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**Social Welfare Analysis of Income  
Distributions: Ranking Income  
Distributions with Crossing Generalised  
Lorenz Curves**

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Food and Agriculture Organization of the United Nations (FAO)

1 November 2005

Online at <https://mpra.ub.uni-muenchen.de/30115/>  
MPRA Paper No. 30115, posted 13 Apr 2011 08:22 UTC

# Social Welfare Analysis of Income Distributions

## Ranking Income Distributions with Crossing Generalised Lorenz Curves



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### Ranking Income Distributions with Crossing Generalised Lorenz Curves

by

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## 1 SUMMARY

This module illustrates how **Crossing Generalised Lorenz (GL) curves** can be used to identify the best income distribution on social welfare grounds within a set of alternative income distributions generated by different policy options.

It starts by illustrating two alternative income distributions resulting from policy changes that lead to income increases for some individuals and decreases for others. GL curves are then calculated for the alternative distributions to rank them on welfare grounds on the basis of **Shorrocks' Theorem**. After observing that Shorrocks' Theorem is not applicable, because GL curves cross once, necessary additional conditions, such as restrictions on the features of the **Social Welfare Function (SWF)** and the shape of income distributions, are set and discussed. Subsequently, a step-by-step procedure to use GL curves to infer welfare judgments when GL cross once, is provided and illustrated with some simple numerical examples.

## 2 INTRODUCTION

This module belongs to a set of modules which discuss how to rank different income distributions on welfare grounds that are generated by alternative policy options, such as: private investment support, input subsidies, output protection. This module, is useful in situations where the analyst has to provide information about the likely impact of a policy measure such as a tax/benefit reform, infrastructural investment policy, a specific sectoral or sub-sectoral policy on the distribution of income, more specifically, to answer policy questions such as whether the policy measure under investigation leads to a social welfare improvement or not.

### Objectives

The specific objective of this module is to illustrate how GL curves can be used to rank income distributions on welfare grounds even if the GL curves of the two distributions cross each other once. The user will learn how to make use of **Generalised Lorenz dominance** with crossing GL curves, to draw conclusions on the most preferred income distribution within a set of possible income distributions generated by alternative policy options. He will also learn more about the limitations of GL curves for welfare considerations.

### Target audience

This module targets different categories of users in different contexts, for example:

- **trainers** can use this module in capacity development activities e.g. to teach policy analysts how to use household data in policy work;
- **policy analysts** can use this module as reference material when carrying out their on-the-job tasks;

- **lecturers** in academic courses can use this material to support under-graduate courses in welfare economics, economic policy, development economics and related fields;
- **other users**, such as NGOs, political parties, professional organizations or consulting firms that are willing to enhance their expertise in analyzing welfare impacts of policies by means of analyzing changes in income distributions.

### Required background

The trainer is strongly recommended to verify the suitability of the trainees' background, in particular, the trainees must be familiar with:

- Concepts of policy impact simulations.
- Concepts of income distribution.
- Concepts and technicalities of Lorenz curves and generalized Lorenz curves.
- Concepts of social welfare and social welfare functions.

If this background is weak or missing, the trainer may consider delivering other modules beforehand, as highlighted in the introduction. Trainees should also know basic concepts of welfare economics, statistics, elementary mathematics and, possibly, basic principles of calculus.

To find relevant materials in these areas, the reader can follow the links included in the text to other EASYPol modules or references<sup>1</sup>. A set of useful links to related EASYPol modules is provided in a section at the end of the document.

## 3 CONCEPTUAL BACKGROUND

This section sets the problem of crossing GL curves and presents the conceptual background required to use GL curves for welfare ranking when curves cross each other once, i.e. when Shorrocks' Theorem is not applicable.

The core of this section is the discussion on:

- the restrictions to be imposed on the SWF when GL curves cross once in order to obtain unanimous judgments on the ranking of income distributions on welfare grounds (the so-called "principle of diminishing transfers");
- the conditions about the variances of the distributions to be compared (the "variance" condition) when two distributions have the same mean income;
- the need to rule out some "extreme" SWF that bend toward inequality neutrality to get a unanimous consensus when comparing a more egalitarian distribution with a

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<sup>1</sup> EASYPol hyperlinks are shown in blue, as follows:

- a) training paths are shown in **underlined bold font**;
- b) other EASYPol modules or complementary EASYPol materials are in ***bold underlined italics***;
- c) links to the glossary are in **bold**; and
- d) external links are in *italics*

- lower mean income, or when comparing a less egalitarian distribution with a higher mean income (Rawlsian toward Utilitarian preferences);
- the “mean-variance” conditions to select the “extreme” SWF to be ruled out in such cases.

### 3.1 Setting the problem: crossing of GL curves

In the module Ranking income distributions with Generalised Lorenz curves, there is a case here two GL curves cross each other. For ease of reference, you will find this example reported here below in Table 1 and Figure 1, respectively.

Distribution I is the result of a policy, the net impact of which results in mixed shifts of income from richer to poorer, i.e. one income unit from individual 3 to individual 1, and from poorer to richer, i.e. one unit of income from individual 4 to individual 5.

