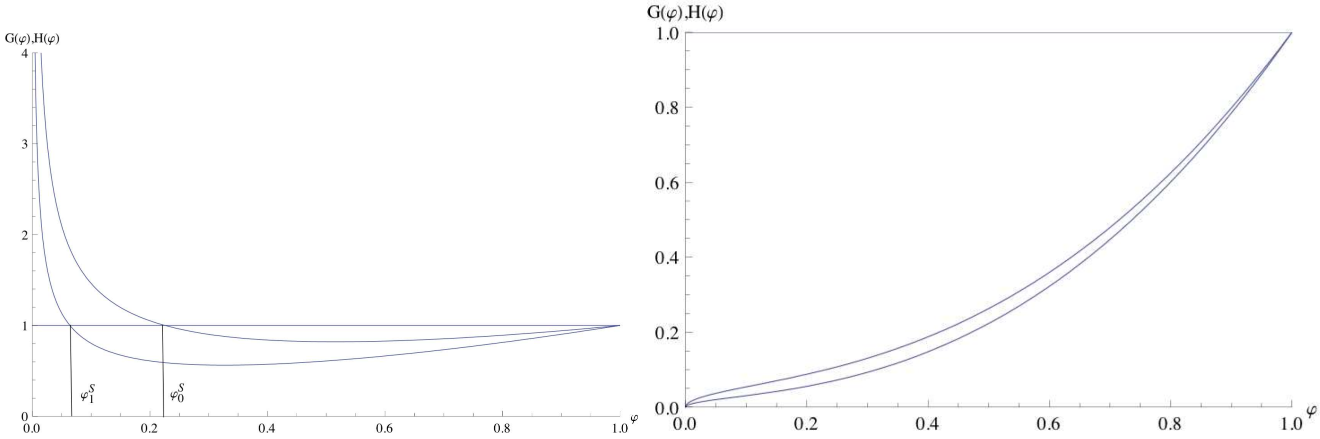


Figure 11: Sustain points: No-Black-Hole and Black-Hole cases

4.2 Interior Long-run Equilibria

Interior long-run equilibria may be defined as interior steady states $\lambda^0 \in (0,1)$ of the same *ad hoc* dynamic equation

$$\dot{\lambda} = M(\lambda) = \lambda\,(1-\lambda)\,(V_H(\lambda) - V_F(\lambda))\,.$$

which implies the equality of industrial real wages in both regions $V_H(\lambda^0) = V_F(\lambda^0)$. Stability conditions for this type of dynamics may be standardly expressed as $\frac{\partial M}{\partial \lambda}(\lambda^0) < 0$ which is equivalent to inequalities $\frac{\partial}{\partial \lambda}\frac{V_H}{V_F}(\lambda^0) < 0$. Note that these inequalities may be also interpreted without any differential equations. It means that the further immigration in Home country causes deceasing of relative welfare and leads to backward migration.

Recall the definition of break point

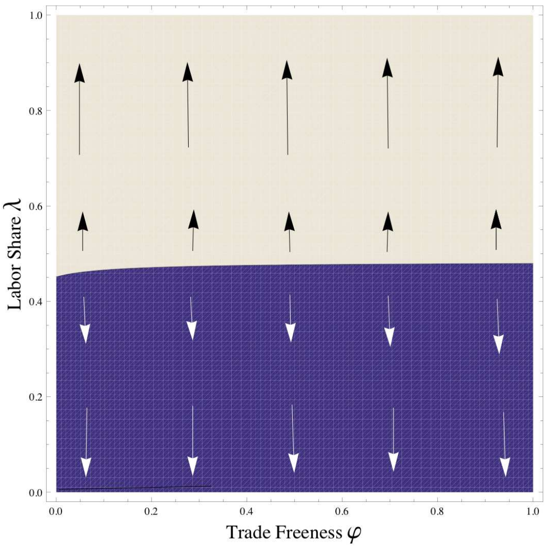
$$\varphi^B = \frac{\rho-\mu}{\rho+\mu}\sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}}$$

Proposition 8. *i) Let Black-Hole Condition $\rho \leqslant \mu$ hold. Then there is at most one single interior long run equilibrium and ,in case of existence, it is always unstable.*

ii) Let No-Black-Hole Condition $\rho > \mu$ holds and $\varphi \geqslant \varphi^B$. Then there is at most one interior long-run equilibrium and ,in case of existence, it is always unstable.

iii) Let No-Black-Hole Condition $\rho > \mu$ holds and $0 < \varphi < \varphi^B$. Then there is at most three interior long-run equilibria.

Numerical simulation show that for appropriate values of the parameters, each of the cases in "not more than n" is possible, i.e. for $\varphi \geqslant \varphi^B$ there are examples of exactly one or without any interior equilibrium, and for $\varphi < \varphi^B$ there are examples of exactly three, two, one and none of interior equilibria. In fact, Figures 6, 7, 8 show us all cases of interior equilibria mentioned above. More exactly, Figure 7 shows single unstable Black Hole equilibria for $\mu = 0.8$ and $\mu = 0.9$ ($\mu > \rho = 0.75$), Figure 8, left-hand side, shows absence of interior equilibria for No-Black-Hole case and $\varphi = 0.7 > \varphi^B = 0.402985$, Figure 6 shows the cases of one, two and three interior equilibria for various values of $\varphi < \varphi^B = 0.0740226$. Note that indication of equilibrium stability is intersection "from above", while inverse direction of intersection characterizes unstable equilibria. For each specified value of freeness φ all possible cases of number and stability type of interior equilibria are described in Lemma 16 in Appendix B. Here we classify the *synthetic* patterns of long run equilibria, including agglomerated and interior ones.



Pattern 0: No Break-Point, Sustain Points $\varphi_0^S = \varphi_1^S = 0$

Figure 12: Black Hole map: $\mu = 0.5 > \rho = 0.25$, $\theta = 0.6$

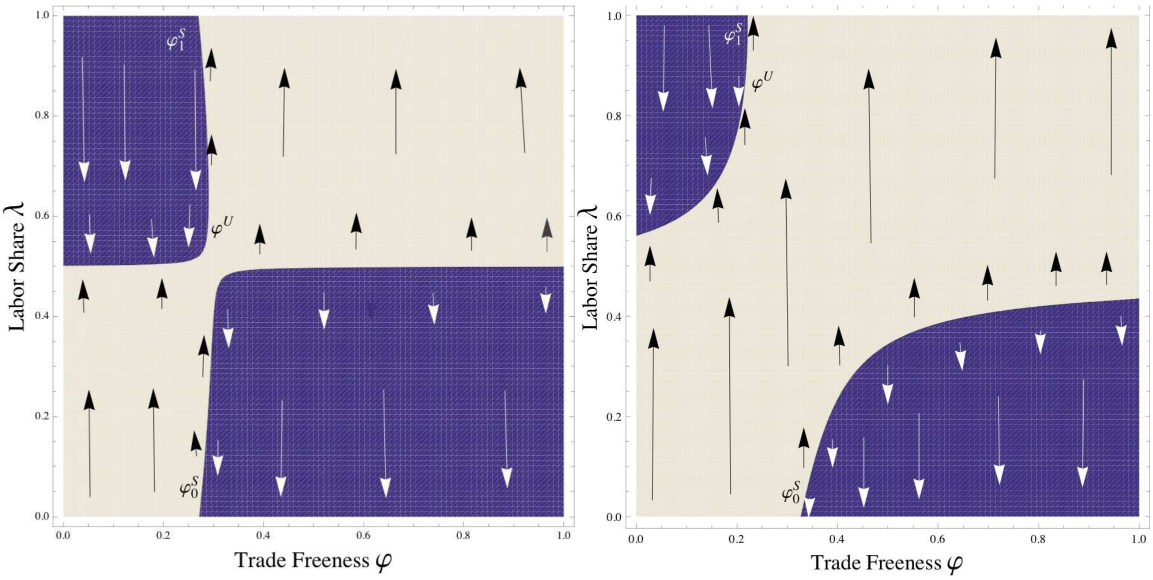
These patterns are represented by simulation results for specified values of ρ, μ, θ and for all values of industrial labor share λ and trade freeness φ. Dark area (referred as "sea") represents cases where relative welfare $\frac{V_H}{V_F}(\varphi,\lambda) < 1$, light area (referred as "shore") corresponds to case $\frac{V_H}{V_F}(\varphi,\lambda) > 1$ and "coastline" $\frac{V_H}{V_F}(\varphi,\lambda) = 1$ represents set of all interior equilibria. Consider a "map" of Black Hole case as an example. We see that all "southern" (i.e. for $\lambda = 0$) points are "in the sea" $\frac{V_H}{V_F}(\varphi,\lambda) < 1$ and thus stable, all "northern" (i.e. for $\lambda = 1$) points are "on the shore" $\frac{V_H}{V_F}(\varphi,\lambda) > 1$ and thus stable, and "coastline" consists of unstable interior equilibria, because going north we cross it from sea to shore, i.e. $\frac{V_H}{V_F}(\varphi,\lambda)$ increases with respect to λ. Figure 12 represent all Black Hole cases, the only difference in other Black Hole cases concerns an exact form of coastline. Arrows show directions of industrial labor migration, more exactly, the upward arrow indicates *immigration* into Home country, the downward one — the opposite process.

The rest classification concerns No-Black-Hole cases only. Let's define one more threshold value for trade freeness φ, the "turn point", that will be useful for our classification

$$\varphi^T(\theta,\rho,\mu) = \sqrt{\max\left\{0, \frac{(1-\theta)(\theta\rho-(1+\theta\rho)\mu)}{\theta((1-\theta)\rho+(1+\theta\rho)\mu)}\right\}}.$$

It distinguish the possible cases of relative welfare behavior in agglomerated state $\lambda = 1$. Namely, relative welfare $\frac{V_H}{V_F}$ increases in $\lambda = 1$ if and only if $\varphi > \varphi^T$ (see for details Lemma 11 in Appendix A). Note that in perfectly competitive case (i.e. $\mu = 0$) this value $\varphi^T(\theta,\rho,0) = 1$ and for sufficiently large weight of monopolistic competitive sector $\mu \geqslant \frac{\theta\rho}{1+\theta\rho}$ we obtain $\varphi^T(\theta,\rho,\mu) \equiv 0$.

Consider the class of "asymmetric tomahawks" (see Proposition 7-iii, case of $\theta < \frac{\rho+\mu}{2(1-\mu)\rho}$ which is equivalent to $\alpha = 2\theta - 1 < \frac{(1-\rho)\mu}{(1-\mu)\rho}$) or in new "geographic" interpretation, class of "maps with two coastlines", northern and southern ones. It may be divided into two subclasses by distinction $\varphi^T < \varphi_1^S$ and $\varphi^T \geqslant \varphi_1^S$. The first case is characterized by "hooked" northern coastline, i.e. it goes first eastwards and then turns westwards. In other words, northern coastline contains *U-turn point* characterized by "longitude" φ^U. Figure 13 illustrates this case by two ex-



Pattern 1
$\mu = 0.25,\ \rho = 0.75,\ \theta = 0.501$

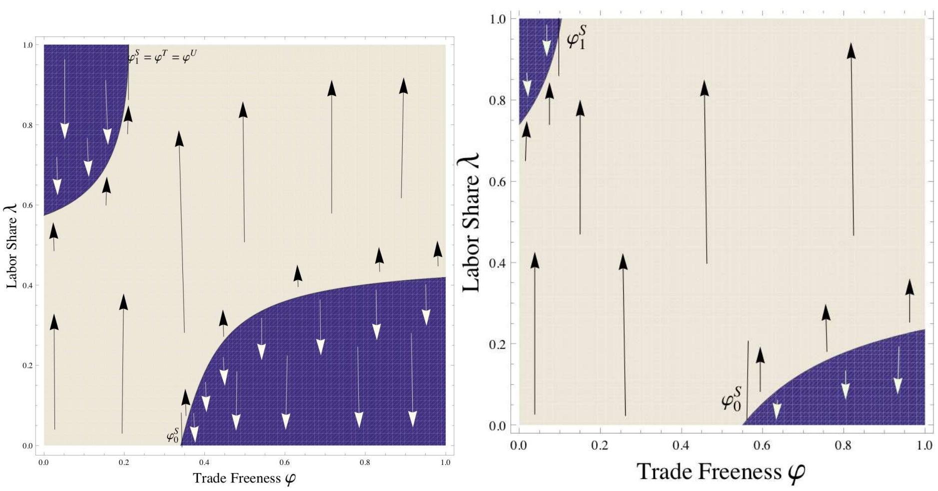
Pattern 2
$\mu = 0.25,\ \rho = 0.75,\ \theta = 0.55$

Figure 13: Patterns for "hooked coastline" $\varphi^T < \varphi_1^S$

amples: in the left-hand side plot $\varphi^T = 0$ and hook is obvious, in right-hand one $\varphi^T = 0.209278 < \varphi_1^S = 0.226391$ and counter motion of coastline is almost invisible, but still exists. There is a small difference between two "hooked" patterns in Figure 13: for left-hand side case we have $\varphi_0^S < \varphi^E$, while for right-hand side one an inequality $\varphi^E < \varphi_0^S$ holds.

Remark 1. Note that usually $\varphi^E \neq \varphi^T$, i.e. *U-turn point* is not the same as *turn point*, yet inequality $\varphi^T < \varphi_1^S$ holds if and only if U-turn point exists. We use turn point φ^T instead of U-turn point φ^E for classification purposes because U-turn point has no analytic presentation and its numerical calculation requires to solve quite tedious system of equations. For technical details see Appendix B, subsection **Existence of U-turn Point**.

The case $\varphi^T \geqslant \varphi_1^S$ will be considered as "map with regular coastlines". We consider threshold case $\varphi^T = \varphi_1^S$ as regular one for reasons to be explained in Policy Implications.



Pattern 3
$\mu = 0.25,\ \rho = 0.75,\ \theta = 0.560885,\ \varphi^T \approx \varphi_1^S$

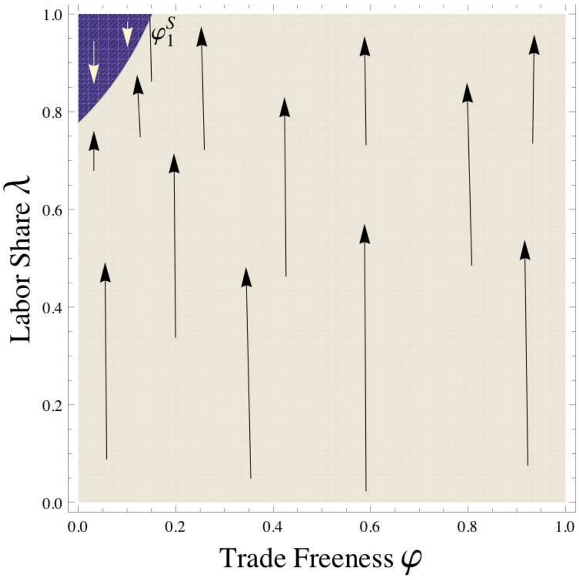
$\mu = 0.25,\ \rho = 0.75,\ \theta = 0.7,\ \varphi^T > \varphi_1^S$

Figure 14: Patterns for "regular coastlines" $\varphi^T \geqslant \varphi_1^S$

As for classification of "axes" (see Proposition 7-iii, case of $\theta \geqslant \frac{\rho+\mu}{2(1-\mu)\rho} \iff \mu \leqslant \frac{(2\theta-1)\rho}{1+2\theta\rho}$) or in new "geographic" interpretation, class of "maps with one coastline" there is no need to divide it into subclasses. Direct calculation show that inequality $\mu \leqslant \frac{(2\theta-1)\rho}{1+2\theta\rho}$ implies that

$$\varphi^T = \sqrt{\frac{(1-\theta)(\theta\rho-(1+\theta\rho)\mu)}{\theta((1-\theta)\rho+(1+\theta\rho)\mu)}} \geqslant \varphi^{**} = \sqrt{\frac{(1-\theta)(1-\mu)(\rho-\mu)}{(1-(1-\theta)(1-\mu))(\rho+\mu)}} > \varphi_1^S.$$

It means that the single (former northern) coastline always is regular.



