Social Welfare Analysis of Income Distributions: Ranking Income Distributions with Lorenz Curves

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ANALYTICAL TOOLS Module 001

Social Welfare Analysis of IncomeDistributions

Ranking Income Distributions with Lorenz Curves

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Ranking Income Distributions with Lorenz Curves

by

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FOOD AND AGRICULTURE ORGANIZATION OF THE UNITED NATIONS

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This modules is part of the resource package Analysis and monitoring of socio-economic impacts of policies.

Table of Contents

1 SUMMARY

This module illustrates how **Lorenz Curves** can be used to identify the best income distribution on social welfare grounds, within a set of alternative income distributions generated by different policy options.

After highlighting some drawbacks of using specific functional forms of the **Social Welfare Function (SWF)** to infer welfare judgments, the rationale for using Lorenz Curves to rank income distributions is provided in a step-by-step procedure and is illustrated with some simple numerical examples. This module also points out the limitations of Lorenz dominance and highlights how, in some circumstances, it is necessary to use **Generalised Lorenz (GL) Curves**[1].

2 INTRODUCTION

This module belongs to a set of modules which discuss how to rank different income distributions on welfare grounds that are generated by alternative policy options, such as: private investment support, input subsidies, output protection. this module, is useful in situations where the analyst has to provide information about the likely impact of a policy measure such as a tax/benefit reform, infrastructural investment policy, a specific sectoral or sub-sectoral policy on the distribution of income, more specifically, to answer policy questions such as whether the policy measure under investigation leads to a social welfare improvement or not.

Objectives

The main objective of this module is to illustrate how Lorenz Curves can be used to rank income distributions on welfare grounds. The user will learn how to make use of **Lorenz dominance** to draw conclusions on the most preferred income distribution within a set of possible income distributions generated by alternative policy options. He will also learn about the limitations of Lorenz Curves in some cases.

Target audience

This module targets different categories of users in different contexts, for example:

- **trainers** can use this module in capacity development activities e.g. to teach policy analysts how to use household data in policy work;
- **policy analysts** can use this module as reference material when carrying out their on-the-job tasks;
- **lecturers** in academic courses can use this material to support under-graduate courses in welfare economics, economic policy, development economics and related fields;

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[1] Generalised Lorenz curves as tools to rank income distributions are discussed in the EASYPol Module 002: ***Social Welfare Analysis of Income Distributions: Ranking Income Distributions with Generalised Lorenz Curves.***

- **other users,** such as NGOs, political parties, professional organizations or consulting firms that are willing to enhance their expertise in analyzing welfare impacts of policies by means of analyzing changes in income distributions.

Required background

The trainer is strongly recommended to verify how adequate the trainees' background is, notably their understanding of the concepts of "income distribution" and "social welfare" and, possibly, basic principles of calculus. In particular, the user must be familiar with concepts of:

- policy impact simulations;
- income distribution;
- Lorenz Curves and technicalities;
- social welfare and social welfare functions.

If their background is weak or missing, the trainer may consider delivering other EASYPol modules beforehand, as highlighted in the introduction. Other technicalities present in this module should be understood by all people with an elementary knowledge of basic mathematics and statistics.

Throughout the text, where relevant, you will also find references to applications in a real country case, references to complementary EASYPol modules, notes for trainers and complementary capacity building facilities[2]. A complete set of useful links to related EASYPol modules is provided at the end of the document.

3 CONCEPTUAL BACKGROUND

This section highlights the analogies and differences between welfare ranking with specific functional forms of the SWF or with Lorenz dominance.

3.1 SWF and Lorenz dominance

By ranking distributions on the basis of their Lorenz dominance**,** an inequality-averse decision-maker can rank income distributions on welfare grounds by exploiting some properties of the Lorenz Curves.

Levels of welfare for any given income distribution can be calculated by specifying an SWF. In this way, given a set of income distributions, we can reduce any income distribution in this set to a single number, thereby generating a «complete ranking» of

[2] EASYPol hyperlinks are shown in blue, as follows:
a) resource packages are shown in **underlined bold font**;
b) other EASYPol modules or complementary EASYPol materials are in ***bold underlined italics***;
c) links to the glossary are in **bold**; and
d) external links are in *italics*

the set. However, in order to obtain this complete ranking, we have to specify the mathematical relationship between individual incomes and social welfare.

