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Abstract

This article shows that, for any transferable utility game in coalitional form with nonempty coalition structure core, the number of steps required to switch from a payoff configuration out of the coalition structure core to a payoff configuration in the coalition structure core is less than or equal to $(n^2 + 4n)/4$, where n is the cardinality of the player set. This number considerably improves the upper bound found so far by Kóczy and Lauwers [6].

Keywords: Coalition structure core, excess function, payoff configuration, outsider independent domination.
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1 Introduction

The core (Gillies [3]) is the solution concept for transferable utility games in coalitional form (henceforth TU-games) that collects the efficient allocations satisfying the coalitional rationality property, *i.e.* the total payoff obtained by each coalition of players is at least as large as its worth. The core is a very popular solution concept despite the fact that its nonemptiness is not always guaranteed. In many TU-games with empty cores, the players can reorganize themselves into a partition that generates a total worth larger than the worth of the grand coalition. Even though such TU-games are not superadditive, they can be economically meaningful (see Aumann [1] for examples constructed from bankruptcy problems). Thus this casts some doubt on the notion of efficiency of core allocations and raises the question of which coalitions will form.

In order two take these two features into account, the coalition structure core (cs-core henceforth) has been studied by Shenoy [10], Sengupta and Sengupta [7], Greenberg [4] and Kóczy and Lauwers [6]. A payoff configuration is constituted by an allocation and a partition such that the total payoff of each coalition in the partition is equal to its worth. The cs-core collects the payoff configurations that satisfy the coalitional rationality property. As such, the classical condition of efficiency used for the core is replaced by an efficiency property over all possible partitions, *i.e* the total payoff of a payoff vector in the cs-core is equal to the maximal total worth that a partition of the players can achieve. By considering payoff configurations, the cs-core enables to tackle the question of which coalitions will form.

Conditions for which the cs-core is nonempty are provided by Greenberg [4], but they leave two interesting questions unanswered: if one starts from an arbitrary payoff configuration, can

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the players alter the initial payoff configuration by forming "improving" coalitions in order to eventually reach a cs-core payoff configuration, and how long can this procedure last? Kóczy and Lauwers [6] answer the first question by proving that the cs-core is accessible, *i.e.* for each payoff configuration, there exists a finite sequence of "improving" payoff configurations that terminates in the cs-core. This article answers the second question by showing that such a sequence contains at most $(n^2 + 4n)/4$ elements.

The sequence of improving payoff configurations should be understood as a dynamic bargaining procedure among the players. An arbitrary payoff configuration is given and then the players are free to form coalitions and to bargain with the restriction that if a coalition of players forms, then it cannot claim more than its worth. Given that constraint and a status quo of the bargaining, a coalition is improving if it can redistribute its worth in such a way that none of its members is worse off compared to the status quo and such that at least one of its members is better off. In order to account for the formation of the improving coalition, the status quo partition is reorganized on three aspects: any of its coalitions that has a nonempty intersection with the improving coalition splits into singletons, any of its coalitions that has an empty intersection with the improving coalition remains in the reorganized partition, and of course the improving coalition belongs to the reorganized partition. The improving payoff configuration is composed of the improving coalition together with the associated reorganized partition.

The procedure used by Kóczy and Lauwers [6] possesses the following desirable property, which we preserve in the present article. If a coalition of the current partition has an empty intersection with the deviating coalition, then it belongs to the next partition in the sequence and the payoffs of its members remain unchanged. Also, as in Kóczy and Lauwers [6], a cs-core payoff configuration is used as a target in the sense that the players try to get closer and closer to this cs-core payoff configuration.

Our article continues the line of research on the accessibility of the core-related solution concepts initiated by Sengupta and Sengupta [8], who show that the core is accessible. Bounds on the number of steps required to access the core have been provided by Kóczy [5], Yang [11] and Béal, Rémila and Solal [2].

The rest of the article is organized as follows. Section 2 contains the definitions of TU-games and of the cs-core. Section 3 states the main result. It also contains the construction of the sequence of payoff configurations on which the proof of the main result is based as well as the proofs of intermediary results. Finally, section 4 concludes.

2 Definitions

Let $\subseteq$ denote weak set inclusion and $\subset$ denote proper set inclusion. We use $|S|$ to denote the number of elements in a finite set S.