Pattern 4

Figure 15: Pattern for "single regular coastline" $\mu = 0.15,\ \rho = 0.75,\ \theta = 0.75,\ \varphi_0^S = 1$

Let's call for short patterns from Figures 12-15 as follows: *Pattern 0* for very special Black Hole case, *Pattern 1* and *Pattern 2* (hooked ones) are represented in Figure 13, *Pattern 3* covers both cases in Figure 14 (there is no need to separate an intermediate case $\varphi^T = \varphi_1^S$), and the *Pattern 4* is the last one from Figure 15.

The following statement is not quite "strictly mathematical", because it uses the fuzzy notion of "the same pattern", though it is obvious what is meant.

Proposition 9. *i) Let Black Hole condition $\mu \geqslant \rho$ holds. Then for all values of asymmetry measure $\alpha = 2\theta - 1$ the same Pattern 0 is obtained.*

ii) Let No-Black-Hole condition $\mu < \rho$ and $\mu \geqslant \frac{\rho}{1+\rho}$ hold, then for sufficiently small values of α Pattern 1 comes out, while for larger ones it transforms into Pattern 2.

iii) Let No-Black-Hole condition $\mu < \rho$ and $\frac{\rho}{1+2\rho} < \mu < \frac{\rho}{1+\rho}$ hold, then increasing α from 0 *to* 1 *changes the pattern gradually from Pattern 1 through Pattern 2 to Pattern 3.*

iv) Finally, let No-Black-Hole condition $\mu < \rho$ and $\mu \geqslant \frac{\rho}{1+2\rho}$ hold, then increasing α from 0 *to* 1 *changes the pattern gradually from Pattern 1 through Patterns 2 and 3 to Pattern 4.*

This Proposition follows from "comparative statics" of threshold values φ_0^S, φ_1^S, φ^B and φ^T with respect to α (see Lemmas 13, 15, 16 in Appendix B).

5 Summary

In present paper, a complete study of Krugman's CP model with agricultural labor asymmetry is carried out. Statements are proved which generalize the known results of Mossay (2006) on existence and uniqueness of short run equilibrium for asymmetric CP model, as well as the results of Robert-Nicoud (2005) on number and stability of long run equilibria. The comparative statics of equilibrium wages, price indices and welfare (nominal and relative) with respect to agricultural and industrial labor allocations were studied. All typical patterns of agglomeration, generalizing the well-known "tomahawk diagram" for symmetric CP model, were found and studied. From substantive point of view it was shown that the statement of Berliant and Kung (2009) on non-robustness of conclusions and policy implications based on symmetric CP model is not quite true: for sufficiently small degree of asymmetry all substantial qualitative conclusions stay true. Only under sufficiently large asymmetry considerable changes do occur.

All possible cases of comparative statics, number and stability of long-run equilibria may be classified in terms of relations among series of critical or threshold values: sustain points, break point, U-turn point, etc. Analytical results of presented paper allow to calculate these values for particular model parameters (i.e. trade costs, labor shares, consumption characteristics). Comparing the values of these critical points one can identify actual Pattern of the economy and predict, in some extent, some implications of varius policy options.

Policy Implications

One can consider the following types of economic policy, i.e. purposeful influence of government upon economy, which is described by CP model: "trade policy" — changing (for example, increasing) of trade freeness, measured by φ, "industral labor policy" — changing (for example, increasing) of industrial labor supply, measured by λ, and "agricultural labor policy" — changing of agricultural population share θ. The first two policies are conducted in the range of fixed Pattern, while the last one may *change* initial Pattern, as shown in Proposition 9. However, conducting of "agricultural policy", i.e. resettlement of *immobile* population is very hard task. So we focus ourselves on discussion of first two policies.

Trade policy

What is the substantial difference between the regular and hooked patterns? Suppose that we start slowly moving from autarky $\varphi = 0$ toward full trade liberalization $\varphi = 1$ being in stable long run equilibrium state. In regular case we shall move along "northern" coastline until meeting sustain point φ_1^S and then staying in stable agglomerated equilibrium $\lambda = 1$. There is no abrupt junctions on our way. In contrast to this, on the hooked coastline in the most eastern point we face the "catastrophe" — sharp leap from interior equilibrium to the agglomerated one.

In other words, under sufficiently large asymmetry tariff decreasing unambiguously leads to agglomeration in big country, leaving no chance to the small one. An the only way for government to change situation is to increase local demand, which may be considered as substitution for local immobile population. One can say that asymmetry is more favorable for bigger country. On the other hand, under moderate asymmetry trade policy allows for smaller country to obtain the favorable conditions for efficient application of industral labor policy.

Industrial labor policy

Suppose that government's purpose is to become an industrial Core. The taken measures and their result depend on economy Pattern and starting point.

It is obvious that in case of Pattern 4 any policy is ineffective — for any given φ there is unique long run equilibrium. The same holds for Pattern 3 in the case sufficiently small trade freeness, more exactly, $\varphi \leqslant \varphi_0^S$. For larger freeness there are two stable long run equilibria — the agglomerated ones, and "historical choice" of one of them is *path dependent*.

Pattern 2 is more sophisticated. We have well-known path dependency for sufficiently large trade freeness $\varphi > \varphi_0^S$, and full determinacy for all $\varphi \leqslant \varphi_1^S$ and $\varphi^U \leqslant \varphi \leqslant \varphi_0^S$ where any industrial labor policy is useless. As for values $\varphi_1^S < \varphi < \varphi^U$ result depends on agricultural asymmetry. The bigger country (Home in our case) could reach the industrial Core position using relatively small policy energies (especially if φ is sufficiently close to U-turn point φ^U), while any policy of smaller (i.e. Foreign) country is ineffective.

Finally, for Pattern 1 all conclusions of symmetric CP model still hold. The only inessential difference is that sustain points for each country may be different $\varphi_0^S \neq \varphi_1^S$. More exactly, for sufficiently large trade freeness $\varphi > \varphi^U$ we obtain full agglomeration with path dependency, for $\varphi_0^S < \varphi \leqslant \varphi^U$ both countries may reach Core position, though the smaller country needs substantially more political energies. In the case $\varphi_1^S < \varphi \leqslant \varphi_0^S$ only Home (i.e. larger country) may be efficient, finally, in the case $\varphi \leqslant \varphi_1^S$ result is predetermined any policy is ineffective.

In other words, under sufficiently large asymmetry an industral labor policy turns out inefficient as well as trade one. As for industral labor policy under moderate asymmetry it is also more favorable to bigger country, however, if trade barriers are sufficient small, the industral labor policy of smaller country may be effective, but it may be too expensive. On the other hand, under sufficiently large tariff barriers industral labor policy is inefficient for both countries. Under intermediate tariff barriers situation is ambiguous but more favorable to bigger country.

Moreover, when the competing countries are almost equal, the ambiguity of direction enables an attempt to redirect the agglomeration process towards your region. Then two remarks are in place:

(a) in some cases the critical mass of these efforts is needed: small measures do not pay, but big ones do;

(b) in contrast, sometimes propaganda of future development without any real investment is sufficient to determine the direction and self-fulfilling expectations make the process self-enforcing.

A Appendix A

A.1 Short Run Equilibrium: Existence and Uniqueness

We get the following two equations (6), (7), connecting nominal wages with exogenous parameters

$$\mu\left(\frac{w_H\cdot\lambda+\theta\cdot\frac{L_a}{L}}{\lambda w_H^{1-\sigma}+(1-\lambda)\varphi\cdot {w_F}^{1-\sigma}}+\frac{\varphi\left(w_F(1-\lambda)+(1-\theta)\frac{L_a}{L}\right)}{\lambda\varphi {w_H}^{1-\sigma}+(1-\lambda)w_F^{1-\sigma}}\right)=w_H^{\sigma},$$

$$\lambda w_H+(1-\lambda)w_F=\frac{\mu}{1-\mu}\cdot\frac{L_a}{L}.$$

Using the second equation we can substitute $\dfrac{1-\mu}{\mu}(\lambda w_H+(1-\lambda)w_F)$ instead of $\frac{L_a}{L}$ and after simplifying transformations we obtain the following equation:

$$\begin{aligned}(1-\lambda)\left[1-((1-\theta)+\mu\theta)(1-\varphi^2)-\varphi\cdot\left(\tfrac{w_H}{w_F}\right)^{\sigma}\right]-\\ -\lambda\left[\left(1-((1-\mu)\theta+\mu)\left(1-\varphi^2\right)\right)\tfrac{w_H}{w_F}-\varphi\cdot\left(\tfrac{w_H}{w_F}\right)^{1-\sigma}\right]=0\end{aligned}\tag{11}$$

defining equilibrium relative wage rate $\frac{w_H}{w_F}$. This equation is analytically insolvable (except the case $\sigma=2$ with cubic equation).

Denote $x=\left(\frac{w_H}{w_F}\right)^{\sigma}$, $A(\mu,\varphi,\theta)=1-((1-\theta)+\mu\theta)(1-\varphi^2)$, $B(\mu,\varphi,\theta)=1-((1-\mu)\theta+\mu)(1-\varphi^2)$. Note that $0<A(\mu,\varphi,\theta)<1$, $0<B(\mu,\varphi,\theta)<1$. Now we can rewrite the previous equation as follows

$$F(x;\mu,\varphi,\theta,\lambda)=(1-\lambda)(A(\mu,\varphi,\theta)-\varphi\cdot x)-\lambda x^{1-\rho}(B(\mu,\varphi,\theta)-\varphi x^{-1})=0.\tag{12}$$

For fixed values of parameters $(\mu,\varphi,\theta,\lambda)$ we shall simplify notion of left-hand side to $F(x)$.

Lemma 1. ***(Existence and uniqueness)***

i) Function $F(x)$ decreases with respect to x.

ii) For all admissible values of $(\mu,\varphi,\theta,\lambda)$ there exists the unique positive root $\bar{x}$ of equation $F(x)=0$ which belongs to interval (φ,φ^{-1}).

iii) This root $\bar{x}>1$ if and only if $\theta>\lambda+\frac{\varphi(1-2\lambda)}{(1+\varphi)(1-\mu)}$. Analogously, $\bar{x}<1$ if and only if $\theta<\lambda+\frac{\varphi(1-2\lambda)}{(1+\varphi)(1-\mu)}$.

iv) For all admissible values of $(\mu,\varphi,\theta,\lambda)$ there exists the unique short-run equilibrium.

Proof. Note that

$$\frac{\partial F}{\partial x}=-\left((1-\lambda)\varphi+(1-\rho)B(\mu,\varphi,\theta)\lambda x^{-\rho}+\rho\cdot\lambda\varphi x^{-\rho-1}\right)<0,$$

for all admissible values of parameters and $x>0$, i.e. $F(x)$ is a strictly decreasing function w.r.t. $x>0$. On the other hand,

$$F(\varphi)=(1-\lambda)(1-\mu)\theta(1-\varphi^2)+\lambda\varphi^{1-\rho}((1-\mu)\theta+\mu)(1-\varphi^2)>0,$$

$$F(\varphi^{-1})=-(1-\lambda)((1-\theta)+\mu\theta)(1-\varphi^2)-\lambda\varphi^{\rho-1}(1-\mu)(1-\theta)(1-\varphi^2)<0$$

It implies that there exists a unique root $\bar{x}$ of equation $F(x)=0$ which belongs to interval (φ,φ^{-1}).

This interval always contains the value 1 that distinguishes between cases $\bar{x}<1$ and $\bar{x}>1$. From decreasing of function $F(x)$ follows that for equilibrium values of wage rates $1>\bar{x}\iff F(1)<F(\bar{x})=0$ and $F(1)>F(\bar{x})=0$ analogously. On the other hand,

$$F(1)=(1-\lambda)(A(\mu,\varphi,\theta)-\varphi)-\lambda(B(\mu,\varphi,\theta)-\varphi)=(1-\varphi)\left((1+\varphi)(1-\mu)(\theta-\lambda)-\varphi(1-2\lambda)\right).$$

It implies that

$$\bar{x} < 1 \iff \theta < \lambda + \frac{\varphi(1-2\lambda)}{(1+\varphi)(1-\mu)}, \; \bar{x} > 1 \iff \theta > \lambda + \frac{\varphi(1-2\lambda)}{(1+\varphi)(1-\mu)}.$$

The root $\bar{x}$ defines the equilibrium *relative* wage as follows $\frac{w_H}{w_F} = (\bar{x})^{1/\sigma}$. On the other hand, the following equation

$$\lambda w_H + (1-\lambda) w_F = \frac{\mu}{1-\mu} \cdot \frac{L_a}{L}$$

allows to obtain the nominal wages as follows:

$$w_F = \frac{\mu}{1-\mu} \cdot \frac{L_a}{L\left(\lambda \frac{w_H}{w_F} + (1-\lambda)\right)} = \frac{\mu}{1-\mu} \cdot \frac{L_a}{L\left(\lambda \bar{x}^{1/\sigma} + (1-\lambda)\right)},$$

$$w_H = w_F \cdot \frac{w_H}{w_F} = \frac{\mu}{1-\mu} \cdot \frac{L_a \cdot \bar{x}^{1/\sigma}}{L\left(\lambda \bar{x}^{1/\sigma} + (1-\lambda)\right)}$$

The rest of equilibrium values, i.e. prices, demand and number of firms, may be explicitly expressed via wages and model parameters (see (2)-(5)). □

A.2 Comparative Statics of Wages

Considering root of equation (12) as implicit function $x(\theta, \lambda, \varphi, \mu, \sigma)$ we study its comparative statics, i.e. reaction on changes in certain parameters. Note that sign of implicit function derivative

$$\operatorname{sign}\frac{\partial x}{\partial v} = \operatorname{sign}\left(-\left(\frac{\partial F}{\partial v}\right) / \left(\frac{\partial F}{\partial x}\right)\right) = \operatorname{sign}\frac{\partial F}{\partial v} \tag{13}$$

for all $v \in \{\theta, \lambda, \varphi, \mu\}$, because $\dfrac{\partial F}{\partial x} < 0$. Besides $\frac{w_H}{w_F} = x^{\frac{1}{\sigma}} = x^{1-\rho}$ strictly increase with respect to x, thus comparative statics of x and relative wage $\frac{w_H}{w_F}$ are equivalent.