Therefore, a natural question to ask is: how do you choose among many functional forms? In addition, there is no guarantee that the same ranking also holds for alternative functional forms of the SWF, even if all of them satisfy the two general requirements that the SWF should be increasing in income and concave[3].

3.2 Lorenz dominance and welfare: Atkinson's Theorem

However, it is not always necessary to specify the functional form of the SWF to identify the best distribution in terms of welfare as it is sometimes sufficient to identify the Lorenz dominating distribution. Lorenz Curves have already been discussed as a tool to detect inequality in income distributions[4]. In this module, Lorenz Curves will be used to rank income distributions by level of welfare. What is of interest now, is to understand whether you can pass from inequality to welfare considerations simply by using the information contained in the Lorenz Curves.

The **Atkinson Theorem**: In 1970, Atkinson established a well-known theorem covering the passage from inequality to welfare.

Box 1 - Atkinson's Theorem (1970)

If the following three conditions are satisfied:

a) The Lorenz Curve of distribution **Y** dominates the Lorenz Curve of distribution **X**.

b) the distributions have an equal mean income or the Y mean is greater than the X mean income.

c) The decision-maker is income-seeking and inequality-averse (i.e. the SWF has positive first derivative and negative second derivative with respect to individual incomes).

then, social welfare is higher in **Y** than in **X**.

The Atkinson Theorem states that, from the point of view of an income-seeking and inequality-averse decision-maker, Lorenz-dominace is a necessary and sufficient condition to detect welfare superiority in the dominating distribution, provided that it has the same or higher mean than that of the dominated distribution (see Box 1, above). This is an important result, because it is independent of the exact functional form of the SWF. Its only requirement is that it increases in income and that it is concave (inequality averse). Thus, in these cases Lorenz dominance implies welfare superiority[5].

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[3] In mathematical terms these requirements imply positive first derivative and negative second derivative of the SWF with respect to individual incomes: $W'>0$ and $W''\leq 0$.

[4] Refer to the EASYPol Module 000: ***Charting Income Inequality: The Lorenz Curve***.

[5] Proof of the Atkinson Theorem for continuous SWFs and continuous income distributions is provided in: Lambert, 1993, pp. 62 to 66.

When using Lorenz Curves to rank income distributions on welfare grounds, one of the following cases will occur:

the dominating distribution has a higher or equal mean, therefore, it is welfare - superior;

the dominating distribution has a lower mean. No conclusive judgement about welfare superiority is possible;

there is no domination of one distribution over the other (Lorenz Curves cross). No conclusive judgement about welfare superiority is possible

If case i) occurs, the Atkinson Theorem can be applied, thus distributions can be ranked. If either cases ii) or iii) occur, further analysis with other tools is required[6].

4 A STEP-BY-STEP PROCEDURE TO RANK INCOME DISTRIBUTIONS WITH LORENZ DOMINANCE

Figure 1 below illustrates how to perform welfare comparisons to rank income distributions by applying Atkinson's Theorem.

Figure 1 - A step-by-step procedure to rank income distributions using Lorenz dominance

STEP	OPERATIONAL CONTENT
1	Sort income distributions by income level
2	Check whether income distributions have different mean incomes
3	Build Lorenz Curves for each income distribution
4	Plot Lorenz Curves and check for dominance
5	Conclusion: if there is Lorenz dominance and the dominating distribution has equal or higher mean income, the dominating distribution is welfare superior

[6] Refer to the EASYPol Module 002: ***Social Welfare Analysis of Income Distributions: Ranking Income Distributions with Generalised Lorenz Curves,*** and related EASYPol modules.

5 STEP-BY-STEP EXAMPLES FOR WELFARE COMPARISON WITH LORENZ DOMINANCE

To demonstrate the above step-by-step procedure, consider the following simple example that is based on an illustrative income distribution.