**Table 1 - Mixed transfers from richer to poorer and from poorer to richer: a case of crossing GL curves**

Individuals (a)	Cum.share of p (hor.axis L/GL) (b)	Distribution A			Distribution I			Diff.cum. aver.Y I-A (i)
		Income (Y) (c)	Cum.share Y% (vert.axis of L) (d)	Cum.aver.Y (vert.axis GL) (e)	Income (Y) (f)	Cum.sh.Y% (vert.axis of L) (g)	Cum.aver.Y (vert.axis GL) (h)	
1	20.0%	3	6.7%	0.6	4	8.9%	0.8	0.2
2	40.0%	6	20.0%	1.8	6	22.2%	2.0	0.2
3	60.0%	9	40.0%	3.6	8	40.0%	3.6	0.0
4	80.0%	12	66.7%	6.0	11	64.4%	5.8	-0.2
5	100.0%	15	100.0%	9.0	16	100.0%	9.0	0.0
Total income		45.0			45.0			
Mean income		9.0			9.0			

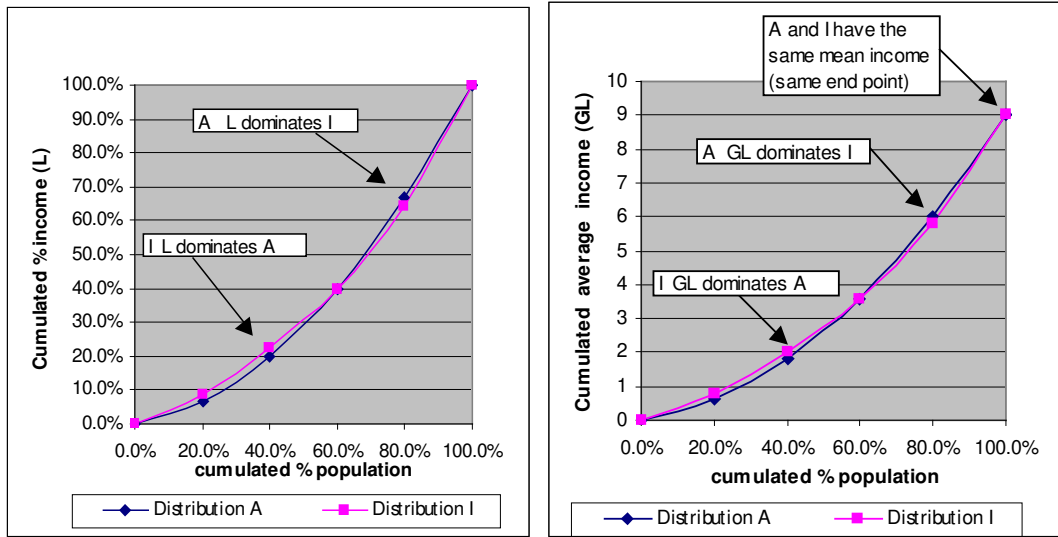
I L dominates A for the first 60% of the population but A L dominates I for greater cumulated shares of the population, i.e. I presents lower cumulated shares of income in the lower part of the distribution and higher cumulated shares in the higher part of the distribution. Therefore, L curves cross.

Note that the GL do cross because the difference between the ordinates of I and A are positive in the lower part of the curves and negative in the upper part.

In this case, L curves cross, as is apparent from Figure 1a (and columns d and g) and also GL curves cross. This is not surprising indeed, because distributions A and I have the same mean income<sup>2</sup>. Note that the Shorrocks’ Theorem presented in the above-mentioned module cannot be applied to rank the two income distributions on welfare grounds, because it requires that the GL curves of one distribution dominate the GL curves of the other. So far, no conclusive judgement can be reached.

<sup>2</sup> Remember, from EASYpol Module 002: [Social Welfare Analysis of Income Distributions: Ranking Income Distributions with Generalised Lorenz Curves](#), that when two distributions have the same mean, the GL curves are simply up-scaled versions of the Lorenz curves.