Throughout this article, we consider a fixed set $N = \{1, \ldots, n\}$ of n players. Nonempty subsets of N are called *coalitions*. A *partition* is a set of disjoint coalitions whose union is N. Denote by $\Pi(N)$ the set of all partitions of N. For a partition $\mathcal{Q}$ and a coalition S, we define the *partners' set* of S in $\mathcal{Q}$ as

$$P(S, \mathcal{Q}) = \bigcup_{T \in \mathcal{Q} : T \cap S \neq \emptyset} T.$$

A *characteristic function* on N is a function $v : 2^N \longrightarrow \mathbb{R}$ that assigns a worth $v(S)$ to each $S \in 2^N$, with the convention that $v(\emptyset) = 0$. A pair (N, v) is a *TU-game* and since N is fixed, we shall refer to the TU-game (N, v) as v. An *individually rational payoff configuration* or simply a *payoff configuration* for the TU-game v is a pair $(x, \mathcal{Q})$ where $\mathcal{Q}$ is a partition (of N) and $x \in \mathbb{R}^n$ is a payoff vector such that

$$\forall i \in N,\ x_i \geq v(\{i\}) \text{ and } \forall S \in \mathcal{Q},\ x(S) = v(S),$$

where $x(S)$ stands for $\sum_{i \in S} x_i$. Two payoff configurations $(x, \mathcal{Q})$ and $(y, \mathcal{R})$ are *payoff equivalent* if $x = y$. Given a payoff configuration $(x, \mathcal{Q})$, the *excess* $e(x, S)$ of a coalition S is given by

$$e(x, S) = v(S) - x(S).$$

The cs-core of the TU-game v is the set $C(v)$ of payoff configurations $(x, \mathcal{Q})$ that satisfy coalitional rationality, *i.e.* for each coalition S we have $e(x, S) \leq 0$. We denote by $\Gamma(N)$ the set of all TU-games on N and with a nonempty cs-core. A first claim provides a property of the payoff configurations in the cs-core.

Claim 1 *Let*

$$v^* = \max_{\mathcal{Q} \in \Pi(N)} \sum_{S \in \mathcal{Q}} v(S)$$

denote the maximal total worth achieved in v among the partitions of N. Then, for each $(x, \mathcal{Q}) \in C(v)$, it holds that $v^* = x(N) = \sum_{S \in \mathcal{Q}} x(S)$.

Proof. Consider any payoff configuration $(x, \mathcal{Q}) \in C(v)$. On the one hand, for each coalition T, we have $v(T) \leq x(T)$, which implies $\sum_{T \in \mathcal{R}} v(T) \leq \sum_{T \in \mathcal{R}} x(T) = x(N)$ for each partition $\mathcal{R} \in \Pi(N)$. Thus $v^* \leq x(N)$. On the other hand, $\sum_{S \in \mathcal{Q}} v(S) = \sum_{S \in \mathcal{Q}} x(S) = x(N)$, which gives $v^* \geq x(N)$. ■

A payoff configuration $(y, \mathcal{R})$ *outsider-independently dominates* or simply *o.i.-dominates* a payoff configuration $(x, \mathcal{Q})$ by a coalition S if:

1. $y(S) > x(S)$ and $y_i \geq x_i$ for each $i \in S$;
2. $S \in \mathcal{R}$ and $T \in \mathcal{R}$ for each $T \in \mathcal{Q}$ such that $T \subseteq N \backslash P(S, \mathcal{Q})$;
3. $y_i = x_i$ for each $i \in N \backslash P(S, \mathcal{Q})$.

A payoff configuration $(y, \mathcal{R})$ is *accessible* from a payoff configuration $(x, \mathcal{Q})$ if:

(i) $(y, \mathcal{R})$ and $(x, \mathcal{Q})$ are payoff equivalent, or

(ii) $(y, \mathcal{R})$ sequentially o.i.-dominates $(x, \mathcal{Q})$, *i.e.* there is a finite sequence of payoff configurations $((x^0, \mathcal{Q}^0), (x^1, \mathcal{Q}^1), \ldots, (x^m, \mathcal{Q}^m))$ with $(x^0, \mathcal{Q}^0) = (x, \mathcal{Q})$ and $(x^m, \mathcal{Q}^m) = (y, \mathcal{R})$, and a sequence of coalitions $(S^1, \ldots, S^m)$ such that for each $k \in \{1, \ldots, m\}$, $(x^k, \mathcal{Q}^k)$ o.i.-dominates $(x^{k-1}, \mathcal{Q}^{k-1})$ by coalition S^k. The integer m is called the number of steps of the accessibility.