Comparative statics with respect to θ

Lemma 2. *Relative wage $\frac{w_H}{w_F}$ increases with respect to θ.*

Proof. Calculating partial derivative we obtain

$$\frac{\partial F}{\partial \theta} = (1-\mu)(1-\varphi^2)((1-\lambda) + \lambda x^{1-\rho}) > 0$$

for all admissible parameter values and $x > 0$, therefore $\dfrac{\partial x}{\partial \theta} > 0$ due to (13). □

Lemma 3. *Nominal wage w_H in home region H increases with respect to agricultural labor share θ, while w_F decreases.*

Proof. Consider a nominal wage w_H as an implicit function of parameters. From equation (7)

$$\lambda w_H + (1-\lambda) w_F = \frac{\mu}{1-\mu} \cdot \frac{L_a}{L}$$

we obtain

$$w_H = w_H(y) = \frac{\mu}{1-\mu} \cdot \frac{L_a}{L \cdot \left(\lambda + \dfrac{(1-\lambda)}{y}\right)}$$

where $y = \frac{w_H}{w_F}$ is a relative wage. It is obvious that $w_H(y)$ increases with respect to y, i.e.

$$\frac{\partial w_H}{\partial y} = \frac{\mu}{1-\mu} \cdot \frac{L_a}{L} \cdot \frac{1-\lambda}{(\lambda y + (1-\lambda))^2} > 0$$

and $\dfrac{\partial y}{\partial \theta} > 0$ (see Lemma 2). Thus $\dfrac{\partial w_H}{\partial \theta} = \dfrac{\partial w_H}{\partial y}\dfrac{\partial y}{\partial \theta} > 0$ for all admissible values of parameters. □

Comparative statics with respect to λ

We assume (without loss of generality) that home region H posses the greater share of agricultural labor than foreign F, i.e. $\theta \geqslant 1/2$. Consider the following value $\alpha = 2\theta - 1 \in [0,1)$ interpreted as *measure of asymmetry* in agricultural population.

Lemma 4. *1. Relative wage $\frac{w_H}{w_F}$ strictly increases with respect to λ if and only if $\varphi > \sqrt{\frac{1-\alpha^2}{\left(\frac{1+\mu}{1-\mu}\right)^2 - \alpha^2}}$. Respectively, $\frac{w_H}{w_F}$ strictly decreases with respect to λ if and only if $\varphi < \sqrt{\frac{1-\alpha^2}{\left(\frac{1+\mu}{1-\mu}\right)^2 - \alpha^2}}$.*

2. Threshold value $\sqrt{\frac{1-\alpha^2}{\left(\frac{1+\mu}{1-\mu}\right)^2 - \alpha^2}}$ strictly decreases with respect to asymmetry α.

3. If $\varphi > \dfrac{1-\mu}{1+\mu}$, relative wage $\frac{w_H}{w_F}$ increases with respect to λ regardless of asymmetry measure.

Proof. Due to (13) it is sufficient to determine sign of $\dfrac{\partial F}{\partial \lambda}$ in short run equilibrium $\bar{x}$. Represent function $F(x)$ as follows $F(x) = (1-\lambda)F_1(x) - \lambda F_2(x)$ where $F_1(x) = A(\mu,\varphi,\theta) - \varphi x$, $F_2(x) = x^{1-\rho}(B(\mu,\varphi,\theta) - \varphi x^{-1})$. Note that $\dfrac{\partial F_1}{\partial x} < 0$, $\dfrac{\partial F_2}{\partial x} > 0$ for all $x > 0$. Let $\bar{x}$ be a root of equation $F(x) = 0$, then $F_2(\bar{x}) = \dfrac{1-\lambda}{\lambda}F_1(\bar{x})$ and consequently

$$\frac{\partial F}{\partial \lambda}(\bar{x}) = -(F_1(\bar{x}) + F_2(\bar{x})) = -\frac{1}{\lambda}F_1(\bar{x}).$$

It remains to determine the sign of $F_1(\bar{x})$. Define

$$x^* = \frac{A(\mu,\varphi,\theta)}{\varphi},\, x^{**} = \frac{\varphi}{B(\mu,\varphi,\theta)},$$

which are the positive roots of equations $F_1(x) = 0$ and $F_2(x) = 0$ respectively. Note that $\bar{x} \in [\min\{x^*, x^{**}\}, \max\{x^*, x^{**}\}]$ since $F_1(\bar{x})$ and $F_2(\bar{x})$ should have the same sign. Direct calculations show that

$$x^* > x^{**} \iff \varphi < \sqrt{\frac{(1-\mu)^2\theta(1-\theta)}{(1-\mu)^2\theta(1-\theta)+\mu}} = \sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2 - (2\theta-1)^2}}.$$

Substitute $\alpha = 2\theta - 1$ and consider all of possible cases:

$\varphi < \sqrt{\frac{1-\alpha^2}{\left(\frac{1+\mu}{1-\mu}\right)^2 - \alpha^2}}$, it implies $x^* > x^{**}$, thus $\bar{x} < x^*$ and $F_1(\bar{x}) > F_1(x^*) = 0$, because $\dfrac{\partial F_1}{\partial x} < 0$. In this case $\dfrac{\partial F}{\partial \lambda}(\bar{x}) < 0$.

$\varphi > \sqrt{\frac{1-\alpha^2}{\left(\frac{1+\mu}{1-\mu}\right)^2 - \alpha^2}}$, it implies $x^{**} > x^*$, thus $\bar{x} > x^*$ and $F_1(\bar{x}) < F_1(x^*) = 0$, because $\dfrac{\partial F_1}{\partial x} < 0$. In this case $\dfrac{\partial F}{\partial \lambda}(\bar{x}) > 0$.

$\varphi = \sqrt{\frac{1-\alpha^2}{\left(\frac{1+\mu}{1-\mu}\right)^2 - \alpha^2}}$, it implies $\bar{x} = x^* = x^{**}$ and $F_1(\bar{x}) = F(\bar{x}) = 0$. In this case $\dfrac{\partial F}{\partial \lambda}(\bar{x}) = 0$.

It is obvious that threshold value $\sqrt{\frac{1-\alpha^2}{\left(\frac{1+\mu}{1-\mu}\right)^2 - \alpha^2}}$ decreases with respect to α, thus it reaches maximum value $\dfrac{1-\mu}{1+\mu}$ under $\alpha = 0$ (or $\theta = 1/2$). Consequently, for all $\varphi > \dfrac{1-\mu}{1+\mu}$ relative wage $\frac{w_H}{w_F}$ increases w.r.t λ. □

A.3 Comparative Statics of Price Indices

Consumer's Price Index in Home region (see, for example, Combes et al, 2008, 3.1.1.3) are equal to

$$P_H = \left(\int_0^{N_H} p_{HH}^{1-\sigma}(j)\,\mathrm{d}j + \int_0^{N_F} p_{FH}^{1-\sigma}(j)\,\mathrm{d}j \right)^{\frac{1}{1-\sigma}}.$$

Analogously, price index in region F is equal to

$$P_F = \left(\int_0^{N_H} p_{HF}^{1-\sigma}(j)\,\mathrm{d}j + \int_0^{N_F} p_{FF}^{1-\sigma}(j)\,\mathrm{d}j \right)^{\frac{1}{1-\sigma}}.$$

Substituting equilibrium values of prices (4), (5), and numbers of firms $N_H = \dfrac{\lambda L}{f\sigma}$, $N_F = \dfrac{(1-\lambda)L}{f\sigma}$ we obtain

$$P_H = \left(N_H \left(\frac{w_H m\sigma}{\sigma-1} \right)^{1-\sigma} + N_F \left(\frac{w_F m\sigma\tau}{\sigma-1} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}} = \frac{m\cdot\sigma}{\sigma-1} \left(\frac{L}{f\sigma} \right)^{\frac{1}{1-\sigma}} \left(\lambda w_H^{1-\sigma} + \varphi\cdot(1-\lambda) w_F^{1-\sigma} \right)^{\frac{1}{1-\sigma}},$$

$$P_F = \frac{m\cdot\sigma}{\sigma-1} \cdot \left(\frac{L}{f\sigma} \right)^{\frac{1}{1-\sigma}} \left(\lambda\varphi w_H^{1-\sigma} + (1-\lambda) w_F^{1-\sigma} \right)^{\frac{1}{1-\sigma}}.$$

Comparative statics with respect to θ

Lemma 5. *Price index P_H increases with respect to θ while P_F decreases, thereby relative price index $\dfrac{P_H}{P_F}$ increases with respect to agricultural population share θ.*

Proof. From equation (7)

$$\lambda w_H + (1-\lambda) w_F = \frac{\mu}{1-\mu} \cdot \frac{L_a}{L}$$

we obtain

$$w_F = \frac{\mu}{1-\mu} \cdot \frac{L_a}{L\left(\lambda \frac{w_H}{w_F} + (1-\lambda) \right)}.$$

Therefore

$$P_H = \frac{m\cdot\sigma}{\sigma-1} \cdot \left(\frac{L}{f\cdot\sigma} \right)^{\frac{1}{1-\sigma}} \left({w_F}^{1-\sigma} \cdot \left(\lambda \left(\frac{w_H}{w_F} \right)^{1-\sigma} + (1-\lambda)\varphi \right) \right)^{\dfrac{1}{1-\sigma}} =$$

$$= \frac{m\cdot\sigma}{\sigma-1} \cdot \left(\frac{L}{f\cdot\sigma} \right)^{\frac{1}{1-\sigma}} \frac{\mu L_a}{(1-\mu)L} \cdot \left(\left(\lambda + (1-\lambda) \left(\frac{w_H}{w_F} \right)^{-1} \right)^{\sigma-1} \cdot \left(\lambda + (1-\lambda)\varphi \left(\frac{w_H}{w_F} \right)^{\sigma-1} \right) \right)^{\dfrac{1}{1-\sigma}}$$

Recall that $\sigma > 1$, therefore P_H increases if and only if

$$\left(\lambda + (1-\lambda) \left(\frac{w_H}{w_F} \right)^{-1} \right)^{\sigma-1} \cdot \left(\lambda + (1-\lambda)\varphi \left(\frac{w_H}{w_F} \right)^{\sigma-1} \right)$$

decreases. Equilibrium relative wage $\frac{w_H}{w_F}$ is increasing function with respect to θ (see Lemma 2), thus is is sufficient to prove that the following function

$$G(y) = \left(\lambda + (1-\lambda)\varphi y^{\sigma-1} \right) \left(\lambda + \frac{1-\lambda}{y} \right)^{\sigma-1}.$$

decreases with respect to y. Note that

$$\frac{\partial G}{\partial y} = (1-\lambda)\varphi(\sigma-1)y^{\sigma-2}\left(\lambda+\frac{1-\lambda}{y}\right)^{\sigma-1} - (\sigma-1)\left(\lambda+(1-\lambda)\varphi y^{\sigma-1}\right)\left(\lambda+\frac{1-\lambda}{y}\right)^{\sigma-2}\frac{1-\lambda}{y^2} =$$

$$= (1-\lambda)(\sigma-1)\left(\lambda+\frac{1-\lambda}{y}\right)^{\sigma-2}\left(\varphi y^{\sigma-2}\left(\lambda+\frac{1-\lambda}{y}\right) - \frac{(\lambda+(1-\lambda)\varphi y^{\sigma-1})}{y^2}\right) =$$

$$= \frac{\lambda(1-\lambda)(\sigma-1)}{y^2}\left(\lambda+\frac{1-\lambda}{y}\right)^{\sigma-2}(\varphi y^{\sigma}-1) < 0$$

if $y^{\sigma} < \varphi^{-1}$. On the other hand, for equilibrium relative wage $y^{\sigma} = \left(\frac{w_H}{w_F}\right)^{\sigma} \in (\varphi, \varphi^{-1})$ (see Lemma 1-ii). Finally

$$\frac{\partial}{\partial \theta}G(y(\theta)) = \frac{\partial G}{\partial y}\cdot\frac{\partial y}{\partial \theta} > 0,$$

i.e. price index always increases with respect to θ. Analogously price index in foreign region

$$P_F = \frac{m\cdot\sigma}{\sigma-1}\cdot\left(\frac{L}{f\cdot\sigma}\right)^{\frac{1}{1-\sigma}}\left(\lambda\varphi w_H^{1-\sigma}+(1-\lambda)w_F^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

increases with respect to its share of agricultural labor $1-\theta$ and therefore decreases with respect to θ. Increasing of relative price index $\dfrac{P_H}{P_F}$ is a trivial consequence of these two statements. $\square$

Comparative statics with respect to λ

Lemma 6. *Relative price index $\dfrac{P_H}{P_F}$ decreases with respect to industrial labor share λ.*

Proof. Consider relative price index $\dfrac{P_H}{P_F}(\lambda)$ as a function of industrial labor share λ. Note that

$$\frac{P_H}{P_F}(\lambda) = \left(\frac{\lambda w_H^{1-\sigma}(\lambda)+(1-\lambda)\varphi {w_F}^{1-\sigma}(\lambda)}{\lambda\varphi w_H^{1-\sigma}(\lambda)+(1-\lambda){w_F}^{1-\sigma}(\lambda)}\right)^{\frac{1}{1-\sigma}}$$

which is equivalent to

$$\left(\frac{P_H}{P_F}(\lambda)\right)^{\sigma-1} = \frac{\lambda\cdot\varphi+(1-\lambda)\left(\frac{w_H}{w_F}(\lambda)\right)^{\sigma-1}}{\lambda+(1-\lambda)\varphi\left(\frac{w_H}{w_F}(\lambda)\right)^{\sigma-1}}.$$

Moreover, $x = \left(\dfrac{w_H}{w_F}(\lambda)\right)^{\sigma}$ is a root of equation

$$(1-\lambda)(A-\varphi\cdot x) - \lambda x^{1-\rho}(B-\varphi x^{-1}) = 0$$

where $A = 1-((1-\theta)+\mu\theta)(1-\varphi^2)$, $B = 1-((1-\mu)\theta+\mu)(1-\varphi^2)$.