**Figures 1a and 1b - Mixed transfers from richer to poorer and poorer to richer: a case of crossing GL curves**



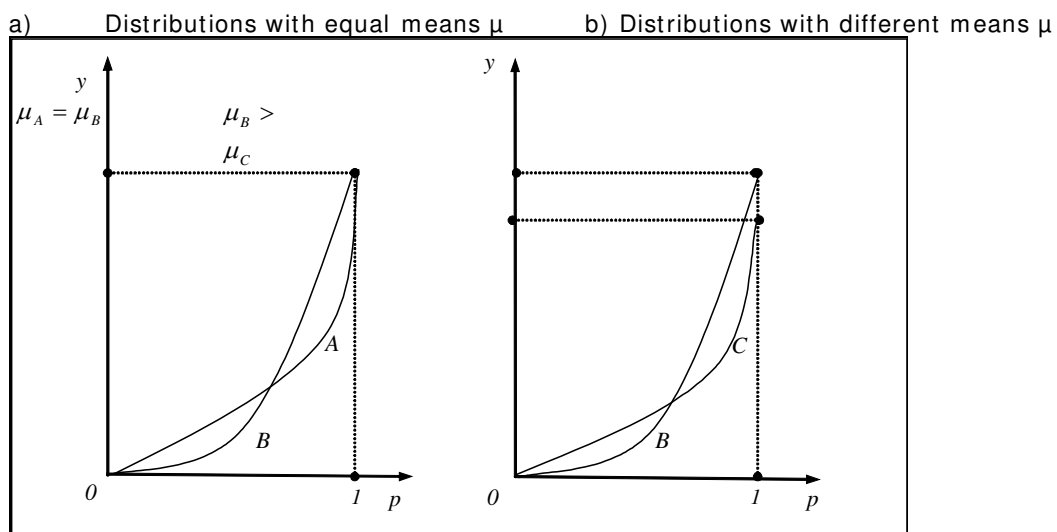
individual (individual 3), against the “inequality-increasing” policy impact, that pushes an individual even further above the mean income (individual 5) by transferring income to him from a worse-off individual (individual 4).

### 3.2 GL curves crossing once: Rawlsian versus utilitarian preferences

In the example reported above, crossing GL curves occurred with equal mean income distributions. In general, however, when GL curves cross once, two possible cases arise: i) mean incomes are equal; ii) mean incomes are different.

The two cases are illustrated in Figure 2, graphs a) and b), below:

**Figure 2 - GL curves crossing once: the two possible cases:**





In the above Figure, GL curves for three distributions A, B, and C, and pair wise comparisons are illustrated. The vertical axis illustrates income  $y$  and the horizontal axis illustrates the proportion of population  $p$ . Note that for distribution A in graph 2a, GL curves dominate distribution B in the lower part, but GL curves are dominated by B in the upper part. On the other hand, the two distributions have the same mean income<sup>3</sup>. Whereas, in graph 2b, GL curves dominate distribution B in the lower part, but distribution C ends up with a lower mean<sup>4</sup>.

The polar concepts of Utilitarian and Rawlsian welfare preferences, are useful here to highlight where and how trade offs between equity and efficiency are reported by crossing GL curves:

**Utilitarian preferences** amount to inequality neutrality, i.e., only the mean income matters. In terms of GL curves, it means that Utilitarians look at the end-point of the distributions: which one is higher and thus preferred. In Figure 2a, above, distributions A and B would be indifferent from a utilitarian point of view. Whereas, in graph 2b, Utilitarians would definitely prefer B to C, because the average income is greater.

**Rawlsian preferences** amount to infinite inequality aversion, i.e. only the poorest income matters. In terms of GL curves, it means that Rawlsians look at the starting point of the income distribution. The distribution whose GL curve dominates the lowest incomes, would be preferred. In Figure 2a and 2b respectively, Rawlsians would definitely prefer A to B and C to B.

Therefore, to compare the two curves in Figure 2a, you would have to focus only on equity, because they are equivalent on efficiency grounds. Rawlsians and Utilitarians would both agree to choose distribution A and discard distribution B. On the other hand, non-extreme decision-makers may have different points of view about which distribution to choose. Restrictions and conditions will be needed on the SWF and on the shape of the distributions.

To compare the two curves in Figure 2b, i.e. where distributions have different means, it is not possible to achieve a unanimous consensus about the “best” distribution, because Rawlsians will always oppose Utilitarians. In such cases, “extreme Utilitarians” will have to be ruled out in order to achieve a unanimous consensus on the “best” distribution among all the other decision-makers.

### 3.3 The principle of “diminishing transfers”

However as specified in the module Ranking income distributions with Lorenz curves, the features of the preferences of the decision-maker embodied in the SWF, i.e. the fact that, other things being equal, the decision-maker is an “income-seeker” and inequality-

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<sup>3</sup> This is the same case as presented in Figure 1, above.

<sup>4</sup> Note that for the case where, for example, distribution A, GL dominates distribution B for lower incomes, i.e. A crosses B from above, GL curves cross once and A has a higher mean than B, this is simply not possible. If the end point of A, i.e. the mean income, were superior to that of B, the GL curves would need to cross each other at least twice

averse, thus favours transfers from richer to poorer and dislikes transfers from poorer to richer individuals, do not convey any insight on how he/she would trade off inequality increasing impacts versus inequality decreasing impacts included in the same “policy package”. Distribution I would be preferred to A, only if the gain in social welfare, obtained by transferring income to the poorer, should more than compensate the loss in social welfare incurred by transferring income to the richer. Broadly speaking, if the decision-maker “likes reducing inequality to the advantage of the poorer more than he dislikes increasing inequality to the advantage of the richer”, he would probably prefer distribution I to A. More formally, a pre-condition for the decision-maker to prefer I to A is that, other things being equal, he/she accepts the so called principle of diminishing transfers, i.e. the third derivative of the SWF has to be positive, as explained in the section below.