3 The result

The main result of this article consists in providing a bound on the accessibility of the cs-core, *i.e.* a maximal number of steps necessary for a payoff configuration in the cs-core to be accessible from any payoff configuration. This result can be stated as follows.

Theorem 1 *Let $v \in \Gamma(N)$ be a TU-game. Then the cs-core is accessible in at most $(n^2 + 4n)/4$ steps.*

From now on, fix any TU-game with a nonempty cs-core $v \in \Gamma(N)$ and $(c, \mathcal{P})$ an element of $C(v)$. Let $p = |\mathcal{P}|$ be the number of coalitions if the partition $\mathcal{P}$, where $p \in \{1, \dots, n\}$. We introduce several notions with respect to $(c, \mathcal{P})$ that will be used in the remainder of the article. A payoff configuration $(x, \mathcal{Q})$ is *$\mathcal{P}$-efficient* if for each $S \in \mathcal{P}$, it holds that $x(S) = v(S)$, and $\mathcal{P}$-inefficient otherwise. For a payoff configuration $(x, \mathcal{Q})$, a player $i \in N$ is *overpaid* if $x_i > c_i$ and *underpaid* if $x_i < c_i$. Also let $O(x) = \{i \in N : x_i > c_i\}$ and $U(x, S) = \{i \in S : x_i < c_i\}$ be the set of overpaid players in $(x, \mathcal{Q})$ and the set of players in a coalition S who are underpaid in $(x, \mathcal{Q})$ respectively. Without any loss of generality, the players in $U(x, S)$ are labeled $i_1, \dots, i_{|U(x,S)|}$ in such a way that for each $q = 1, \dots, |U(x, S)| - 1$, it holds that

$$i_q < i_{q+1} \Longleftrightarrow c_{i_q} - x_{i_q} \leq c_{i_{q+1}} - x_{i_{q+1}}.$$

A second claim provides a useful property of $\mathcal{P}$-inefficient payoff configurations.

Claim 2 *For any $\mathcal{P}$-inefficient payoff configuration $(x, \mathcal{Q})$, there exists a coalition $S \in \mathcal{P}$ such that $e(x, S) > 0$.*

Proof. On the one hand, we have $\sum_{S \in \mathcal{Q}} v(S) = \sum_{S \in \mathcal{Q}} x(S) = x(N) = \sum_{S \in \mathcal{P}} x(S)$. On the other hand, from Claim 1, we have $\sum_{S \in \mathcal{Q}} v(S) \leq \sum_{S \in \mathcal{P}} v(S)$. Thus we have $\sum_{S \in \mathcal{P}} x(S) \leq \sum_{S \in \mathcal{P}} v(S)$, *i.e.*

$$\sum_{S \in \mathcal{P}} e(x, S) \geq 0.$$

Since $(x, \mathcal{Q})$ is $\mathcal{P}$-inefficient, there is some $S \in \mathcal{P}$ such that $e(x, S) \neq 0$. This means that there exists $T \in \mathcal{P}$ (possibly equal to coalition S) such that $e(x, T) > 0$. ■

The proof of the theorem can be described as follows. Firstly, we show that for any payoff configuration $(x, \mathcal{Q})$, there exists a $\mathcal{P}$-efficient payoff configuration which is accessible from $(x, \mathcal{Q})$ in at most p steps. Then we prove that for any $\mathcal{P}$-efficient payoff configuration $(x, \mathcal{Q})$ not in the cs-core and any coalition S such that $e(x, S) > 0$, there exists a player in the partners' set $P(S, \mathcal{Q})$ of S in $\mathcal{Q}$ who is overpaid in $(x, \mathcal{Q})$. We use this feature to show that $(x, \mathcal{Q})$ is o.i.-dominated by a $\mathcal{P}$-efficient payoff configuration with less overpaid players. Finally, it is enough to repeat the above steps to construct a payoff configuration without overpaid players, *i.e.* a payoff configuration belonging to the cs-core.