Consider the special case $A\cdot B = \varphi^2$, then the root $x = \dfrac{A}{\varphi} = \dfrac{\varphi}{B}$ and

$$\left(\frac{P_H}{P_F}(\lambda)\right)^{\sigma-1} = \frac{\lambda\cdot\varphi+(1-\lambda)\left(\frac{A}{\varphi}\right)^{\frac{\sigma-1}{\sigma}}}{\lambda+(1-\lambda)\varphi\left(\frac{A}{\varphi}\right)^{\frac{\sigma-1}{\sigma}}} = \frac{\left(\frac{A}{\varphi}\right)^{\frac{\sigma-1}{\sigma}}+\lambda\cdot\left(\varphi-\left(\frac{A}{\varphi}\right)^{\frac{\sigma-1}{\sigma}}\right)}{\varphi\left(\frac{A}{\varphi}\right)^{\frac{\sigma-1}{\sigma}}+\lambda\cdot\left(1-\varphi\left(\frac{A}{\varphi}\right)^{\frac{\sigma-1}{\sigma}}\right)}.$$

Note that

$$\frac{\partial}{\partial\lambda}\left(\frac{\left(\frac{A}{\varphi}\right)^{\frac{\sigma-1}{\sigma}}+\lambda\cdot\left(\varphi-\left(\frac{A}{\varphi}\right)^{\frac{\sigma-1}{\sigma}}\right)}{\varphi\left(\frac{A}{\varphi}\right)^{\frac{\sigma-1}{\sigma}}+\lambda\cdot\left(1-\varphi\left(\frac{A}{\varphi}\right)^{\frac{\sigma-1}{\sigma}}\right)}\right)=-\frac{\left(\frac{A}{\varphi}\right)^{\frac{\sigma-1}{\sigma}}(1-\varphi^2)}{\left(\varphi\left(\frac{A}{\varphi}\right)^{\frac{\sigma-1}{\sigma}}+\lambda\cdot\left(1-\varphi\left(\frac{A}{\varphi}\right)^{\frac{\sigma-1}{\sigma}}\right)\right)^2}<0,$$

i.e. in this case $\left(\frac{P_H}{P_F}(\lambda)\right)^{\sigma-1}$ strictly decreases as well as $\frac{P_H}{P_F}(\lambda)$.

Let $A\cdot B\neq\varphi^2$ then from equation (8) we obtain

$$\frac{\lambda}{(1-\lambda)}=\frac{A-\varphi x}{x^{1-\rho}(B-\varphi x^{-1})}=\frac{x^\rho(A-\varphi x)}{Bx-\varphi},$$

thus

$$\left(\frac{P_H}{P_F}(\lambda)\right)^{\sigma-1}=\frac{\frac{\lambda}{(1-\lambda)}\cdot\varphi+x^\rho}{\frac{\lambda}{(1-\lambda)}+\varphi\cdot x^\rho}=\frac{\frac{x^\rho(A-\varphi x)}{Bx-\varphi}\cdot\varphi+x^\rho}{\frac{x^\rho(A-\varphi x)}{Bx-\varphi}+\varphi\cdot x^\rho}=\frac{(A-1)\cdot\varphi+(B-\varphi^2)x}{(A-\varphi^2)+(B-1)\cdot\varphi\cdot x}=S(x).$$

Note that relative price index $\frac{P_H}{P_F}(\lambda)$ increases if and only if $S(x(\lambda))$ increases, moreover, $\frac{\partial S}{\partial\lambda}=\frac{\partial S}{\partial x}\cdot\frac{\partial x}{\partial\lambda}$. The first multiplicand

$$\frac{\partial S}{\partial x}=\frac{\partial}{\partial x}\left(\frac{(A-1)\cdot\varphi+(B-\varphi^2)x}{(A-\varphi^2)+(B-1)\cdot\varphi\cdot x}\right)=\frac{(1-\varphi^2)\cdot(AB-\varphi^2)}{((A-\varphi^2)+(B-1)\cdot\varphi\cdot x)^2},$$

therefore $\frac{\partial S}{\partial x}>0$ if and only if $A\cdot B>\varphi^2$. On the other hand, it is easily to see that

$$A\cdot B>\varphi^2\iff\varphi<\sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}}$$

and $\frac{\partial x}{\partial\lambda}<0$ (see proof of Lemma 3). The last implication follows from Lemma 4. Analogously,

$$A\cdot B<\varphi^2\iff\varphi>\sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}}$$

and $\frac{\partial x}{\partial\lambda}>0$. It implies that in case $A\cdot B\neq\varphi^2$ derivative $\frac{\partial S}{\partial\lambda}(x(\lambda))=\frac{\partial S}{\partial x}\cdot\frac{\partial x}{\partial\lambda}<0$, i.e. relative price index $\frac{P_H}{P_F}(\lambda)$ decreases for all admissible values of parameters. □

Lemma 7. *Price indicies P_H and P_F are equal if and only if the following identity holds*

$$\frac{1-((1-\theta)+\mu\theta)(1-\varphi)}{1-((1-\mu)\theta+\mu)(1-\varphi)}=\left(\frac{\lambda}{1-\lambda}\right)^{\frac{\sigma}{\sigma-1}}.$$

Inequality $P_H>P_F$ holds if and only if

$$\theta>\frac{(1-\mu)+\mu\varphi}{(1-\mu)(1-\varphi)}-\frac{(1-\mu)+(1+\mu)\varphi}{(1-\mu)(1-\varphi)\left(1+\left(\frac{\lambda}{1-\lambda}\right)^{\frac{\sigma}{\sigma-1}}\right)}$$

or, equivalently,

$$\frac{\lambda}{1-\lambda} < \left(\frac{1-((1-\theta)+\mu\theta)(1-\varphi)}{1-((1-\mu)\theta+\mu)(1-\varphi)}\right)^{\rho}.$$

Proof. Note that equation

$$\frac{P_H}{P_F} = \left(\frac{\lambda w_H^{1-\sigma}+(1-\lambda)\varphi {w_F}^{1-\sigma}}{\lambda\varphi w_H^{1-\sigma}+(1-\lambda){w_F}^{1-\sigma}}\right)^{\frac{1}{1-\sigma}} = 1$$

is equivalent to $\left(\frac{w_H}{w_F}\right)^{\sigma-1} = \frac{\lambda}{1-\lambda}$. Substituting this term twice into equation (11)

$$(1-\lambda)\left[1-((1-\theta)+\mu\theta)(1-\varphi^2)-\varphi\cdot\left(\frac{w_H}{w_F}\right)^{\sigma-1}\cdot\left(\frac{w_H}{w_F}\right)\right]-$$
$$-\lambda\left[\left(1-((1-\mu)\theta+\mu)\left(1-\varphi^2\right)\right)\frac{w_H}{w_F}-\frac{\varphi}{\left(\frac{w_H}{w_F}\right)^{\sigma-1}}\right]=0,$$

leaving term $\frac{w_H}{w_F}$ untouched, we obtain the following identity $\frac{w_H}{w_F} = \frac{1-\lambda}{\lambda}\cdot\frac{1-((1-\theta)+\mu\theta)(1-\varphi)}{1-((1-\mu)\theta+\mu)(1-\varphi)}$. From the other hand, identity $\left(\frac{w_H}{w_F}\right)^{\sigma-1} = \frac{\lambda}{1-\lambda}$ implies $\frac{w_H}{w_F} = \left(\frac{\lambda}{1-\lambda}\right)^{\frac{1}{\sigma-1}}$, therefore

$$\frac{1-((1-\theta)+\mu\theta)(1-\varphi)}{1-((1-\mu)\theta+\mu)(1-\varphi)} = \left(\frac{\lambda}{1-\lambda}\right)^{\frac{\sigma}{\sigma-1}}$$

is an necessary and sufficient condition of price indices equalization. Consider this identity as as an equation with respect to θ. Then its solution is

$$\theta^* = \frac{(1-\mu)+\mu\varphi}{(1-\mu)(1-\varphi)} - \frac{(1-\mu)+(1+\mu)\varphi}{(1-\mu)(1-\varphi)\left(1+\left(\frac{\lambda}{1-\lambda}\right)^{\frac{\sigma}{\sigma-1}}\right)}.$$

Relative price index $\dfrac{P_H}{P_F}$ increases with respect to θ therefore $\theta > \theta^* \iff \frac{P_H}{P_F} > 1 \iff P_H > P_F$ and $\theta < \theta^* \iff P_H < P_F$. Analogously, equalization condition may be considered as equation with respect to λ withe the unique root λ^* satisfying

$$\frac{\lambda^*}{1-\lambda^*} = \left(\frac{1-((1-\theta)+\mu\theta)(1-\varphi)}{1-((1-\mu)\theta+\mu)(1-\varphi)}\right)^{\rho}.$$

Note that function $\dfrac{\lambda}{1-\lambda}$ increases with respect to λ, while relative price index $\dfrac{P_H}{P_F}$ decreases, therefore

$$\frac{P_H}{P_F}(\lambda) > 1 \iff \lambda < \lambda^* \iff \frac{\lambda}{1-\lambda} < \frac{\lambda^*}{1-\lambda^*}$$

. □

A.4 Comparative Statics of Industrial Welfares

Now consider the welfare (or *real wage*) of industrial workers in both regions

$$V_H = \frac{w_H}{P_H^{\mu}} = \left(\frac{\sigma-1}{m\cdot\sigma}\right)^{\mu}\cdot\left(\frac{L}{f\cdot\sigma}\right)^{\frac{\mu}{\sigma-1}}\cdot\frac{w_H}{\left(\lambda w_H^{1-\sigma}+(1-\lambda)\varphi {w_F}^{1-\sigma}\right)^{\frac{\mu}{1-\sigma}}}.$$

Analogously,

$$V_F = \frac{w_F}{P_F^{\mu}} = \left(\frac{\sigma-1}{m\cdot\sigma}\right)^{\mu} \cdot \left(\frac{L}{f\cdot\sigma}\right)^{\frac{\mu}{\sigma-1}} \cdot \frac{w_F}{\left(\lambda\varphi w_H^{1-\sigma} + (1-\lambda) {w_F}^{1-\sigma}\right)^{\frac{\mu}{1-\sigma}}}.$$

Comparative statics with respect to θ

Lemma 8. *Welfare $V_H(\theta)$ increases with respect to θ, while $V_F(\theta)$ decreases, thus relative welfare $\dfrac{V_H}{V_F}(\theta)$ increases with respect to θ.*

Proof. Representing welfare formula as follows

$$V_H(\theta) = \left(\frac{\sigma-1}{m\cdot\sigma}\right)^{\mu} \cdot \left(\frac{L}{f\cdot\sigma}\right)^{\frac{\mu}{\sigma-1}} \cdot w_H^{1-\mu}(\theta) \left(\lambda + (1-\lambda)\varphi \cdot \left(\frac{w_H}{w_F}(\theta)\right)^{\sigma-1}\right)^{\frac{\mu}{\sigma-1}}$$

and recalling that both relative wage $\frac{w_H}{w_F}(\theta)$ and nominal wage $w_H(\theta)$ increase with respect to θ (see Lemma 2 and Lemma 4), we obtain that welfare $V_H(\theta)$ increases with θ. Analogously, the foreign welfare

$$V_F(\theta) = \left(\frac{\sigma-1}{m\cdot\sigma}\right)^{\mu} \cdot \left(\frac{L}{f\cdot\sigma}\right)^{\frac{\mu}{\sigma-1}} \cdot w_F^{1-\mu}(\theta) \left(\lambda\varphi + (1-\lambda) \cdot \left(\frac{w_H}{w_F}(\theta)\right)^{1-\sigma}\right)^{\frac{\mu}{\sigma-1}}.$$

decreases with respect to θ. Therefore, relative welfare $\dfrac{V_H}{V_F}(\theta)$ increases with respect to θ. □

Comparative statics with respect to λ

The relative welfare considered as function of industrial labor share is equal to

$$\frac{V_H}{V_F}(\lambda) = \frac{w_H}{w_F}(\lambda) \cdot \frac{\left(\lambda\varphi w_H^{1-\sigma}(\lambda) + (1-\lambda) {w_F}^{1-\sigma}(\lambda)\right)^{\frac{\mu}{1-\sigma}}}{\left(\lambda w_A^{1-\sigma}(\lambda) + (1-\lambda)\varphi {w_B}^{1-\sigma}(\lambda)\right)^{\frac{\mu}{1-\sigma}}} =$$

$$= \left(\left(\frac{w_H}{w_F}(\lambda)\right)^{\frac{\sigma-1}{\mu}} \cdot \frac{\frac{\lambda}{(1-\lambda)} + \left(\frac{w_H}{w_F}(\lambda)\right)^{\sigma-1}\varphi}{\frac{\lambda}{(1-\lambda)}\varphi + \left(\frac{w_H}{w_F}(\lambda)\right)^{\sigma-1}}\right)^{\frac{\mu}{\sigma-1}}.$$

Recall that $x = \left(\frac{w_H}{w_F}\right)^{\sigma}$ is a root of equation

$$(1-\lambda)(A - \varphi\cdot x) - \lambda x^{1-\rho}(B - \varphi x^{-1}) = 0,$$

where $A = 1 - ((1-\theta) + \mu\theta)(1-\varphi^2)$, $B = 1 - ((1-\mu)\theta + \mu)(1-\varphi^2)$, $0 < A < 1$, $0 < B < 1$.

Suppose that $A\cdot B \neq \varphi^2$ or $\varphi \neq \sqrt{\dfrac{1-(2\theta-1)^2}{\left(\dfrac{1+\mu}{1-\mu}\right)^2 - (2\theta-1)^2}}$, then

$$\frac{\lambda}{1-\lambda} = \frac{A-\varphi x}{x^{1-\rho}(B-\varphi x^{-1})} = \frac{x^{\rho}(A-\varphi x)}{Bx-\varphi}$$

and

$$\frac{V_H}{V_F}(\lambda,\varphi) = \left(R(x(\lambda),\varphi)\right)^{\frac{\mu}{\sigma-1}}, \tag{14}$$

where $x(\lambda)$ is implicit function defined by equation mentioned above and

$$R(x,\varphi)=x^{\frac{\rho}{\mu}}\frac{\frac{x^{\rho}(A-\varphi x)}{Bx-\varphi}+\varphi x^{\rho}}{\frac{x^{\rho}(A-\varphi x)}{Bx-\varphi}\varphi+x^{\rho}}=x^{\frac{\rho}{\mu}}\frac{(1-\mu)\theta-((1-\mu)\theta+\mu)\varphi x}{(1-\mu)(1-\theta)x-((1-\theta)+\mu\theta)\varphi}.$$

Note that $\frac{\mu}{\sigma-1}>0$ thus $\operatorname{sign}\frac{\partial}{\partial\lambda}\frac{V_H}{V_F}(\lambda)=\operatorname{sign}\frac{\partial}{\partial\lambda}R(x(\lambda),\varphi)=\operatorname{sign}\frac{\partial R}{\partial x}\operatorname{sign}\frac{\partial x}{\partial\lambda}$. Let $a=(1-\mu)(1-\theta)$, $b=((1-\theta)+\mu\theta)\varphi$, $c=(1-\mu)\theta$, $d=((1-\mu)\theta+\mu)\varphi$, then

$$\frac{\partial R}{\partial x}=x^{\frac{\rho}{\mu}-1}\cdot\frac{\frac{\rho}{\mu}\cdot(ax-b)(c-dx)+(bd-ac)x}{(ax-b)^2}=\frac{x^{\frac{\rho}{\mu}-1}}{(ax-b)^2}\cdot Q(x)$$

where $Q(x)=\frac{\rho}{\mu}\cdot(ax-b)(c-dx)+(bd-ac)x$ and $\operatorname{sign}\frac{\partial R}{\partial x}=\operatorname{sign}Q(x)$.