The principle of diminishing transfers<sup>5</sup> states that the increase in the social welfare generated by a transfer of a given amount of income from a richer to a poorer individual, both of whom are in the lower part of the distribution, increases the social welfare more than a transfer of the same amount from a richer to a poorer individual, both of whom are in the upper part of the distribution.

The SWF is:

i) **Increasing in income.** The function  $w=w(y)$  is such that, other things being equal, an increase of the income of any individual  $i$ , at any income level, must lead to a positive variation of welfare. In Figure 3, below, where a welfare function is illustrated in the two dimensional space to highlight the contribution to the social welfare of the  $i^{\text{th}}$  income (i.e. all other things being equal), for example: income increases  $\Delta y$  at the income levels  $y_1$  and  $y_3$ , lead to positive variations of welfare  $\Delta w_1$  and  $\Delta w_3$ , respectively<sup>6</sup>. Mathematically, this property is reflected by the positive first derivative

of the welfare function:  $\frac{\partial w}{\partial y_i} > 0$ .

ii) **Reflecting inequality-aversion (principle of transfers).** The function  $w=w(y)$  is such that, other things being equal, an increase  $\Delta y$  of a richer individual’s income generates a lower welfare variation than the same increase  $\Delta y$  for a poorer individual. In Figure 3, below, for example, an income increase  $\Delta y$  at income level  $y_3$  generates a lower welfare increase  $\Delta w_3$  than the welfare increase  $\Delta w_1$  generated by the income increase  $\Delta y$  at the income level  $y_1$ <sup>7</sup>:

<sup>5</sup> To better understand this principle and to see how it is reflected in the mathematical properties of the SWF, it is worth recalling the assumptions about the SWF provided in EASYPol Module 001: [Social Welfare Analysis of Income Distributions: Ranking Income Distributions with Lorenz Curves](#), and analyzing the principle of diminishing transfers in that context.

<sup>6</sup> Similarly, decreases in income  $-\Delta y$  at income levels  $y_2$  and  $y_4$  lead to negative variations of welfare,  $-\Delta w_2$  and  $-\Delta w_4$  respectively.

<sup>7</sup> Similarly, decreases in income  $-\Delta y$  at income levels  $y_2$  and  $y_4$  generate negative variations of welfare  $-\Delta w_2$  and  $-\Delta w_4$  respectively, such that  $-\Delta w_2 < -\Delta w_4$  -i.e.  $\Delta w_2 > \Delta w_4$ .

$$\Delta w_3 < \Delta w_1$$

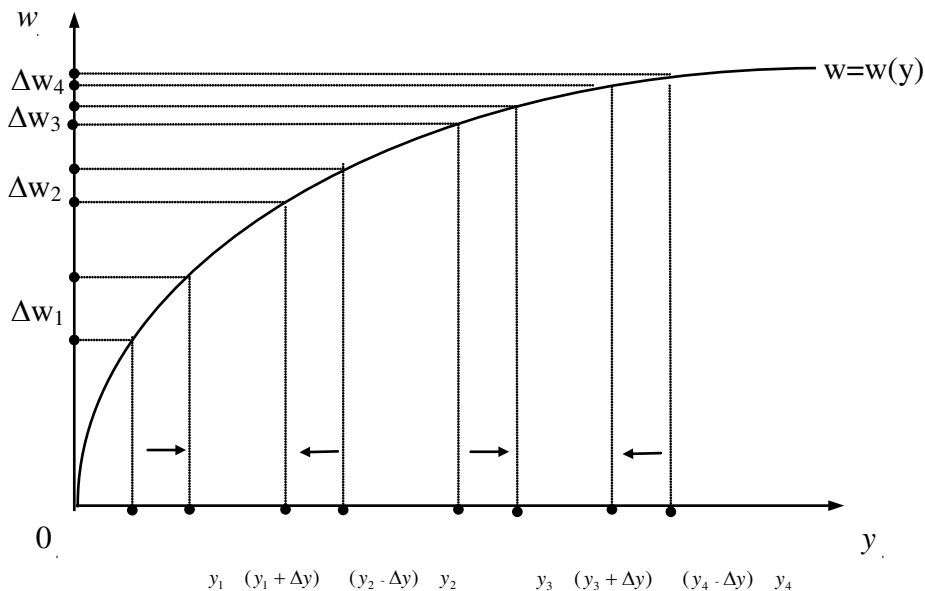
Therefore, the welfare variation decreases as income increases. Or, also:

$$(\Delta w_3 - \Delta w_1) < 0$$

The term in brackets can be considered the “variation of the variation of welfare” as negative income increases. Mathematically, for infinitesimal changes of  $y$ , i.e. for  $\Delta y \rightarrow 0$  this property is reflected by the negative second derivative of the welfare function:

$$\frac{\partial^2 w}{\partial y_i^2} < 0 .$$

**Figure 3 - The principle of diminishing transfers**



iii) **Accepting the principle of diminishing transfers.** The function  $w=w(y)$  is such that it satisfies the principle of diminishing transfers if, for small transfers of income  $\Delta y$ , the gain in welfare, due to a transfer of income from richer to poorer individuals in the lower part of the distribution, say from  $y_2$  to  $y_1$  (as indicated by the arrows), is greater than the gain in welfare due to a transfer of income from richer to poorer individuals in the upper part of the distribution, say from  $y_4$  to  $y_3$ .