All these steps rely on a particular type of o.i.-domination, which we introduce in the following. So let $(x, \mathcal{Q})$ be any payoff configuration not in the cs-core and S any coalition such that $e(S, x) > 0$. The *S-transformation* of $(x, \mathcal{Q})$ is the pair $(y, \mathcal{R})$ such that

$$\mathcal{R} = \{S\} \cup \{\{i\} : i \in P(S, \mathcal{Q}) \setminus S\} \cup \{T \in \mathcal{Q} : T \subseteq N \setminus P(S, \mathcal{Q})\}$$

and for each $i \in N$,

$$y_i = \begin{cases} c_i & \text{if } i = i_t \in U(x,S) \text{ and } \displaystyle\sum_{r=1}^{t} \left(c_{i_r} - x_{i_r}\right) \leq e(x,S) \\ x_i + e(x,S) - \displaystyle\sum_{r=1}^{t-1} \left(c_{i_r} - x_{i_r}\right) & \text{if } i = i_t \in U(x,S) \text{ and} \\ & \displaystyle\sum_{r=1}^{t-1} \left(c_{i_r} - x_{i_r}\right) \leq e(x,S) < \sum_{r=1}^{t} \left(c_{i_r} - x_{i_r}\right), \\ x_i & \text{if } i = i_t \in U(x,S) \text{ and} \\ & \displaystyle\sum_{r=1}^{t-1} \left(c_{i_r} - x_{i_r}\right) > e(x,S) \text{ or } i \in (N \backslash P(S,\mathcal{Q})) \cup (S \backslash U(x,S)), \\ v(\{i\}) & \text{if } i \in P(S,\mathcal{Q}) \backslash S. \end{cases}$$

In words, the deviating coalition S is an element of the new partition $\mathcal{R}$ as well as each coalition T in $\mathcal{Q}$ that has an empty intersection with S. Finally each of the remaining players, *i.e.* each player in the partners' set of S in $\mathcal{Q}$ but not in S, forms a singleton coalition in $\mathcal{R}$. Regarding the new payoff vector y, we first use the excess $e(x,S)$ so as to rise x_i up to c_i for as many underpaid players i in $U(x,S)$ as possible. This is the reason why the players $i \in U(x,S)$ are ordered by the smallest amount they need in order to fill in the gap between x_i and c_i. Then any other player $i \in S$ will keep his original payoff x_i as well as each player who is not in the partners' set of S in $\mathcal{Q}$. Finally, each player in the partners' set of S in $\mathcal{Q}$ but not in S obtains his stand-alone worth $v(\{i\})$.

We continue by two claims about the S-transformation.

Claim 3 *Let $(x,\mathcal{Q})$ be a payoff configuration not in the cs-core and S a coalition such that $e(S,x) > 0$. Then, the configuration $(x,\mathcal{Q})$ is o.i.-dominated by its S-transformation.*

Proof. The proof is divided in two parts. Firstly, we prove that the S-transformation $(y,\mathcal{R})$ is a payoff configuration. By construction we have $y(S) = v(S)$ and $y_i = v(\{i\})$ for each $i \in P(S,\mathcal{Q}) \backslash S$. Furthermore, for each $T \in \mathcal{Q}$ such that $T \subseteq N \backslash P(S,\mathcal{Q})$, it holds that $y(T) = x(T)$ which implies that $y(T) = v(T)$ since $(x,\mathcal{Q})$ is a payoff configuration. Thus, we have $y(T) = v(T)$ for each $T \in \mathcal{R}$. Since $y_i \geq x_i \geq v(\{i\})$ for each $i \in N \backslash (P(S,\mathcal{Q}) \backslash S)$ by construction, we also have $y_i \geq v(\{i\})$ for all $i \in N$. Therefore, $(y,\mathcal{R})$ is a payoff configuration. Secondly, it is easy to check that $(y,\mathcal{R})$ o.i.-dominates $(x,\mathcal{Q})$ via S since $y(S) > x(S)$ and $y_i \geq x_i$ for each $i \in S$. ■

Let us write that $(y,\mathcal{R})$ *trans-dominates* $(x,\mathcal{Q})$ if there exists a coalition S such that $(y,\mathcal{R})$ is the S-transformation of $(x,\mathcal{Q})$. Furthermore, if a payoff configuration $(y,\mathcal{R})$ is accessible from $(x,\mathcal{Q})$ through a chain of trans-dominations, we write that $(y,\mathcal{R})$ is *trans-accessible* from $(x,\mathcal{Q})$. The next claim states the only property of the S-transformation which is used later.