Lemma 9. *Let $\varphi\geqslant\sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}}$ then relative welfare function $\frac{V_H}{V_F}(\lambda,\varphi)$ increases with respect to industrial labor share λ.*

Proof. Note that

$$\frac{V_H}{V_F}=\frac{\frac{w_H}{w_F}(\lambda)}{\left(\frac{P_H}{P_F}(\lambda)\right)^{\mu}}$$

By Lemma 4 for all $\varphi\geqslant\sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}}$ relative wage $\frac{w_H}{w_F}(\lambda)$ is strictly positive non-decreasing function with respect to λ, while relative price index strictly decreases with respect to λ due to Lemma 6. Therefore relative welfare $\frac{V_H}{V_F}$ increases with respect to λ. □

Define the following threshold value that will be called "break-point"

$$\varphi^B(\rho,\mu,\theta)=\frac{\rho-\mu}{\rho+\mu}\sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}}$$

Lemma 10. *i) Let Black-Hole condition $\rho\leqslant\mu$ holds, then relative welfare $\frac{V_H}{V_F}$ strictly increases with respect to industrial labor share λ.*

ii) Let No-Black-Hole condition $\rho>\mu$ holds and $\varphi>\varphi^B(\rho,\mu,\theta)$, then relative welfare $\frac{V_H}{V_F}$ strictly increases with respect to industrial labor share λ.

iii) Let No-Black-Hole condition $\rho>\mu$ holds and $\varphi\leqslant\varphi^B(\rho,\mu,\theta)$, then there are real values $0<\lambda_1(\theta)\leqslant\lambda_2(\theta)$ coinciding only for $\varphi=\varphi^B(\theta)$ such that relative welfare $\frac{V_H}{V_F}(\lambda)$ increases for all $\lambda<\lambda_1(\theta)$ and $\lambda>\lambda_2(\theta)$ while for all $\lambda_1(\theta)<\lambda<\lambda_2(\theta)$ it decreases.

Proof. i) The case $\varphi\geqslant\sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}}$ was considered in Lemma 9. Now assume that

$$\varphi<\sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}}\iff bd-ac<0\iff\frac{\partial x}{\partial\lambda}<0.$$

Then

$$Q(x) = \frac{\rho}{\mu} \cdot (ax-b)(c-dx) + (bd-ac)x \leqslant (ax-b)(c-dx) + (bd-ac)x = -(adx^2 - 2bdx + bc) < 0,$$

due to Black-Hole condition $\frac{\rho}{\mu} \leqslant 1$ and the fact that discriminant $D = 4b^2d^2 - 4(ad)(bc) = 4bd(bd-ac) < 0$. It implies that $\frac{\partial R}{\partial x} < 0$, $\frac{\partial x}{\partial \lambda} < 0$, thus $\frac{\partial}{\partial \lambda}\frac{V_H}{V_F} > 0$.

ii) Let $\sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}} > \varphi > \frac{\rho-\mu}{\rho+\mu}\sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}}$, then $bd-ac<0$ and $\frac{\partial x}{\partial \lambda} < 0$. Moreover quadratic form

$$Q(x) = \frac{\rho}{\mu} \cdot (ax-b)(c-dx) + (bd-ac)x = -\frac{\rho}{\mu}adx^2 + (\frac{\rho-\mu}{\mu}ac + \frac{\rho+\mu}{\mu}bd)x - \frac{\rho}{\mu}bc$$

is strictly negative if and only if its discriminant

$$D = \frac{abcd}{\mu^2}\left((\rho-\mu)^2\frac{ac}{bd} + (\rho+\mu)^2\frac{bd}{ac} - 2(\rho^2+\mu^2)\right) < 0.$$

It is light to see that this takes place if and only if $1 < \frac{ac}{bd} < \left(\frac{\rho+\mu}{\rho-\mu}\right)^2$. The first inequality is valid due to assumption $bd-ac<0$ and the second one

$$\frac{ac}{bd} = \frac{(1-\mu)^2\theta(1-\theta)}{\varphi^2(\mu+(1-\mu)^2\theta(1-\theta))} = \frac{1-(2\theta-1)^2}{\varphi^2\left(\left(\frac{1+\mu}{1-\mu}\right)^2 - (2\theta-1)^2\right)} < \left(\frac{\rho+\mu}{\rho-\mu}\right)^2$$

is equivalent to $\varphi > \frac{\rho-\mu}{\rho+\mu}\sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}}$. Also for all $\frac{\rho-\mu}{\rho+\mu}\sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}} < \varphi < \sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}}$ the following inequalities hold: $\frac{\partial R}{\partial x} < 0$ and $\frac{\partial x}{\partial \lambda} < 0$ therefore $R(x(\lambda))$ strictly increases with respect to λ as well as $\frac{V_H}{V_F}(\lambda)$.

iii) Consider the following quadratic equation

$$Q(x) = -\frac{\rho}{\mu}adx^2 + (\frac{\rho-\mu}{\mu}ac + \frac{\rho+\mu}{\mu}bd)x - \frac{\rho}{\mu}bc = 0.$$

Previous considerations imply that under assumptions $\rho > \mu$ and $\varphi \leqslant \frac{\rho-\mu}{\rho+\mu}\sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}}$ its discriminant is non-negative and there exists two real roots $\xi_2 \leqslant \xi_1$ of equation $Q(x) = 0$ coinciding each other only for $\varphi = \frac{\rho-\mu}{\rho+\mu}\sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}}$. Note that $Q(x) > 0$ for all $\xi_2 < x < \xi_1$ while for $x < \xi_2$ or $x > \xi_1$ imply $Q(x) < 0$. Moreover by definition signs of $Q(x)$and derivative $\frac{\partial R}{\partial x}$ coincide.

In considered case $\varphi \leqslant \frac{\rho-\mu}{\rho+\mu}\sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}} < \sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}}$ which implies $\frac{\partial x}{\partial \lambda} < 0$ (see Lemma 3). Therefore inverse function $\lambda(x)$ (that may be explicitly obtained from equation (8)) decreases with respect to x. Define $\lambda_1 = \lambda(\xi_1)$, $\lambda_2 = \lambda(\xi_2)$, then $\lambda_1 \leqslant \lambda_2$ and they are equal iff $\varphi = \frac{\rho-\mu}{\rho+\mu}\sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}}$. Moreover, derivative $\frac{\partial R}{\partial \lambda}(x(\lambda),\varphi) = \frac{\partial R}{\partial x}\frac{\partial x}{\partial \lambda} < 0$ if and only if $\lambda_1 < \lambda < \lambda_2$ and it is positive for $\lambda < \lambda_1$ or $\lambda > \lambda_2$. Note that in the case $\varphi = \dfrac{\rho-\mu}{\rho+\mu}\sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}}$ function $R(x(\lambda),\varphi)$ is strictly increasing because $\lambda_1=\lambda_2$ and interval of decreasing is empty.

It is not guaranteed that both $\lambda_1,\lambda_2\in(0,1)$ and numerical simulations show that, for example, case $1<\lambda_1<\lambda_2$ is quite possible. Note that, in $\lambda=0$ relative welfare always increases (see Lemma 11 below) which implies either $0<\lambda_1$ or $\lambda_2<0$. However, the second case is impossible because in No-Black-Hole case for sufficiently small $\varphi>0$ we have $\dfrac{V_H}{V_F}(0)>1$ and $\dfrac{V_H}{V_F}(1)<1$ (see Lemma 13 in Appendix B), i.e. relative welfare should somewhere decrease between $\lambda=0$ and $\lambda=1$. $\square$

Let's define one more threshold value that could be called "turn point"

$$\varphi^T(\theta)=\sqrt{\max\left\{0,\frac{(1-\theta)(\theta\rho-(1+\theta\rho)\mu)}{\theta((1-\theta)\rho+(1+\theta\rho)\mu)}\right\}}.$$

Note that for all $\mu\geqslant\dfrac{\theta\rho}{1+\theta\rho}$ identity $\varphi^T(\theta)\equiv 0$ holds.

Lemma 11. *For all $0<\varphi<1$ relative welfare $\dfrac{V_H}{V_F}(\lambda,\varphi)$ increases in $\lambda=0$, i.e. $\dfrac{\partial}{\partial\lambda}\dfrac{V_H}{V_F}(0,\varphi)>0$. Relative welfare $\dfrac{V_H}{V_F}(\lambda,\varphi)$ increases in $\lambda=1$, i.e. $\dfrac{\partial}{\partial\lambda}\dfrac{V_H}{V_F}(1,\varphi)>0$ if and only if $\varphi>\varphi^T(\theta)$, in particular, $\dfrac{\partial}{\partial\lambda}\dfrac{V_H}{V_F}(1,\varphi)>0$ for all φ in the case of $\mu\geqslant\dfrac{\theta\rho}{1+\theta\rho}$.*

Proof. Due to Lemma 9 it is sufficient to consider the case $\varphi<\sqrt{\frac{1-(2\theta-1)^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-(2\theta-1)^2}}$ only. In this case $\dfrac{\partial x}{\partial\lambda}<0$ and $\dfrac{V_H}{V_F}(\lambda,\varphi)$ increases if and only if $\dfrac{\partial R}{\partial x}<0$. In turn, $x=x(\lambda)$ is a root of equation

$$(1-\lambda)(A-\varphi\cdot x)-\lambda x^{1-\rho}(B-\varphi x^{-1})=0,$$

where $A=1-((1-\theta)+\mu\theta)(1-\varphi^2)$, $B=1-((1-\mu)\theta+\mu)(1-\varphi^2)$. Consider two special cases $\lambda=0$ and $\lambda=1$, then the corresponding roots are $x=\dfrac{A}{\varphi}$ and $x=\dfrac{\varphi}{B}$. Recall that $\dfrac{\partial R}{\partial x}<0$ if and only if $Q(x)<0$ where

$$Q(x)=\frac{\rho}{\mu}\cdot(ax-b)(c-dx)+(bd-ac)x,$$

$a=(1-\mu)(1-\theta)$, $b=((1-\theta)+\mu\theta)\varphi$, $c=(1-\mu)\theta$, $d=((1-\mu)\theta+\mu)\varphi$. Due to assumption $\theta\geqslant 1/2$ we consider an asymmetry measure $\alpha=2\theta-1\in[0,1)$. Substituting $\dfrac{1+\alpha}{2}$ instead of θ we obtain, after some transformations, that

$$Q\left(\frac{A}{\varphi}\right)=\frac{(1-\mu)^2(1-\alpha^2)-\left((1+\mu)^2-(1-\mu)^2\alpha^2\right)\varphi^2}{16\mu\varphi}f(\alpha,\rho,\mu,\varphi),$$

where

$$f(\alpha,\rho,\mu,\varphi)=-[(1-\mu)(1+\alpha)(2+(1-\alpha)\rho)\mu-(1-\alpha)\rho)+((1-\alpha)+(1+\alpha)\mu)((1+\alpha)\rho+(2+(1-\alpha)\rho)\mu\varphi^2)].$$

Note that the first multiplicand is positive due to assumption $\varphi^2<\dfrac{1-\alpha^2}{\left(\dfrac{1+\mu}{1-\mu}\right)^2-\alpha^2}=\dfrac{(1-\alpha^2)(1-\mu)^2}{(1+\mu)^2-(1-\mu)^2\alpha^2}$.

Moreover,

$$f(\alpha,\rho,\mu,\varphi) < -(1+\alpha)[(1-\mu)((2+(1-\alpha)\rho)\mu-(1-\alpha)\rho)+((1-\alpha)+(1+\alpha)\mu)\rho]$$

because $\varphi^2 > 0$, thus $Q\left(\dfrac{A}{\varphi}\right) < 0$ if and only if

$$g(\alpha,\rho,\mu) = (1-\mu)((2+(1-\alpha)\rho)\mu-(1-\alpha)\rho)+((1-\alpha)+(1+\alpha)\mu)\rho > 0.$$

Routine calculations show that under assumptions $0 \leqslant \alpha \leqslant 1, 0 \leqslant \mu \leqslant 1, 0 \leqslant \rho \leqslant 1$ minimum value of polinomial function $g(\alpha,\rho,\mu)$ is equal to 0 and it is reached on the set of points $(\alpha,\rho,0)$ for arbitrary $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \rho \leqslant 1$. Therefore for all *admissible* values (e.g. $\mu > 0$) the term $g(\alpha,\rho,\mu)$ is strictly positive. The case $\lambda = 0$ is considered.

Now let $\lambda = 1$ and study the sign of $Q\left(\frac{\varphi}{B}\right)$. After some transformations we get

$$Q\left(\frac{\varphi}{B}\right) = \frac{(1-\mu)\varphi\left[(1-\mu)^2(1-\alpha^2)-\left((1+\mu)^2-(1-\mu)^2\alpha^2\right)\varphi^2\right]}{4\mu((1-\alpha)(1-\mu)+(1+\mu+(1-\mu)\alpha)\varphi^2)^2} h(\alpha,\rho,\mu,\varphi),$$

where

$$h(\alpha,\rho,\mu,\varphi) = (1-\alpha)((1+\alpha)(1-\mu)\rho-2\mu)-(1+\alpha)((1-\alpha)\rho+(2+(1+\alpha)\rho)\mu)\varphi^2.$$

Note that the first multiplicand is strictly positive due to assumption

$$\varphi^2 < \frac{1-\alpha^2}{\left(\dfrac{1+\mu}{1-\mu}\right)^2-\alpha^2} = \frac{(1-\alpha^2)(1-\mu)^2}{(1+\mu)^2-(1-\mu)^2\alpha^2}.$$

On the other hand, $h(\alpha,\rho,\mu,\varphi)$ may positive or negative depending on parameter's relations. More exactly, $h(\alpha,\rho,\mu,\varphi) < 0$ if and only if

$$\varphi^2 > \frac{(1-\alpha)((1+\alpha)\rho-(2+(1+\alpha)\rho)\mu)}{(1+\alpha)((1-\alpha)\rho+(2+(1+\alpha)\rho)\mu)}.$$

Reverse substitution $\alpha = 2\theta - 1$ transforms this inequality into

$$\varphi^2 > \frac{(1-\theta)(\theta\rho-(1+\theta\rho)\mu)}{\theta((1-\theta)\rho+(1+\theta\rho)\mu)}.$$

Note that under condition $\mu \geqslant \dfrac{\theta\rho}{1+\theta\rho}$ numerator is negative and this inequality holds for all φ. $\square$

B Appendix B

B.1 Stability of agglomerated equilibria

Direction of labor migration flow depends on welfare levels in both regions. For example, $V_H(\lambda) > V_F(\lambda)$ for some $0 < \lambda < 1$ causes in-flow, i.e. increasing of λ and vice versa. As for limit cases $\lambda = 0$ or $\lambda = 1$ we interpret condition $V_H(0) < V_F(0)$ (or $\dfrac{V_H}{V_F}(0) < 1$) as *stable* state, because $\lambda = 0$ cannot be decreased further, while opposite inequality characterizes *unstable* situation. Analogously, $\lambda = 1$ is *stable* if $V_H(1) > V_F(1) \iff \dfrac{V_H}{V_F}(1) > 1$. Note that the same result may be obtained by linearizing of *ad-hoc* equation $\dot{\lambda} = \lambda \cdot (1-\lambda)(V_A(\lambda) - V_B(\lambda))$ in neighborhoods of steady states $\lambda = 0$ and $\lambda = 1$.

Recall that by convention the home agricultural labor share satisfies condition $\theta \geqslant 1/2$. Consider the following value $\alpha = 2\theta - 1 \in [0,1)$ that may be interpreted as *measure of asymmetry* in agricultural labor.