In other words, we can deduce from Figure 3, that this amounts to:

$$(\Delta w_1 - \Delta w_2) > (\Delta w_3 - \Delta w_4)$$

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This implies that, after rearranging the equation:  $(\Delta w_4 - \Delta w_3) > (\Delta w_2 - \Delta w_1)$

i.e. the variation of the variation of income increases as income increases<sup>8</sup>.

or also:  $(\Delta w_4 - \Delta w_3) - (\Delta w_2 - \Delta w_1) > 0$

i.e. the variation of the “variation of the variation” of welfare is positive due to small changes of income. In mathematical terms, for infinitesimal changes of  $y$ , i.e. for  $\Delta y \rightarrow 0$ , this property is reflected by the positive third derivative of the welfare function:

$$\frac{\partial^3 w}{\partial y_i^3} > 0.$$

Similarly, with reference to Figure 3, above, for such decision-makers, an increase in the social welfare generated by a transfer of a given amount of income from a richer to poorer individual, both in the lower part of the distribution, more than offsets the loss of welfare generated by the transfer of the same amount from a poorer to a richer individual, both in the upper part of the distribution.

### 3.4 Requirements for ranking distributions if GL curves cross once

The first consequence for welfare rankings when GL curves cross, as in the above case, is that unanimous welfare prescriptions can no longer be achieved for all SWF, such

that:  $\frac{\partial w}{\partial y_i} > 0$  and  $\frac{\partial^2 w}{\partial y_i^2} < 0$ . It can be shown, however<sup>9</sup>, that welfare prescriptions are

possible in some cases when GL curves cross, but only for those SWF satisfying the “principle of diminishing transfers” i.e. those SWF that have the third derivative with

respect to individual incomes greater than zero:  $\frac{\partial^3 w}{\partial y_i^3} > 0$ .

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<sup>8</sup> Note that the two equations in brackets are both negative, but the equation on the left is “less negative” than the one on the right.

<sup>9</sup> Mathematical proof that restrictions on the SWF and conditions on the means and variances of the distributions, are required when GL curves cross, are sketched out in Lambert, 1993, p. 75.

### Box 1 - The principle of diminishing transfers

Preliminary requirement: the SWF must reflect the “principle of diminishing transfers”

When GL curves cross, unanimous welfare prescriptions can in some cases be obtained by using GL curves, only if we restrict the class of admissible SWF to those having:

$$\frac{\partial w}{\partial y_i} > 0, \quad \frac{\partial^2 w}{\partial y_i^2} < 0 \quad \text{and} \quad \frac{\partial^3 w}{\partial y_i^3} > 0$$

We can also demonstrate that, in addition to the above mentioned restrictions on the SWF, further requirements have to be fulfilled in order to use GL curves for ranking distributions.

These requirements depend on the relationship between the mean incomes of the distributions to be compared, as discussed in section 3.2., above, i.e. whether mean incomes are equal or different.

Let us start with the first case: **GL CURVES CROSS ONCE AND MEAN INCOMES ARE EQUAL**<sup>10</sup>. Here, Utilitarian SWFs are indifferent as mean incomes are equal. Rawlsian SWFs definitely prefer income distributions that dominate the lower part of the graph. But to have unanimous welfare prescriptions on the dominating distribution in the lower part of the graph, the following condition must be satisfied:

### Box 2 - Additional requirements when distributions have equal mean incomes

If two income distributions **Y** and **X** have the same mean and the following two conditions are verified:

- a) the GL curve of **Y** crosses the GL curve of **X** from above; and
- b) the variance of **Y** is lower than the variance of **X** ( $\sigma_Y^2 \leq \sigma_X^2$ ) – (VARIANCE CONDITION);

then **Y** would be preferred by all SWF satisfying the “principle of diminishing transfers”.

Let us now consider the second case: **GL CURVES CROSS ONCE AND MEAN INCOMES ARE UNEQUAL**. In this case, to have unanimous welfare prescriptions on the dominating distribution in the lower part, the following condition must be satisfied:

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<sup>10</sup> As mean incomes are equal, this case implies that standard Lorenz curves cross, otherwise welfare rankings could be made applying Atkinson’s Theorem as presented in EASYPol Module 001: [Social Welfare Analysis of Income Distributions: Ranking Income Distributions with Lorenz Curves](#).

### Box 3 - Additional requirements when distributions have different mean incomes

If these three conditions are verified:

- a) the GL curve of an income distribution **Y** crosses the GL curve of an income distribution **X** from above;
- b) the mean income of **Y** is lower than the mean income of **X** ( $\bar{y} < \bar{x}$ ); and
- c)  $\sigma_Y^2 < \sigma_X^2 - (\bar{x} - \bar{y})(2z - \bar{x} - \bar{y})$ , where  $z$  is the maximum income of the two distributions (MEAN-VARIANCE CONDITION);

then **Y** would be preferred by all SWF satisfying the “Principle of Diminishing Transfers”.