Claim 4 *The S-transformation $(y,\mathcal{R})$ of $(x,\mathcal{Q})$ satisfies $O(y) \subseteq O(x)$.*

Proof. Consider a player $i \in N$ such that $i \in N \backslash O(x)$, *i.e.* i is not overpaid in $(x,\mathcal{Q})$. If $i \in U(x,S)$ then either $y_i = c_i$ for those players for which the gap $c_i - x_i$ is filled up by the excess $e(x,S)$ or $y_i < c_i$ for the other members of $U(x,S)$ since it is assumed that $i \in N \backslash O(x)$. In both cases, $i \in N \backslash O(y)$. If $i \in (N \backslash P(S,\mathcal{Q})) \cup (S \backslash U(x,S))$, then $y_i = x_i \leq c_i$ and thus $i \in N \backslash O(y)$. Finally, if $i \in P(S,\mathcal{Q}) \backslash S$ then $y_i = v(\{i\}) \leq c_i$, which means once again that $i \in N \backslash O(y)$. ■

Lemma 1 *For any $\mathcal{P}$-inefficient payoff configuration $(x, \mathcal{Q})$, there exists a $\mathcal{P}$-efficient payoff configuration, which is trans-accessible from $(x, \mathcal{Q})$ in at most p steps.*

Proof. Fix any $\mathcal{P}$-inefficient payoff configuration, which we denote by $(x^0, \mathcal{Q}^0)$ for the sake of notations. Let us construct a sequence of $m+1$ payoff configurations $((x^0, \mathcal{Q}^0), (x^1, \mathcal{Q}^1), \ldots, (x^m, \mathcal{Q}^m))$ and a sequence of m coalitions $(S^1, \ldots, S^m)$ as follows. For each $k \in \{1, \ldots, m\}$ such that $(x^{k-1}, \mathcal{Q}^{k-1})$ is a $\mathcal{P}$-inefficient payoff configuration, we can choose $S^k \in \mathcal{P}$ such that $x^{k-1}(S^k) < v(S^k)$ by Claim 2. Then define $(x^k, \mathcal{Q}^k)$ as the S^k-transformation of $(x^{k-1}, \mathcal{Q}^{k-1})$. By Claim 3, for each $k \in \{1, \ldots, m\}$, the payoff configuration $(x^k, \mathcal{Q}^k)$ o.i.-dominates $(x^{k-1}, \mathcal{Q}^{k-1})$. It follows that $(x^m, \mathcal{Q}^m)$ is trans-accessible from $(x^0, \mathcal{Q}^0)$. It remains to prove that $(x^m, \mathcal{Q}^m)$ is $\mathcal{P}$-efficient after $m \leq p$ steps. Recall that for each $k \in \{1, \ldots, m\}$, the coalition S^k is chosen such that $S^k \in \mathcal{P}$ and $x^{k-1}(S^k) < v(S^k)$. In addition, $x^k(S^k) = v(S^k)$ and the payoff of any player in S^k will not be altered in the subsequent steps since for each $S^q \in \mathcal{P}$, $k < q \leq m$, $S^k \cap P(S^q, \mathcal{Q}^q) = \emptyset$. This implies that once S^k has been chosen, it cannot be chosen in a subsequent step, *i.e.* the sequence $(S^1, \ldots, S^m)$ contains m distinct coalitions belonging to $\mathcal{P}$. Because $\mathcal{P}$ contains p coalitions, the sequence of coalitions $(S^1, \ldots, S^m)$ contains at most p elements. Therefore, there exists some integer $m \leq p$ such that we cannot choose $S^{m+1} \in \mathcal{P}$ such that $e(x^m, S^{m+1}) > 0$. This means that $x^m(T) \geq v(T)$ for each $T \in \mathcal{P}$ and thus that $\sum_{T \in \mathcal{P}} x^m(T) \geq \sum_{T \in \mathcal{P}} v(T) = v^*$. By definition of v^*, this implies that for each $S \in \mathcal{P}$, we have $x^m(S) = v(S)$. We conclude that $(x^m, \mathcal{Q}^m)$ is a $\mathcal{P}$-efficient payoff configuration. ■