Lemma 12. *In the case of $\mu \geqslant \rho$ **(Black-Hole condition)** both agglomerated states $\lambda = 0$ and $\lambda = 1$ are stable regardless of other parameter values.*

Proof. Note that

$$\frac{V_H}{V_F}(\lambda) = \frac{w_H}{w_F}(\lambda) \cdot \frac{\left(\lambda \varphi w_H^{1-\sigma}(\lambda) + (1-\lambda) {w_F}^{1-\sigma}(\lambda)\right)^{\frac{\mu}{1-\sigma}}}{\left(\lambda w_H^{1-\sigma}(\lambda) + (1-\lambda)\varphi {w_F}^{1-\sigma}(\lambda)\right)^{\frac{\mu}{1-\sigma}}} =$$

$$= \left(\left(\frac{w_H}{w_F}(\lambda)\right)^{\frac{\sigma-1}{\mu}} \cdot \frac{\left(\lambda \left(\frac{w_H}{w_F}(\lambda)\right)^{1-\sigma} + (1-\lambda)\varphi\right)}{\left(\lambda\varphi \left(\frac{w_H}{w_F}(\lambda)\right)^{1-\sigma} + (1-\lambda)\right)} \right)^{\frac{\mu}{\sigma-1}}$$

where $\frac{w_H}{w_F}(\lambda)$ is a root equation

$$(1-\lambda)\left((1-\mu)\theta + ((1-\theta)+\mu\theta)\varphi^2\right) - \left(1 - ((1-\mu)\theta+\mu)\left(1-\varphi^2\right)\right)\lambda \cdot \tfrac{w_H}{w_F} -$$
$$-(1-\lambda)\varphi \cdot \left(\tfrac{w_H}{w_F}\right)^{\sigma} + \lambda\varphi \cdot \left(\tfrac{w_H}{w_F}\right)^{1-\sigma} = 0.$$

This equation allows explicit solution in the cases $\lambda = 0$ or $\lambda = 1$.

For $\lambda = 0$ we obtain that

$$\frac{V_H}{V_F}(0) = \lim_{\lambda \to 0} \frac{V_H}{V_F}(\lambda) = \left(\theta(1-\mu)\varphi^{\frac{\mu-\rho}{\rho}} + \varphi^{\frac{\mu+\rho}{\rho}}(1-\theta(1-\mu))\right)^{\frac{1}{\sigma}},$$

where $\rho = \frac{\sigma-1}{\sigma}$. Analogously,

$$\frac{V_H}{V_F}(1) = \left((1-\theta)(1-\mu)\varphi^{\frac{\mu-\rho}{\rho}} + \varphi^{\frac{\mu+\rho}{\rho}}(1-(1-\theta)(1-\mu))\right)^{-\frac{1}{\sigma}}.$$

Consider the following functions

$$G(\mu,\rho,\varphi,\theta) = \theta(1-\mu)\varphi^{\frac{\mu-\rho}{\rho}} + (1-\theta(1-\mu))\varphi^{\frac{\mu+\rho}{\rho}},$$

$$H(\mu,\rho,\varphi,\theta) = (1-\theta)(1-\mu)\varphi^{\frac{\mu-\rho}{\rho}} + (1-(1-\theta)(1-\mu))\varphi^{\frac{\mu+\rho}{\rho}}$$

then agglomerated equilibrium $\lambda = 0$ is stable if and only if $G(\mu,\rho,\varphi,\theta) < 1$ and $\lambda = 1$ is stable if and only if $H(\mu,\rho,\varphi,\theta) < 1$. Under Black-Hole condition $\mu \geqslant \rho$ both function G and H increase with respect to φ. Moreover $G(\mu,\rho,1,\theta) = H(\mu,\rho,1,\theta) = 1$ thus $G(\mu,\rho,\varphi,\theta) < 1$ and $H(\mu,\rho,\varphi,\theta) < 1$ for all $0 < \varphi < 1$. □

Lemma 13. *Suppose that* ***No-Black-Hole condition*** $\mu < \rho$ *holds.*

1. Let $\mu \geqslant \dfrac{\rho}{1+2\rho}$ *then for all* $\alpha \in [0,1)$ *there exist* $\varphi_0^S(\alpha)$, $\varphi_1^S(\alpha) \in (0,1)$ *such that* $\varphi_0^S(0) = \varphi_1^S(0)$, *for all* $\alpha \in (0,1)$ *are differentiable and* $\dfrac{\partial \varphi_0^S}{\partial \alpha} > 0$, $\dfrac{\partial \varphi_1^S}{\partial \alpha} < 0$, *in particular,* $0 < \varphi_1^S(\alpha) < \varphi_0^S(\alpha) < 1$. *For any given* α *agglomerated equilibrium* $\lambda = 0$ *is stable if and only if* $\varphi > \varphi_0^S(\alpha)$ *and* $\lambda = 1$ *is stable if and only if* $\varphi > \varphi_0^S(\alpha)$.

2. Let $\mu < \dfrac{\rho}{1+2\rho}$ *and* $\alpha^* = \dfrac{(1+\rho)\mu}{(1-\mu)\rho}$. *Then for all* $\alpha \in [0,\alpha^*)$ *there exist* $\varphi_0^S(\alpha)$, $\varphi_1^S(\alpha) \in (0,1)$ *such that* $\varphi_0^S(0) = \varphi_1^S(0)$, *for all* $\alpha \in (0,1)$ *are differentiable and* $\dfrac{\partial \varphi_0^S}{\partial \alpha} > 0$, $\dfrac{\partial \varphi_1^S}{\partial \alpha} < 0$, *in particular,* $0 < \varphi_1^S(\alpha) < \varphi_0^S(\alpha) < 1$. *For all* $\alpha \in [\alpha^*,1)$ *values* $\varphi_0^S(\alpha) \equiv 1$ *while* $\varphi_1^S(\alpha) < 1$. *In any case, for any given* α *agglomerated equilibrium* $\lambda = 0$ *is stable if and only if* $\varphi > \varphi_0^S(\alpha)$ *and* $\lambda = 1$ *is stable if and only if* $\varphi > \varphi_0^S(\alpha)$.

Proof. Note that under No-Black-Hole condition $\rho > \mu$ we have $\lim\limits_{\varphi\to 0} G(\mu,\rho,\varphi,\theta) = +\infty$, $G(\mu,\rho,1,\theta) = 1$. The first derivative

$$\frac{\partial G}{\partial \varphi} = \theta(1-\mu)\frac{\mu-\rho}{\rho}\varphi^{\frac{\mu-2\rho}{\rho}} + \frac{\mu+\rho}{\rho}(1-\theta(1-\mu))\varphi^{\frac{\mu}{\rho}}$$

is equal to zero in the single point $\varphi^* = \sqrt{\frac{\theta(1-\mu)(\rho-\mu)}{(1-\theta(1-\mu))(\rho+\mu)}}$. Moreover, the second derivative

$$\frac{\partial^2 G}{\partial \varphi^2} = \theta(1-\mu)\frac{\mu-\rho}{\rho}\cdot\frac{\mu-2\rho}{\rho}\varphi^{\frac{\mu-3\rho}{\rho}} + \frac{\mu+\rho}{\rho}\cdot\frac{\mu}{\rho}(1-\theta(1-\mu))\varphi^{\frac{\mu-\rho}{\rho}} > 0,$$

i.e. G is a convex function with respect to φ and φ^* is a minimum point for any fixed θ, μ and $\rho > \mu$. The same holds for function $H(\varphi)$ except that the minimum point value is equal to $\varphi^{**} = \sqrt{\frac{(1-\theta)(1-\mu)(\rho-\mu)}{(1-(1-\theta)(1-\mu))(\rho+\mu)}}$.

Recall that we assume $\theta \geqslant 1/2$ and consider $\alpha = 2\theta - 1$ as a measure of agricultural population' asymmetry. Substituting $\theta = \dfrac{1+\alpha}{2}$ we obtain the following terms of minimum points φ^*, φ^{**} as functions of asymmetry measure α

$$\varphi^*(\alpha) = \sqrt{\frac{(1+\alpha)(1-\mu)(\rho-\mu)}{((1+\mu)-(1-\mu)\alpha)(\rho+\mu)}},\ \varphi^{**} = \sqrt{\frac{(1-\alpha)(1-\mu)(\rho-\mu)}{((1+\mu)+(1-\mu)\alpha)(\rho+\mu)}}$$

Note that for $\theta = 1/2$ (or $\alpha = 0$) functions $G(\mu,\rho,\varphi,1/2) = H(\mu,\rho,\varphi,1/2)$ and consequently $\varphi^{**}(0) = \varphi^*(0)$ while in case of non-zero asymmetry $\alpha = 2\theta - 1 > 0$ (or $\theta > 1/2$) the strict inequality $\varphi^{**}(\alpha) < \varphi^*(\alpha)$ holds.

Note that $\varphi^*(\alpha)$ increases with respect to α thus for all $\alpha < 1$ following inequalities hold

$$\varphi^*(\alpha) = \sqrt{\frac{(1+\alpha)(1-\mu)(\rho-\mu)}{((1+\mu)-(1-\mu)\alpha)(\rho+\mu)}} < \varphi^*(0) = \sqrt{\frac{(1-\mu)(\rho-\mu)}{\mu(\rho+\mu)}}.$$

In the case $\frac{(1-\mu)(\rho-\mu)}{\mu(\rho+\mu)} \leqslant 1$ or equivalently $\mu \geqslant \dfrac{\rho}{1+2\rho}$ value $\varphi^*(\alpha) < 1$ is a unique minimum point of function $G(\mu,\rho,\varphi,\dfrac{1+\alpha}{2})$ therefore $G(\mu,\rho,\varphi^*(\alpha),\dfrac{1+\alpha}{2}) < G(\mu,\rho,1,\dfrac{1+\alpha}{2}) \equiv 1$ while $\lim\limits_{\varphi\to 0} G(\varphi) = +\infty$ and function G strictly decreases with respect to φ in interval $(0,\varphi^*(\alpha))$ consequently there exist the unique point $\varphi_0^S(\alpha) \in (0,\varphi^*(\alpha))$ such that $G(\mu,\rho,\varphi_0^S(\alpha),\dfrac{1+\alpha}{2}) = 1$. Moreover for all $0 < \varphi < \varphi_0^S(\alpha)$ inequality $G(\mu,\rho,\varphi,\dfrac{1+\alpha}{2}) > 1$ holds while $\varphi_0^S(\alpha) < \varphi < 1$ implies $G(\mu,\rho,\varphi,\dfrac{1+\alpha}{2}) < 1$. Analogously, there exists the unique value $\varphi_1^S(\alpha) \in (0,\varphi^{**}(\alpha))$ such that $H(\varphi) < 1$ if and only if $\varphi_1^S(\alpha) < \varphi < 1$.

Now consider sustain points φ_0^S, φ_1^S as functions of asymmetry measure $\alpha = 2\theta - 1 \in [0,1)$. Note that $\varphi_0^S(0) = \varphi_1^S(0)$ because functions $G(\mu,\rho,\varphi,\dfrac{1+\alpha}{2})$ and $H(\mu,\rho,\varphi,\dfrac{1+\alpha}{2})$ coincide for $\alpha = 0$. For all $\alpha \in (0,1)$

and fixed $\rho > \mu$ sustain point function $\varphi_0^S(\alpha)$ is an implicit function defined by equation

$$G(\mu,\rho,\varphi,\frac{1+\alpha}{2}) = \frac{\alpha+1}{2}(1-\mu)\varphi^{\frac{\mu-\rho}{\rho}} + (1-\frac{\alpha+1}{2}(1-\mu))\varphi^{\frac{\mu+\rho}{\rho}} = 1.$$

Note that $\varphi_0^S(\alpha)$ belongs to interval $(0,\varphi^*(\alpha))$ where G decreases with respect to φ i.e. $\dfrac{\partial G}{\partial \varphi}(\varphi_0^S(\alpha)) < 0$. On the other hand,

$$\frac{\partial G}{\partial \alpha} = \frac{(1-\mu)}{2}\left(\varphi^{\frac{\mu-\rho}{\rho}} - \varphi^{\frac{\rho+\mu}{\rho}}\right) = \frac{(1-\mu)}{2}\left((\varphi^{\frac{\mu-\rho}{\rho}}-1)+(1-\varphi^{\frac{\rho+\mu}{\rho}})\right) > 0$$

because $\varphi \in (0,1)$, $\dfrac{\mu-\rho}{\rho} < 0$, $\dfrac{\mu+\rho}{\rho} > 0$. It implies that an implicit function derivative $\dfrac{\partial \varphi_0^S}{\partial \alpha} = -\dfrac{\partial G}{\partial \alpha} \Big/ \dfrac{\partial G}{\partial \varphi} > 0$, i.e. φ_0^S increases with respect to α. Analogous considerations show that φ_1^S decreases with respect to α.

Now consider the case $\mu < \dfrac{\rho}{1+2\rho}$ which is equivalent to $\dfrac{(1+\rho)\mu}{(1-\mu)\rho} < 1$. Note that inequality $\alpha = 2\theta - 1 < \alpha^* = \dfrac{(1+\rho)\mu}{(1-\mu)\rho}$ is equivalent to $\dfrac{(1+\alpha)(1-\mu)(\rho-\mu)}{((1+\mu)-(1-\mu)\alpha)(\rho+\mu)} < 1$ that implies

$$\varphi^*(\alpha) = \sqrt{\frac{(1+\alpha)(1-\mu)(\rho-\mu)}{((1+\mu)-(1-\mu)\alpha)(\rho+\mu)}} < 1.$$

Thus for $\alpha < \alpha^*$ all of the previous considerations still hold. In case of $\alpha \geqslant \alpha^*$ an inequality $\varphi^*(\alpha) = \sqrt{\dfrac{(1+\alpha)(1-\mu)(\rho-\mu)}{((1+\mu)-(1-\mu)\alpha)(\rho+\mu)}} \geqslant 1$ holds. It means that function $G(\varphi) > 1$ for all $\varphi \in (0,1)$ which implies instability of agglomerated equilibrium $\lambda = 0$ for all $\varphi \in (0,1)$, i.e. $\varphi_0^S(\alpha) = 1$. For $\varphi_1^S(\alpha)$ nothing changes because

$$\varphi^{**}(\alpha) = \sqrt{\frac{(1-\alpha)(1-\mu)(\rho-\mu)}{((1+\mu)+(1-\mu)\alpha)(\rho+\mu)}} \leqslant \varphi^{**}(0) = \sqrt{\frac{(1-\mu)(\rho-\mu)}{(1+\mu)(\rho+\mu)}} < 1$$

for all $\alpha \geqslant 0$. □

B.2 Number and Stability of Interior Long Run Equilibria

Note that industrial labor share $0 < \lambda^0 < 1$ defines the interior long run equilibrium if and only if $V_H(\lambda^0) = V_F(\lambda^0)$ or $\dfrac{V_H}{V_F}(\lambda^0) = 1$.