Hence, in both cases of GL curves crossing once, there is either a variance or a mean-variance condition to satisfy. However when mean incomes are unequal, as in this last case the mean-variance condition is more stringent because the variance of **Y** must be *sufficiently* less than the variance of **X**, not only just less, as the variance condition would prescribe.

If the variance condition or the mean-variance condition do not hold, the case of crossing GL curves cannot be solved, and welfare rankings are simply not possible.

When the mean-variance condition holds, however, we can go a step further to measure the robustness of the welfare ranking to the degree of inequality-aversion. It can be demonstrated that from the mean-variance condition it is indeed possible to calculate the lower limit of inequality-aversion  $b$ , below which welfare prescriptions obtained by GL ranking no longer hold.

The relevant expression is:

$$b = \frac{z(\bar{x} - \bar{y})}{(\sigma_X^2 - \sigma_Y^2) - (\bar{x} - \bar{y})(2z - \bar{x} - \bar{y})}$$

where symbols have the usual meaning. For example, if  $b=2$ , all decision-makers whose SWF includes an inequality-aversion parameter greater than 2 will agree on the result. Those with a lower inequality aversion (e.g. the Utilitarians) may not agree on the welfare ranking. Calculating the lower limit is very useful to understand the robustness of the ranking in terms of consensus across different decision makers with different degrees of inequality aversion.

#### **4 A STEP-BY-STEP PROCEDURE FOR WELFARE RANKINGS IF GL CURVES CROSS ONCE**

The flowchart in Figure 4, below, illustrates the step-by-step procedure for welfare rankings of two income distributions when their GL curves cross once<sup>11</sup>. In actual fact, Steps 1 to 6 are aimed at verifying whether welfare rankings can be solved with either Lorenz domination or with GL domination. If this is not possible because the GL curves cross each other once, the conditions reported in the section above need to be checked. Step 7 first requires you to calculate the variance of each income distribution. then, in step 8, the mean of the two distributions is checked. If the two distributions have equal mean incomes steps 9a to 11a will follow, whereas, if the two distributions have different mean incomes, steps 9b to 11b will follow.

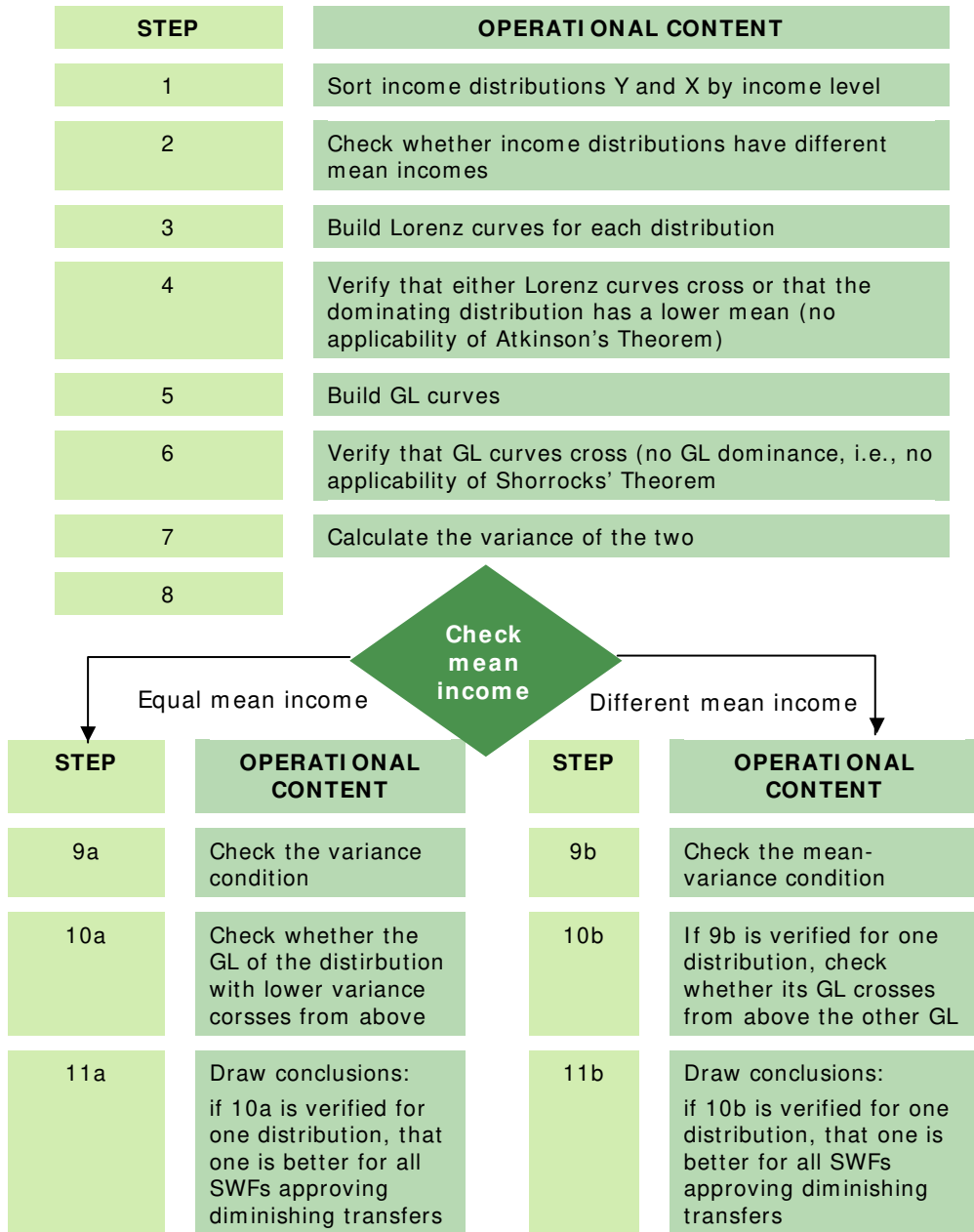
If GL curves cross once and mean incomes of the two distributions are equal, Step 9a, requires that the variance condition be verified. Then step 10a requires that you check which of the two GL curves crosses the other from above. In step 11a conclusions are drawn: if  $GL(Y)$  crosses  $GL(X)$  once from above and, at the same time, the Y variance is lower than the X variance, then income distribution Y will be socially preferred to X by all SWFs satisfying the “principle of diminishing transfers”.