Lemma 2 *Consider any $\mathcal{P}$-efficient payoff configuration $(x, \mathcal{Q})$. For each coalition S such that $e(x, S) > 0$, there exists a player in $P(S, \mathcal{Q}) \backslash S$ who is overpaid in $(x, \mathcal{Q})$.*

Proof. Consider any $\mathcal{P}$-efficient payoff configuration $(x, \mathcal{Q})$ and suppose that there exists some coalition S such that $e(x, S) > 0$. Let us show first that $P(S, \mathcal{Q}) \backslash S$ is a nonempty set, which is equivalent to say that there does not exists a set of coalitions $K \subseteq \mathcal{Q}$ such that $S = \cup_{T \in K} T$. Obviously, it cannot be that $S \in \mathcal{Q}$ since this would mean $x(S) = v(S)$, a contradiction with the positive excess required in the choice of S. So, by way of contradiction, assume that S is the union of at least two elements of $\mathcal{Q}$ and denote by $\{S_1, \ldots, S_k\}$ the set of these $k \in \{2, \ldots, |\mathcal{Q}|\}$ elements. Recall that we have $e(x, S) > 0$ by definition of S. Now, consider the partition $\mathcal{Q}' = (\mathcal{Q} \backslash \{S_1, \ldots, S_k\}) \cup \{\cup_{q=1}^k S_q\}$ or equivalently $\mathcal{Q}' = (\mathcal{Q} \backslash \{S_1, \ldots, S_k\}) \cup \{S\}$. Then $\sum_{T \in \mathcal{Q}'} v(T) > \sum_{T \in \mathcal{Q}} x(T) = \sum_{T \in \mathcal{Q}} v(T) = v^*$ a contradiction with the efficiency of $(x, \mathcal{Q})$.

Next, we prove that the nonempty set $P(S, \mathcal{Q}) \backslash S$ contains a player overpaid in $(x, \mathcal{Q})$. This part of the proof is done by contradiction. So assume that none of the players in $P(S, \mathcal{Q}) \backslash S$ is overpaid in $(x, \mathcal{Q})$. On the one hand, this means that $x(P(S, \mathcal{Q}) \backslash S) \leq c(P(S, \mathcal{Q}) \backslash S)$. Since, in addition we have $x(S) < v(S) \leq c(S)$, we get

$$x(P(S, \mathcal{Q})) < c(P(S, \mathcal{Q})). \tag{1}$$

On the other hand, by definition of a payoff configuration, for each $T \in \mathcal{Q}$, it holds that

$$x(T) = v(T) \leq c(T). \tag{2}$$

The $\mathcal{P}$-efficiency of $(x, \mathcal{Q})$ also implies that

$$\sum_{T \in \mathcal{Q}} x(T) = \sum_{T \in \mathcal{Q}} v(T) = \sum_{T \in \mathcal{Q}} c(T). \tag{3}$$

From expressions (2) and (3), we obtain $x(T) = c(T)$ for each $T \in \mathcal{Q}$. Since $P(S, \mathcal{Q})$ is the union of disjoints elements of $\mathcal{Q}$, we immediately get $x(P(S, \mathcal{Q})) = c(P(S, \mathcal{Q}))$, a contradiction with inequality (1). ■

We are now ready to prove Theorem 1.