> ***Example**: There are five individuals whose incomes, expressed in monetary units, are reported in Table 1, in columns "income distribution A". Let us assume that distribution A is the benchmark reflecting the without policy situation. This base-case will be modified by a redistributive policy, shifting one unit of income from individual 2 to individual 1, and one unit of income from individual 5 to individual 3, thereby generating the new income distribution B. This is a case of transfers from richer to poorer individuals.*

Will the decision maker, who is income-seeking and inequality-averter, decide to implement this policy? In other words, is distribution B welfare-superior to A?
The step-by-step procedure can be applied to answer this question.

Step 1 requires that the income distributions to be compared be sorted by income levels, as already reported in Table 1, below.

Step 2 aims at verifying the mean income distribution. The mean of the two distributions is 9 income units for both the distributions, as reported in Table 1. This policy does not alter the mean income. After checking for dominance, information about the mean will be used to apply the Atkinson Theorem.

Step 3 requires that Lorenz Curves for each income distribution be built[7].

Step 4 requires that Lorenz Curves be plotted and checked for dominance. Lorenz Curves are reported in Figure 2a, below. To facilitate checking for dominance, the difference between the Lorenz Curve ordinates of B and A are calculated and illustrated in Figure 2b. Note that all the differences are positive, implying that B dominates A.

Step 5 requires that the Atkinson's theorem be applied. If there is dominance of one distribution and the dominating distribution has a higher or equal mean, then it is welfare-superior. The two distributions present the same mean income. In addition, as apparent in Figure 2a and 2b, the Lorenz Curve of B dominates A. Therefore, according to the Atkinson Theorem, for all income-seeking and inequality-averse decision-makers, distribution B is better than distribution A (welfare-superior).

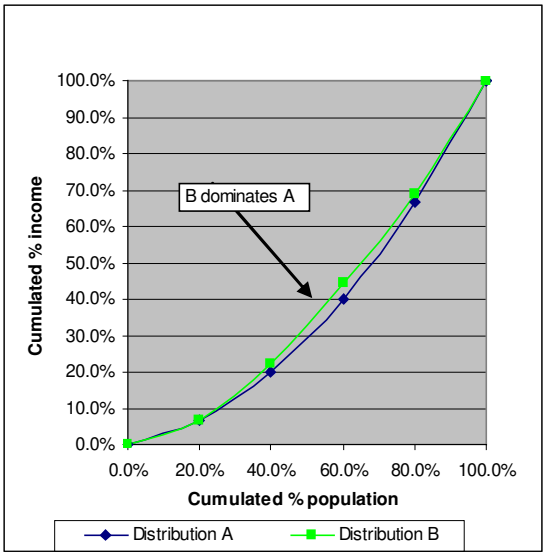
Table 1 - Transfers from richer to poorer: Atkinson's Theorem

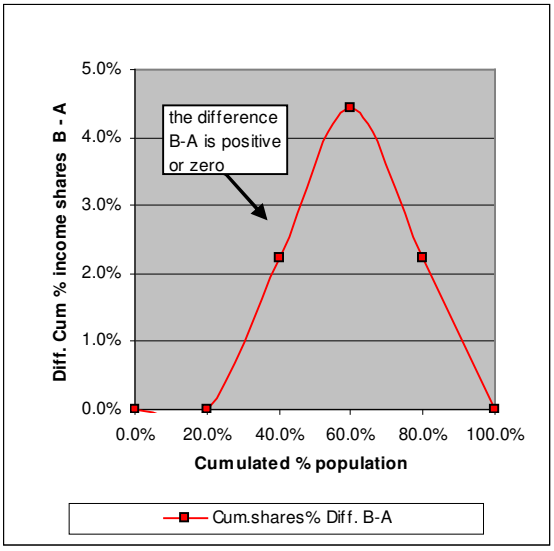
[7] The procedure to build Lorenz curves has already been described in the EASYPol Module 000 ***Charting Income Inequality: The Lorenz Curve***.