If GL curves cross once and mean incomes of the two distributions are different, the mean-variance condition needs to be checked (Step 9b). Then step 10b requires to check which of the two GL curves crosses the other from above. In step 11b Conclusions are drawn: if a) the mean-variance condition is satisfied for Y; and b)  $GL(Y)$  crosses  $GL(X)$  from above, then the Y income distribution is socially preferred to X for all SWF satisfying the “principle of diminishing transfers”.

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<sup>11</sup> If multiple crossing of GL occur, sub-populations need to be analysed. Multiple crossings of GL curves however are quite infrequent in real cases. For the analysis of these cases refer to, e.g. Lambert, 1993, pp. 78 to 80.

**Figure 4 - Flowchart for a step-by step procedure for welfare rankings when GL curves cross once**





## 5 EXAMPLES: RANKING DISTRIBUTIONS WHEN GL CURVES CROSS ONCE

### 5.1 Distributions with equal mean income

The case where two income distributions, A and I, with the same mean income are compared, was discussed in section 3.1 and illustrated in Table 1 and Figure 1, above. Figure 1a depicts what Lorenz curves look like in this case<sup>12</sup>. Since Lorenz curves cross, GL curves are built and illustrated in Figure 1b. Apparently, also GL curves cross.

Hence, in this case, you need to calculate the variance of income distributions.

- The variance of income distribution A is 22.50
- The variance of distribution I is 22.00.

Therefore, considering that:

- the GL curve of distribution I crosses over the top of the A GL curve,
- distribution I has lower variance than A.

I is welfare-superior to A according to all decision-makers whose SWF satisfies the principle of diminishing transfers.

### 5.2 Distributions with a different mean income

On the other hand, Table 3, below, illustrates a case where GL curves cross once and mean incomes are unequal. Distribution L is the result of a policy which, starting from distribution A, leads to an income transfer from the middle of the distribution to the poorer area, thus bringing a net income decrease for the richer. This last impact decreases the mean income of L with respect to A, as indicated in the last row of columns (c) and (f).

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<sup>12</sup> Since income distributions have an equal mean, the only case in which Lorenz curves cannot rank income distributions is when they cross. Refer to EASYPol Module 001: [Social Welfare Analysis of Income Distributions: Ranking Income Distributions with Lorenz Curves](#), where the application of Atkinson's Theorem is presented.

**Table 3 - More equitable but lower mean distribution: GL curves crossing once**

Individuals (a)	Cum.share of p (hor.axis L/GL) (b)	Distribution A			Distribution L			Diff.cum. aver.Y L-A (i)
		Income (Y) (c)	Cum.share Y% (vert.axis of L) (d)	Cum.aver.Y (vert.axis GL) (e)	Income (Y) (f)	Cum.sh.Y% (vert.axis of L) (g)	Cum.aver.Y (vert.axis GL) (h)	
1	20.0%	3	6.7%	0.6	4	9.3%	0.8	0.2
2	40.0%	6	20.0%	1.8	6	23.3%	2.0	0.2
3	60.0%	9	40.0%	3.6	8	41.9%	3.6	0.0
4	80.0%	12	66.7%	6.0	12	69.8%	6.0	0.0
5	100.0%	15	100.0%	9.0	13	100.0%	8.6	-0.4
Total income		45.0			43.0			
Mean income		9.0			8.6			

L dominates A, i.e. A presents lower cumulated shares of income everywhere, but L has lower mean, so the Atkinson's theorem cannot be applied.