Consider the special case $A \cdot B = \varphi^2$.Then an equation (11) has a unique positive root $\bar{x} = \frac{A}{\varphi}$ while $\lambda \in [0,1]$ may be *arbitrary* (see proof of Lemma 6 in Appendix A). It is easy to see that identity $A \cdot B = \varphi^2$ is equivalent to $\varphi = \sqrt{\dfrac{1-\alpha^2}{\left(\dfrac{1+\mu}{1-\mu}\right)^2 - \alpha^2}}$, where $\alpha = 2\theta - 1$ a measure of asymmetry in agricultural labor. On the other hand, equilibrium relative wage $\dfrac{w_H}{w_F} = (\bar{x})^{\frac{1}{\sigma}}$. Substituting it into interior long run equilibrium equation

$$\frac{V_A}{V_B}(\lambda) = \left(\left(\frac{w_H}{w_F}(\lambda)\right)^{\frac{\sigma-1}{\mu}} \cdot \frac{\lambda + (1-\lambda)\left(\frac{w_H}{w_F}(\lambda)\right)^{\sigma-1}\varphi}{\lambda\varphi + (1-\lambda)\left(\frac{w_H}{w_F}(\lambda)\right)^{\sigma-1}}\right)^{\frac{\mu}{\sigma-1}} = 1$$

we obtain the unique solution

$$\lambda^0 = \frac{\bar{x}^{\rho} - \varphi \cdot \bar{x}^{\frac{\rho(1+\mu)}{\mu}}}{\bar{x}^{\rho} - \varphi \cdot \bar{x}^{\frac{\rho(1+\mu)}{\mu}} + \bar{x}^{\frac{\rho}{\mu}} - \varphi}.$$

Note that condition $\lambda^0 \in (0,1)$ is not guaranteed.

Consider the general case $A \cdot B \neq \varphi^2$ then the interior long run equilibrium equation

$$\frac{V_A}{V_B}(\lambda) = \left(\left(\frac{w_H}{w_F}(\lambda) \right)^{\frac{\sigma-1}{\mu}} \cdot \frac{\lambda + (1-\lambda)\left(\frac{w_H}{w_F}(\lambda)\right)^{\sigma-1}\varphi}{\lambda\varphi + (1-\lambda)\left(\frac{w_H}{w_F}(\lambda)\right)^{\sigma-1}} \right)^{\frac{\mu}{\sigma-1}} = 1$$

is equivalent to $R(x(\lambda)) = 1$ where

$$R(x) = x^{\frac{\rho}{\mu}} \frac{(1-\mu)\theta - ((1-\mu)\theta + \mu)\varphi x}{(1-\mu)(1-\theta)x - ((1-\theta) + \mu\theta)\varphi}$$

(see (14)) and for each root x^0 of this equation the corresponding value of λ^0 may be found as a single root of equation

$$(1-\lambda)(A - \varphi \cdot x^0) - \lambda \left(x^0\right)^{1-\rho} (B - \varphi \left(x^0\right)^{-1}) = 0,$$

where $A = 1 - ((1-\theta) + \mu\theta)(1-\varphi^2)$, $B = 1 - ((1-\mu)\theta + \mu)(1-\varphi^2)$. Note that for this value the case $\lambda \notin (0,1)$ is quite possible, thus not each root of equation $R(x) = 1$ corresponds to some interior long run equilibrium.

Lemma 14. *Let **Black-Hole condition** $\mu \geqslant \rho$ holds. Then there exists the unique interior long run equilibrium and it is unstable.*

Proof. Under Black-Hole condition $\rho \leqslant \mu$ relative welfare $\frac{V_A}{V_B}(\lambda)$ strictly increases with respect to λ (see Lemma 11-i in Appendix A) and $\frac{V_H}{V_F}(0) < 1$, $\frac{V_H}{V_F}(1) > 1$ due to Lemma 12 from Appendix A. It implies that there exists the unique value $\lambda^0 \in (0,1)$ such that $V_H(\lambda^0) = V_F(\lambda^0)$. Moreover, $\frac{\partial}{\partial\lambda}\frac{V_H}{V_F}(\lambda^0) > 0$ (as well as for all $\lambda \in [0,1]$). $\square$

No-Black-Hole Case: Stability Patterns for Interior Long Run Equilibria

Black-Hole case was completely described in Lemma 14, thus we further consider Non-Black-Hole case only. Note that Lemma 10 from Appendix A hold for arbitrary values $\lambda \in (0,+\infty)$ of roots of equation $R(x(\lambda)) = 1$. This root defines long run equilibria only if $0 < \lambda < 1$. Anyway, this lemma implies that there exists threshold value of trade freeness

$$\varphi^B(\rho,\mu,\alpha) = \frac{\rho-\mu}{\rho+\mu}\sqrt{\frac{1-\alpha^2}{\left(\frac{1+\mu}{1-\mu}\right)^2 - \alpha^2}} \in (0,1)$$

dividing admissible values set $(\varphi,\lambda) \in (0,1) \times [0,1]$ into two parts. In rectangle $[\varphi^B, 1) \times [0,1]$ relative welfare $\frac{V_H}{V_F}(\varphi,\lambda)$ is continuous function strictly increasing with respect to λ which implies existence of at most one long run equilibrium for each $\varphi \in [\varphi^B, 1)$. On the other hand, in rectangle $(0,\varphi^B) \times [0,1]$ relative welfare $\frac{V_H}{V_F}(\varphi,\lambda)$ is continuous function with dynamics depending on fact if interval $(0,1)$ covers both values of $\lambda_1 < \lambda_2$, only one value or none. Anyways, for all admissible values of $\lambda < \lambda_1$ relative welfare $\frac{V_H}{V_F}(\varphi,\lambda)$ increases, for $\lambda_1 < \lambda < \lambda_2$ it decreases and for $\lambda > \lambda_2$ increases again. Note that in symmetric case $\theta = \frac{1}{2}$ (or $\alpha = 0$) $\varphi^B(\rho,\mu,0) = \frac{\rho-\mu}{\rho+\mu}\sqrt{\frac{1}{\left(\frac{1+\mu}{1-\mu}\right)^2}} = \frac{(\rho-\mu)(1-\mu)}{(\rho+\mu)(1+\mu)} \in (0,1)$ and coincides with symmetric break-point that ends the "helve of tomahawk" (see Figure 1, left-hand plot).

Lemma 15. *Break-point $\varphi^B(\alpha)$ decreases with respect to α and $\varphi^B(1) = 0$. There exists the unique value $\tilde{\alpha} \in (0,1)$ such that $\varphi^B(\tilde{\alpha}) = \varphi_0^S(\tilde{\alpha})$, $\varphi^B(\alpha) > \varphi_0^S(\alpha)$ for $\alpha \in [0,\tilde{\alpha})$ and $\varphi^B(\alpha) < \varphi_0^S(\alpha)$ for $\alpha \in (\tilde{\alpha},1]$. For all $\alpha \in [0,1]$ inequality $\varphi^B(\alpha) > \varphi_1^S(\alpha)$ holds.*

Proof. It is obvious that $\varphi^B(\rho,\mu,\alpha)$ decreases with respect to α and $\varphi^B(\rho,\mu,1)=0$. Moreover sustain point $\varphi_0^S(\alpha)$ increases with respect to α (see Lemma 9 in Appendix A) and in symmetric case we obtain $\varphi^B(\rho,\mu,0)>\varphi_0^S(0)=\varphi_1^S(0)$ (see, for example, Robert-Nicaud, 2005, Proposition 5). Thus there exists the unique value $\tilde{\alpha}$ such that $\varphi^B(\rho,\mu,\tilde{\alpha})=\varphi_0^S(\tilde{\alpha})$ and $\varphi^B(\rho,\mu,\alpha)>\varphi_0^S(\alpha)$ for $\alpha\in[0,\tilde{\alpha})$, $\varphi^B(\rho,\mu,\alpha)<\varphi_0^S(\alpha)$ for $\alpha\in(\tilde{\alpha},1]$.

Note that both $\varphi^B(\rho,\mu,\alpha)$ and $\varphi_0^S(\alpha)$ converge to 0 for $\alpha\to 1$. By definition of sustain point $\varphi>\varphi_1^S(\alpha)$ if and only if relative welfare in agglomerated state $\lambda=1$ satisfies inequality

$$\frac{V_H}{V_F}=\left(\frac{(1-\alpha)}{2}(1-\mu)\varphi^{\frac{\mu-\rho}{\rho}}+(1-\frac{(1-\alpha)}{2}(1-\mu))\varphi^{\frac{\mu+\rho}{\rho}}\right)^{-\frac{1}{\sigma}}>1$$

(see proof of Lemma 13). To accomplish lemma's proof we need to verify that inequality

$$h(\alpha)=\frac{(1-\alpha)}{2}(1-\mu)\left(\frac{\rho-\mu}{\rho+\mu}\sqrt{\frac{1-\alpha^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-\alpha^2}}\right)^{\frac{\mu-\rho}{\rho}}+(1-\frac{(1-\alpha)}{2}(1-\mu))\left(\frac{\rho-\mu}{\rho+\mu}\sqrt{\frac{1-\alpha^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-\alpha^2}}\right)^{\frac{\mu+\rho}{\rho}}<1$$

holds for all $\alpha\in[0,1)$ and $0<\mu<\rho<1$. Note that $\alpha=0$ satisfies this inequality because $\varphi^B(\rho,\mu,0)>\varphi_1^S(0)$. We shall prove that in fact $h(\alpha)$ is *decreasing* function with respect to α, therefore inequality $h(\alpha)<1$ remains true for all $\alpha>0$.

After differentiation and simplifying we obtain that

$$\frac{\partial h}{\partial\alpha}=\frac{2\mu\left(\frac{\rho-\mu}{\rho+\mu}\sqrt{\frac{1-\alpha^2}{\left(\frac{1+\mu}{1-\mu}\right)^2-\alpha^2}}\right)^{\frac{\mu+\rho}{\rho}}\cdot f(\alpha,\rho,\mu)}{(1-\alpha)(1+\alpha)^2(1-\mu)((1-\alpha)+\mu+\alpha\mu)(\rho-\mu)^2\rho}$$

where

$$f(\alpha,\rho,\mu)=-(1-\mu)^3\rho^2\alpha^4+2(1-\mu)^2\mu\rho^2\alpha^3+(1-\mu)\rho(3\mu^2+(2-\rho)\rho)-2\mu^3(1+\rho)^2\alpha-(1+\mu)\rho(1+\rho)(\rho+\mu^2)$$

and sign of derivative $\frac{\partial h}{\partial\alpha}$ obviously coincides with sign of $f(\alpha,\rho,\mu)$. Routine, yet very tedious, analysis shows that polynomial function $f(\alpha,\rho,\mu)$ reaches its maximum on set $S=\{(\alpha,\rho,\mu)|0\leqslant\alpha\leqslant 1,0\leqslant\mu\leqslant\rho\leqslant 1\}$ in the points $(\alpha,0,0)$ for arbitrary $\alpha\in[0,1]$, this maximum value is equal to 0, thus for all *admissible* values $0<\mu<\rho$ function $f(\alpha,\rho,\mu)<0$. □

Consider various stability patterns of agglomerated long run equilibria described in Lemma 13 and match them the corresponding patterns of interior ones. Recall one more definition of "turn point" φ^T represented as a function of $\alpha=2\theta-1$

$$\varphi^T(\alpha)=\sqrt{\max\left\{0,\frac{(1-\alpha)((1+\alpha)\rho-(2+(1+\alpha)\rho)\mu)}{(1+\alpha)((1-\alpha)\rho+(2+(1+\alpha)\rho)\mu)}\right\}}.$$

Lemma 16. *1. Let* $\varphi<\varphi_1^S(\alpha)$, then both agglomerated equilibria $\lambda=0$ and $\lambda=1$ are unstable, *and there exists the unique stable interior equilibrium.*

2. Let $\varphi>\varphi^B(\rho,\mu,\alpha)$ and $\varphi<\varphi_0^S(\alpha)$, then agglomerated equilibrium $\lambda=0$ is unstable, $\lambda=1$ is stable, *and there are no interior equilibria.*

3. *Let* $\varphi>\varphi^B(\rho,\mu,\alpha)$ and $\varphi>\varphi_0^S(\alpha)$, then both agglomerated equilibria $\lambda=0$ and $\lambda=1$ are stable, *and there exists the unique unstable interior equilibrium.*

4. *Let* $\varphi_1^S(\alpha) < \varphi < \varphi^B(\rho,\mu,\alpha)$, $\varphi^T(\alpha) > \varphi_1^S(\alpha)$ and $\varphi < \varphi_0^S(\alpha)$, then agglomerated equilibrium $\lambda = 0$ is unstable, $\lambda = 1$ is stable *and there are no interior equilibria.*

5. Let $\varphi_1^S(\alpha) < \varphi < \varphi^B(\rho,\mu,\alpha)$, $\varphi^T(\alpha) > \varphi_1^S(\alpha)$ and $\varphi > \varphi_0^S(\alpha)$, then both agglomerated equilibria $\lambda = 0$ and $\lambda = 1$ are stable, *and there exists the unique unstable interior equilibrium.*

6. Let $\varphi^T(\alpha) < \varphi_1^S(\alpha) < \varphi < \varphi^B(\rho,\mu,\alpha)$ and $\varphi < \varphi_0^S(\alpha)$, then agglomerated equilibrium $\lambda = 0$ is unstable, $\lambda = 1$ is stable, *and*

6a. if $\lambda_2 \in (0,1)$ *and* $\frac{V_H}{V_F}(\lambda_2) < 1$ *then there are two interior equilibria, stable and unstable,*

6b. if $\lambda_2 \in (0,1)$ *and* $\frac{V_H}{V_F}(\lambda_2) = 1$ *(non-generic case) then there is unique unstable interior equilibrium,*

6c. in all other cases there are no interior equilibria.

7. Let $\varphi^T(\alpha) < \varphi_1^S(\alpha) < \varphi < \varphi^B(\rho,\mu,\alpha)$ and $\varphi > \varphi_0^S(\alpha)$, then both agglomerated equilibria $\lambda = 0$ and $\lambda = 1$ are stable, *and*

7a. if $\lambda_1, \lambda_2 \in (0,1)$, $\frac{V_H}{V_F}(\lambda_1) > 1$ *and* $\frac{V_H}{V_F}(\lambda_2) < 1$ *then there exists three interior equilibria, one stable and two unstable*

7b. if $\lambda_1, \lambda_2 \in (0,1)$, $\frac{V_H}{V_F}(\lambda_1) = 1$ *or* $\frac{V_H}{V_F}(\lambda_2) = 1$ *(non-generic cases) there are two interior equilibria,*

7c. in all other cases there is unique unstable interior equilibrium.

Proof. For any fixed $\varphi, \mu, \theta, \rho$ relative welfare $\frac{V_H}{V_F}(\lambda)$ may be considered as one-variable function and its plot is continuous (even smooth) curve. Interior equilibria are the intersection points of this curve with unit level line $\frac{V_H}{V_F} = 1$.