Individuals	Distribution A		Distribution B		Differences B-A
	Income (Y)	Cum.share Y%	Income	Cum.share Y%	Cum.Share%
1	3.0	6.7%	3	6.7%	0.0%
2	6.0	20.0%	7	22.2%	2.2%
3	9.0	40.0%	10	44.4%	4.4%
4	12.0	66.7%	11	68.9%	2.2%
5	15.0	100.0%	14	100.0%	0.0%
Total income	45.0		45.0		0.0
Mean income	9.0		9.0		0.0

Remark: Cumulated shares of B are greater or equal to those of A: Lorenz dominance

Figures 2a and 2b - Transfers from richer to poorer – Lorenz Curves





In general, distributions obtained by transferring income from richer to poorer, without altering the mean income, as in the above case, will generate Lorenz dominating distributions. This implies, for the Atkinson Theorem, that they are welfare-superior, assuming that the decision-maker is income-seeking and inequality-averse.

6 FURTHER EXAMPLES OF WELFARE COMPARISONS

In addition to transfers from richer to poorer, as illustrated above, and, to further illustrate the Atkinson Theorem, consider the following examples, where two other possible policy scenarios are analysed: a) transfers from poorer to richer; b) additional income accruing to selected individuals.

a. Transfers from poorer to richer

Table 2 and Figure 3, below, report two cases of transfers from poorer to richer. Distribution D is derived from distribution A, by shifting one unit of income from individual 2 to individual 3, as indicated by the arrow. Note, that the cumulated share of income at the income level of individual 2 is lower than in A. This leads to Lorenz dominance of A over D, as plotted in Figure 3a. The Atkinson Theorem enables us to conclude that A is better than D on welfare grounds, assuming that income-seeking and inequality-averse decision–makers reject transfers from poorer to richer.

Table 2 - Lorenz dominance and transfers from poorer to richer

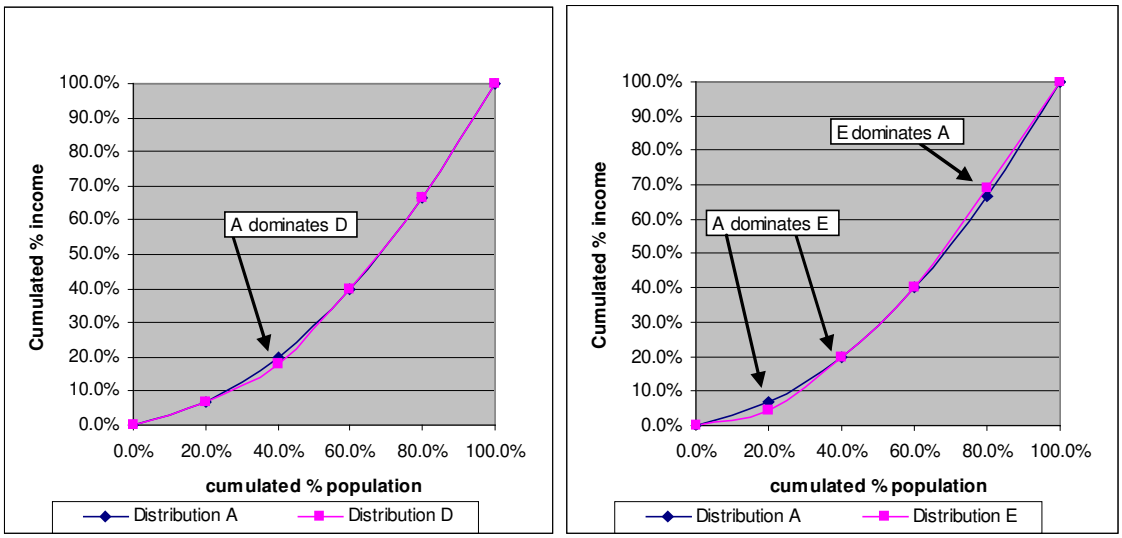
Individuals	Distribution A		Distribution D		Distribution E	
	Income (Y)	Cum.share Y%	Income (Y)	Cum.share Y%	Income	Cum.share Y%
1	3	6.7%	3	6.7%	2	4.4%
2	6	20.0%	5	17.8%	7	20.0%
3	9	40.0%	10	40.0%	9	40.0%
4	12	66.7%	12	66.7%	13	68.9%
5	15	100.0%	15	100.0%	14	100.0%
Total income	45		45		45.0	
Mean income	9.0		9.0		9.0	

Remark: Cumulated shares of D and E at these points are lower than those of A: Lorenz inferiority

b. Mixed transfers

As a second example, see distribution E, derived from A, transferring one unit of income from individual 2 to individual 4 (poorer to richer) and one unit of income from individual 5 to individual 2 (richer to poorer). In this case, the Lorenz Curves of A and E cross each other (see Figure 3b). Therefore, the Atkinson's Theorem cannot be used to define which distribution is better on welfare grounds.