Proof. [Theorem 1] Consider any payoff configuration $(x^0, \mathcal{Q}^0) \notin C(v)$. We construct a sequence of $m+1$ payoff configurations $((x^0, \mathcal{Q}^0), (x^1, \mathcal{Q}^1), \ldots, (x^m, \mathcal{Q}^m))$ and a sequence of m coalitions $(S^1, \ldots, S^m)$ with the following features. For each $k \in \{1, \ldots, m\}$ such that $(x^{k-1}, \mathcal{Q}^{k-1})$ is not in the cs-core, we choose S^k in the set $\{S \in 2^N : e(x^{k-1}, S) > 0\}$ of coalitions with positive excess with respect to x^{k-1}, and if in addition $(x^{k-1}, \mathcal{Q}^{k-1})$ is not $\mathcal{P}$-efficient, the coalition S^k with positive excess is taken in $\mathcal{P}$, which is possible by Claim 2. So consider any integer $k \in \{1, \ldots, m\}$ such that $(x^{k-1}, \mathcal{Q}^{k-1})$ is not in the cs-core. Two possibilities are left. If $(x^{k-1}, \mathcal{Q}^{k-1})$ is $\mathcal{P}$-efficient, then Lemma 2 and the definition of the S^k-transformation of $(x^{k-1}, \mathcal{Q}^{k-1})$ imply that $O(x^k) \subset O(x^{k-1})$ since any player i in the nonempty set $P(S^k, Q^{k-1}) \backslash S^k$ who is overpaid in $(x^{k-1}, \mathcal{Q}^{k-1})$ receives $x_i^k = v(\{i\})$ and thus is no longer overpaid in $(x^k, \mathcal{Q}^k)$. If otherwise $(x^{k-1}, \mathcal{Q}^{k-1})$ is not $\mathcal{P}$-efficient, then Lemma 1 implies that there exists some integer $q \in \{1, \ldots, p\}$ such that $(x^{k-1+q}, \mathcal{Q}^{k-1+q})$ is a $\mathcal{P}$-efficient payoff configuration trans-accessible from $(x^{k-1}, \mathcal{Q}^{k-1})$. Together with Claim 4 and Lemma 2, this implies that $O(x^{k+q}) \subset O(x^{k-1})$. Since the number of overpaid players is less than n, it is obvious that this procedure ends up after some finite number m of steps. The obtained payoff configuration $(x^m, \mathcal{Q}^m)$ can be of two types. Either there is no overpaid player in $(x^m, \mathcal{Q}^m)$, which means that $(x^m, \mathcal{Q}^m)$ belongs to the cs-core since its $\mathcal{P}$-efficiency necessarily means that $x^m = c$, or there exist some overpaid players in $(x^m, \mathcal{Q}^m)$ but they cannot be removed because $(x^m, \mathcal{Q}^m)$ already belongs to the cs-core.

It remains to provide a bound to this number m of steps. Denote by $(x, \mathcal{Q})$ the first $\mathcal{P}$-efficient payoff configuration that is constructed and by l be the number of overpaid players in this payoff configuration. Constructing this first $\mathcal{P}$-efficient payoff configuration $(x, \mathcal{Q})$ from $(x^0, \mathcal{Q}^0)$ requires at most p steps by Lemma 1 and then eliminating its l overpaid players requires at most $l(p+1)$ steps as described in the first part of the proof. The total number of steps $p+l(p+1)$ is equivalent to $(l+1)(p+1)-1$. Next, note that $l \leq n-p$ since each coalition $T \in \mathcal{P}$ contains at least one player who is not overpaid in $(x, \mathcal{Q})$. The maximal number of steps required to achieve the accessibility is thus bounded by $(n+1-p)(p+1)-1$. Recall that if two integers have a constant sum, then their product is maximal when they are equal if the sum is even or consecutive if the sum is odd. Here the sum of the two integer is $n+2$ so that the maximal value of $(n+1-p)(p+1)-1$ is

$$\left(\frac{n+2}{2}\right)^2 - 1 = \frac{n^2+4n}{4}$$

if n is even or

$$\left(\frac{n+3}{2}\right)\left(\frac{n+1}{2}\right) - 1 = \frac{n^2+4n-1}{4}$$

if n is odd. Since $(n^2+4n)/4$ is always an integer number for n even, we conclude that a payoff configuration in the cs-core is accessible from $(x^0, \mathcal{Q}^0)$ in at most $(n^2+4n)/4$ steps. ■

4 Conclusion

The polynomial bound on the accessibility of the cs-core provided in this article strongly improves the existing bound found by Kóczy and Lauwers [6]. Our bound can be lowered in special classes of games. For instance it is well known from Theorem 8 in Shapley [9] that the core on any convex TU-game is a von Neumann Morgenstern stable set, which means that any imputation that is not in the core is dominated by a core imputation. In other words, the core of any convex

TU-game is accessible in only one step. Since $\{N\}$ is the unique partition generating v^* in a convex TU-game (unless v is additive), this property is still true for the cs-core of any convex TU-game. However we do not think that our bound can be lowered enough to become linear for the class of all TU-games with a nonempty cs-core.

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