Case 1. Condition $\varphi < \varphi_1^S(\alpha) \leqslant \varphi_0^S(\alpha)$ implies that $\frac{V_H}{V_F}(0) > 1$ (i.e. $\lambda = 0$ is unstable), $\frac{V_H}{V_F}(1) < 1$ (i.e. $\lambda = 1$ is unstable), thus plot of $\frac{V_H}{V_F}(\lambda)$ intersects at least once and intersection point λ^0 should belong to decreasing interval (λ_1, λ_2) (see Lemma 12 in Appendix A), i.e. λ^0 is stable interior equilibrium. On the other hand, for all $\lambda > \lambda_2$ function $\frac{V_H}{V_F}(\lambda)$ strictly increases, therefore there are no other intersection points $\frac{V_H}{V_F}(\lambda) = 1$.

Case 2. Let $\varphi \geqslant \varphi^B(\rho,\mu,\alpha)$ and $\varphi < \varphi_0^S(\alpha)$, it implies that $\frac{V_H}{V_F}(0) > 1$ (i.e. $\lambda = 0$ is unstable) and $\frac{V_H}{V_F}(\lambda)$ strictly increases for all λ. Thus $\frac{V_H}{V_F}(1) > 1$ (i.e. $\lambda = 1$ is stable) and there are no intersections $\frac{V_H}{V_F}(\lambda) = 1$.

Case 3. Let $\varphi \geqslant \varphi^B(\rho,\mu,\alpha)$ and $\varphi > \varphi_0^S(\alpha)$, it implies that $\frac{V_H}{V_F}(0) < 1$ (i.e. $\lambda = 0$ is stable) and $\frac{V_H}{V_F}(\lambda)$ strictly increases for all λ. Thus there is unique intersection $\frac{V_H}{V_F}(\lambda^0) = 1$ and $\frac{V_H}{V_F}(1) > 1$ (i.e. $\lambda = 1$ is stable). Moreover, $\frac{\partial}{\partial\lambda}\frac{V_H}{V_F}(\lambda^0) > 0$, i.e. λ^0 is unstable.

Case 4. Let $\varphi_1^S(\alpha) < \varphi < \varphi^B(\rho,\mu,\alpha)$, $\varphi^T(\alpha) > \varphi_1^S(\alpha)$ and $\varphi < \varphi_0^S(\alpha)$, it implies that $\frac{V_H}{V_F}(0) > 1$ (i.e. $\lambda = 0$ is unstable), $\frac{V_H}{V_F}(1) > 1$ (i.e. $\lambda = 1$ is stable) $\frac{\partial}{\partial\lambda}\frac{V_H}{V_F}(0) > 0$ and $\frac{\partial}{\partial\lambda}\frac{V_H}{V_F}(1) < 0$. The last inequality implies that $\lambda = 1$ belongs to decreasing interval (λ_1, λ_2) while $\frac{V_H}{V_F}(1) > 1$. It means that there are no intersection points $\frac{V_H}{V_F}(\lambda) = 1$ for $\lambda \in (0,1)$.

Case 5. Let $\varphi_1^S(\alpha) < \varphi < \varphi^B(\rho,\mu,\alpha)$, $\varphi^T(\alpha) > \varphi_1^S(\alpha)$ and $\varphi > \varphi_0^S(\alpha)$, it implies that $\frac{V_H}{V_F}(0) < 1$ (i.e. $\lambda = 0$ is stable), $\frac{V_H}{V_F}(1) > 1$ (i.e. $\lambda = 1$ is stable) $\frac{\partial}{\partial\lambda}\frac{V_H}{V_F}(0) > 0$ and $\frac{\partial}{\partial\lambda}\frac{V_H}{V_F}(1) < 0$. The last inequality implies that $\lambda = 1$ belongs to decreasing interval (λ_1, λ_2) while $\frac{V_H}{V_F}(1) > 1$. It means that there are no intersection points

$\frac{V_H}{V_F}(\lambda) = 1$ for $\lambda \in (\lambda_1, 1)$ and there is unique intersection point $\frac{V_H}{V_F}(\lambda^0) = 1$ for $\lambda^0 \in (0, \lambda_1)$ which is unstable interior equilibrium.

The rest cases are more diversified. Numerical simulations show that all listed sub-cases are possible.

Case 6. Let $\varphi^T(\alpha) < \varphi_1^S(\alpha) < \varphi < \varphi^B(\rho, \mu, \alpha)$ and $\varphi < \varphi_0^S(\alpha)$, it implies that $\frac{V_H}{V_F}(0) > 1$ (i.e. $\lambda = 0$ is unstable), $\frac{V_H}{V_F}(1) > 1$ (i.e. $\lambda = 1$ is stable) $\frac{\partial}{\partial \lambda}\frac{V_H}{V_F}(0) > 0$ and $\frac{\partial}{\partial \lambda}\frac{V_H}{V_F}(1) > 0$. Sub-case 6a provides exactly two intersection points $\lambda^0 \in (\lambda_1, \lambda_2)$ – stable, and $\lambda^{00} \in (\lambda_2, 1)$ – unstable interior equilibrium. In sub-case 6b these two intersection points run into tangent point λ_2 which is unstable (more exactly, half-stable) interior equilibrium. In all other sub-cases ($\lambda_2 \notin (0,1)$ or $\frac{V_H}{V_F}(\lambda_2) > 1$) there are no intersection points.

Case 7. Let $\varphi^T(\alpha) < \varphi_1^S(\alpha) < \varphi < \varphi^B(\rho, \mu, \alpha)$ and $\varphi > \varphi_0^S(\alpha)$, it implies that $\frac{V_H}{V_F}(0) < 1$ (i.e. $\lambda = 0$ is stable), $\frac{V_H}{V_F}(1) > 1$ (i.e. $\lambda = 1$ is stable) $\frac{\partial}{\partial \lambda}\frac{V_H}{V_F}(0) > 0$ and $\frac{\partial}{\partial \lambda}\frac{V_H}{V_F}(1) > 0$. Conditions $\frac{V_H}{V_F}(0) < 1$ and $\frac{V_H}{V_F}(1) > 1$ provide at least one intersection point for all sub-cases. The rest depends on particular sub-case. □

B.3 Existence of U-turn point

Sub-cases 6a and 7a of Lemma 16 define the most sophisticated structure of long run equilibria. Using "geographic" terminology, introduced in subsection 4.2, one can say that in this cases for any given φ^0 there exist *two* interior long run equilibria reside "on northern coastline". Sub-case 7a provides an additional third interior equilibrium "on southern coastline" that is always unstable. As for two 'northern' equilibria, the lesser value λ^0 is stable, i.e. $\frac{\partial}{\partial \lambda}\frac{V_H}{V_F}(\lambda^0) < 0$, and the greater one λ^{00} is unstable, i.e. $\frac{\partial}{\partial \lambda}\frac{V_H}{V_F}(\lambda^{00}) > 0$ (see proof of **Case 6**, Lemma 16). On the other hand, both points (φ^0, λ^0) and $(\varphi^0, \lambda^{00})$ reside on the same unit equiscalar line $\frac{V_H}{V_F} = 1$ and gradients $\nabla^0 = \text{grad}\frac{V_H}{V_F}(\varphi^0, \lambda^0)$, $\nabla^{00} = \text{grad}\frac{V_H}{V_F}(\varphi^0, \lambda^{00})$ of relative welfare function $\frac{V_H}{V_F}(\varphi, \lambda)$ applied to these points directed 'south-eastwards' and 'north-eastwards' respectively. Note that this "hook" is possible only if gradient in sustain point φ_1^S also directed 'north-eastwards', i.e. $\frac{\partial}{\partial \lambda}\frac{V_H}{V_F}(\varphi_1^S, 1) > 0$ or, by definition, $\varphi^T < \varphi_1^S$ (see Figure 16).

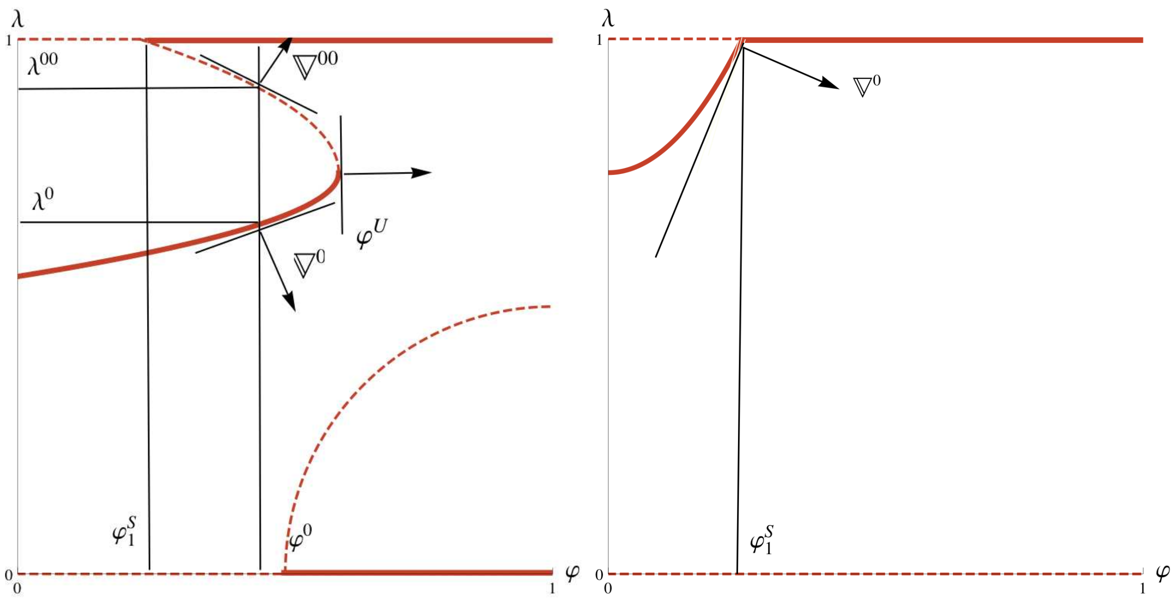


Figure 16: U-turn point vs regular case

U-turn point φ^U is a supremum value for all φ with two 'northern' equilibria $0 < \lambda^0 < \lambda^{00} < 1$. It may be characterized as multiple 'northern' interior equilibrium ($\lambda^0 = \lambda^{00} = \lambda^U$) with "horizontal" gradient, i.e. $\frac{\partial}{\partial\lambda}\frac{V_H}{V_F}(\varphi^U, \lambda^U) = 0$.

Combining all previous conditions we obtain the following system of equations, that allow to calculate U-turn point φ^U for given admissible values of parameters θ, μ, ρ.

1. Short run equilibrium *equation of three variables*

$$F_1(x, \varphi, \lambda) = (1-\lambda)(A - \varphi \cdot x) - \lambda x^{1-\rho}(B - \varphi x^{-1}) = 0,$$

where $x = \left(\frac{w_H}{w_F}\right)^\sigma$, $A = 1 - ((1-\theta) + \mu\theta)(1-\varphi^2)$, $B = 1 - ((1-\mu)\theta + \mu)(1-\varphi^2)$.

2. Interior long run equilibrium condition $V_H(\varphi, \lambda) = V_F(\varphi, \lambda)$, or equivalently,

$$F_2(x, \varphi, \lambda) = x^{\frac{\rho}{\mu}} \cdot \frac{\lambda + (1-\lambda)x^\rho \varphi}{\lambda\varphi + (1-\lambda)x^\rho} - 1 = 0.$$

3. U-turn condition $\frac{\partial}{\partial\lambda}\frac{V_H}{V_F}(\varphi^U, \lambda^U) = 0$ that is equivalent (see proof of Lemma 10 in Appendix A) to the following equation

$$F_3(x, \varphi, \lambda) = \frac{\rho}{\mu} \cdot (ax - b)(c - dx) + (bd - ac)x = 0,$$

where $a = (1-\mu)(1-\theta)$, $b = ((1-\theta) + \mu\theta)\varphi$, $c = (1-\mu)\theta$, $d = ((1-\mu)\theta + \mu)\varphi$.

Solution $(x,^U \varphi^U, \lambda^U)$ of this system (if exists) defines U-turn point in case of its admissibility, i.e. $0 < \varphi^U < 1$, $0 < \lambda^U < 1$. There is no need, however, to solve this system for classification purposes only, because *existence* of U-turn point may be revealed by *analytical* verification of more simple inequality $\varphi^T < \varphi_1^S$.

References

[1] Baldwin, R. and R. Forslid (2000) The Core-Periphery Model and Endogenous Growth: Stabilising and De-Stabilising Integration, *Economica* 67, 307-324.

[2] Baldwin, R. (2001) The Core-Periphery Model with Forward-Looking Expectations, *Regional Science and Urban Economics* 31, 21-49.

[3] Baldwin, R. et al. (2003) *Economic Geography and Public Policy* (Princeton University Press)

[4] Beenstock M., Felsenstein D. (2008) Regional heterogeneity, conditional convergence and regional inequality, *Regional Studies* 42, 4:475–488

[5] Berliant M., Kung, Fan-chin, (2009) *Bifurcations in regional migration dynamics*, Regional Science and Urban Economics 39, 6:714-720.

[6] Cecchetti, S.G. et al (2002) Price Index Convergence among United State Cities, *International Economic Review* 43, 1081-1099.

[7] Combes P.-P. et al. (2008) *Economic Geography* (Princeton University Press)

[8] Cronon, W. (1991) *Nature's Metropolis* (Chicago and the Great West. London: W. W. Norton)

[9] Debreu, G. (1959) *Theory of Value* (Wiley)

[10] Dixit, A. K., and J. E. Stiglitz (1977). Monopolistic competition and optimum product diversity, *American Economic Review* 67, 297–308.

[11] Fujita, M. et al. (1999) *The Spatial Economy: Cities, Regions and International Trade* (Cambridge, MA: MIT Press)

[12] Fujita, M., and J.-F. Thisse (2002) *Economics of Agglomeration: Cities, Industrial Location and Regional Growth* (Cambridge University Press)

[13] Ginsburgh, V. et al. (1985) On existence and stability of spatial equilibria and steady-states, *Regional Science and Urban Economics* 15, 149–158.

[14] Hanson, G.H. (1997). Increasing returns, trade and the regional structure of wages, *Economic Journal* 107, 113–133.

[15] Hanson, G.H. (2005). Market Potential, Increasing Returns, and Geographic Concentration, *Journal of International Economics* 67:1, 1-24

[16] Klaesson, J., H. Larsson (2009). Wages, Productivity and Industry Composition – agglomeration economies in Swedish regions, *CESIS Working Paper No. 203*.

[17] Krugman, P. R. (1980) Scale economies, product differentiation, and the pattern of trade, *American Economic Review* 70, 950–959.

[18] Krugman, P. R. (1991) Increasing Returns and Economic Geography, *Journal of Political Economy* 99, 483-499.

[19] Landes, D. S. (1998) *The Wealth and Poverty of Nations* (London: Abacus)

[20] Mossay, P. (2006) The core–periphery model: a note on the existence and uniqueness of short-run equilibrium, *Journal of Urban Economics* 59, 389–393.

[21] Robert-Nicoud, F. (2005) The structure of simple "New Economic Geography" models, *Journal of Economic Geography* 5, 201–234.

[22] Roos, M.W.M. (2006) Regional price levels in Germany, *Applied Economics* 38, 13:1553-1566

[23] Starrett, D. (1978) Market allocations of location choice in a model with free mobility, *Journal of Economic Theory* 17, 21–37.