Figures 3a and 3b - Transfers from poorer to richer and mixed transfers: Lorenz Curves



c. Additional income accruing to selected individuals

Consider the following examples, illustrated in Table 3, below. Starting from distribution A, distribution F is generated giving two additional income units to individual 2. Nobody is worse off, in absolute terms, because all the other incomes are left unchanged. In addition, the mean income has increased, from 9 to 9.4 monetary units. Yet, Atkinson's theorem does not allow any welfare comparison between distributions A and F, because there is no Lorenz dominance (Lorenz Curves cross each other, see Figure 4a), below. In this case, Lorenz Curves fail to provide a conclusive answer on welfare superiority of one distribution with respect to the other.

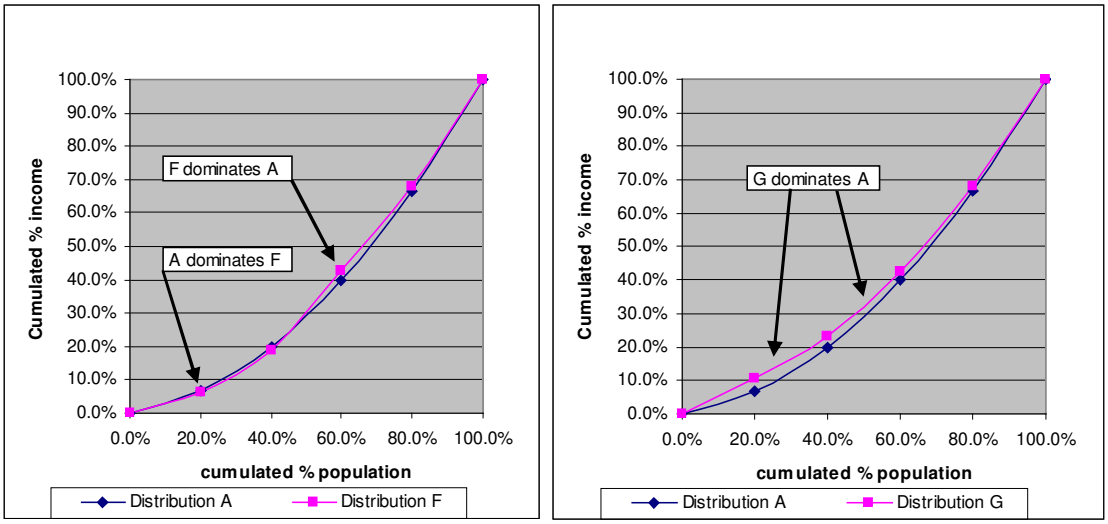
Whereas, in distribution G, additional income is accrued to the poorest individual. Thanks to the Lorenz dominance of G over A, as apparent in Figure 4b, below, the Atkinson Theorem enables us to state that, for an income-seeking and inequality-averse decision maker, this additional income is a welfare improvement.

Table 3 - Lorenz dominance and accruals of additional incomes

Individuals	Distribution A		Distribution F		Distribution G	
	Income (Y)	Cum.share Y%	Income (Y)	Cum.share Y%	Income	Cum.share Y%
1	3	6.7%	3	6.4%	5	10.6%
2	6	20.0%	6	19.1%	6	23.4%
3	9	40.0%	11	42.6%	9	42.6%
4	12	66.7%	12	68.1%	12	68.1%
5	15	100.0%	15	100.0%	15	100.0%
Total income	45		47		47.0	
Mean income	9.0		9.4		9.4	

Remark: cumulated income shares below the level of income increased are lower than in A. Those equal or above are greater. Lorenz curves therefore cross.

Figure 4 - Accrual of additional incomes to selected individuals: Lorenz Curves



7 SYNTHESIS AND CONCLUSIONS

The basic result of this module is that Lorenz Curves are a powerful tool for ranking welfare of different income distributions. However, unlike the case of the complete specification of a SWF, Lorenz Curves may give a «partial ordering», i.e. they may fail to fully rank a set of distributions because, in some cases, i.e. when either Lorenz Curves cross or the dominating distribution has a lower mean, it is not possible to reach any conclusive judgment. In many circumstances, when Lorenz Curves fail to provide a conclusive answer, "Generalised Lorenz Curves" can succeed[8].

Table 4, below, summarises all the results achieved so far, highlighting all outcomes deriving from the combination of the types of relationship between curves and mean incomes of the distribution observed.

Table 4 - Distributional dominance and welfare ranking

	Type of relation	Mean Income	Outcome	Restrictions	Notes
1	L(Y)>L(X)	Y=X	W(Y) > W(X)	Wi'>0; Wi"< 0	
2	L(Y)>L(X)	Y>X	W(Y) > W(X)	Wi'>0; Wi"< 0	
3	L(Y)>L(X)	Y<X	cannot say		Need GL
4	L(Y) and L(X) cross	Y<X	cannot say		Need GL

Legenda
L(Y) = Lorenz Curve of distribution Y
L(X) = Lorenz Curve of distribution X
W(Y) = Social Welfare in Y
W(X) = Social Welfare in X
Wi' and Wi" = first and second derivative respectively of W w.r.t. income of i-th individual
GL = Generalised Lorenz Curves

8 READERS' NOTES

8.1 Time requirements

This module and related discussion may take two to three hours to deliver to an audience already familiar with concepts of policy, policy impact simulations, income and income distributions, Lorenz Curves, social welfare and social welfare functions.

[8] These cases are further explored in the EASYPol Module 002 ***Social Welfare Analysis of Income Distributions: Ranking Income Distribution with Generalised Lorenz Curves***, and related EASYPol modules.

8.2 Frequently asked questions

Frequently asked questions include:

- ✓ **What is the meaning and role of the preferences of the decision-maker?** i.e., what does it mean that the decision-maker is "inequality averse" and an income-seeker? It is important in these cases to refer to the shape of the welfare function imposed by the restrictions on its first and second derivatives.
- ✓ Why do Lorenz Curves fail to rank cases such as the one illustrated in Table 3, distribution F, even if it is apparent that nobody is worse-off? Refer to the fact that Lorenz Curves capture the "shares of income", not income units.
- ✓ Selected trainees who are not familiar with how to build policy scenarios may not understand how, in practical cases, the "with policy" income distribution is generated, i.e. how to logically link the policy proposal to the new income distribution. In addition, preparation and running exercises slightly more complex than the examples provided in the module with real data must be considered.

8.3 EASYPol links

This module belongs to a set of modules which discuss how to provide normative prescriptions when confronting alternative income distributions, i.e. how to identify the best income distribution in terms of social welfare, in a set of alternative income distributions. It is one of the modules composing the resource package **Analysis and monitoring of socio-economic impacts of policies.**

The following EASYPOL modules form a set of materials logically preceding the current module, which can be utilized to strengthen the background of the user:

- EASYPol Module 000: ***Charting Income Inequality: The Lorenz Curve***.

Issues addressed in this module are further expanded in the following modules:

- EASYPol Module 002: ***Social Welfare Analysis of Income Distributions: Ranking Income Distribution with Generalised Lorenz Curves***.
- EASYPol Module 003: ***Social Welfare Analysis of Income Distributions:Ranking Income Distribution with Crossing Generalised Lorenz Curves.***

A case study presenting the use of Lorenz Curves to rank income distributions in the context an agricultural policy impact simulation exercise with real data is reported in the EASYPol Module 042: ***Inequality and Poverty Impacts of Selected Agricultural Policies: The Case of Paraguay.***

9 FURTHER READINGS

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