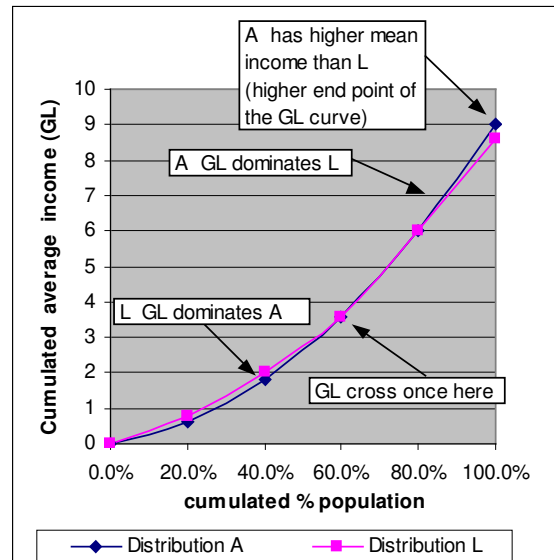
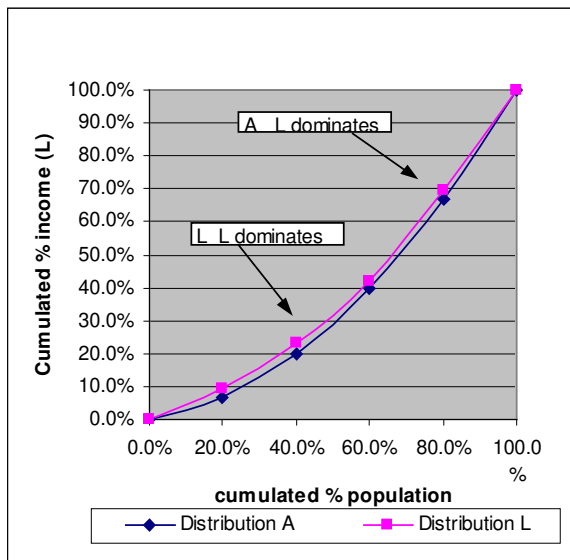
Note that the GL do cross because the difference between the ordinates of L and A are positive in the lower part of the curves and negative in the upper part. In particular, at thendpoint of the curve, A GL dominates L (A has higher mean income).

Now, income distribution L has a lower mean income, but also has a dominating Lorenz curve, as Figure 5a, below, shows.

In this case, GL curves are needed and also the mean-variance condition is required. In Figure 5b you can see that L crosses A towards the top of the curve. To be welfare-superior, the variance of L has to be lower than the threshold set by the mean variance condition, i.e. lower than 19.14.

The variance of Y is actually 14.80, thus implying that the preferred distribution is L and not A on the basis of welfare grounds as all SWF satisfy the following factors  $W' > 0$ ,  $W'' < 0$  and  $W''' > 0$ .

**Figures 5a and 5b - Crossing GL curves with different mean incomes**



## 6 CONCLUSION

To conclude, it is worth summarising the main results achieved so far. The basic result is that Lorenz curves and Generalised Lorenz curves are a powerful tool for welfare ranking of different income distributions. However, unlike the case of the complete specification of a SWF, these tools may give a «partial ordering», as there might be cases where required conditions for welfare ranking are not met .

Table 4, below, summarizes all results achieved so far, highlighting all outcomes deriving from the combination of the type of relation between curves and mean incomes of the distribution observed.

**Table 4 - Distributional dominance and welfare rankings**

	Type of relation	Mean income	Outcome	Validity	Notes
1	$L(Y) > L(X)$	$Y=X$	$W(Y) > W(X)$	$W'' > 0; W''' < 0$	
2	$L(Y) > L(X)$	$Y > X$	$W(Y) > W(X)$	$W'' > 0; W''' < 0$	
3	$L(Y) > L(X)$	$Y < X$	cannot say		need GL
4	$L(Y)$ and $L(X)$ cross	whatever	cannot say		need GL
5	$GL(Y) > GL(X)$	$Y=X$	$W(Y) > W(X)$	$W'' > 0; W''' < 0$	may solve 3 and 4
6	$GL(Y) > GL(X)$	$Y > X$	$W(Y) > W(X)$	$W'' > 0; W''' < 0$	may solve 3 and 4
7	$GL(Y) > GL(X)$	$Y < X$	cannot occur		
8	$GL(Y)$ crosses $GL(X)$ from above once	$Y=X$	$W(Y) R W(X)$	$W'' > 0; W''' < 0; W'''' > 0$	need $\text{var}(Y) < \text{var}(X)$ , otherwise cannot say
9	$GL(Y)$ crosses $GL(X)$ from above once	$Y < X$	$W(Y) R W(X)$	$W'' > 0; W''' < 0; W'''' > 0$	need the mean-variance condition, otherwise cannot say
10	$GL(Y)$ crosses $GL(X)$ from above once	$Y > X$	cannot occur		

Notes:

$L(Y)$  = Lorenz curve of distribution  $Y$

$GL(Y)$  = Generalised Lorenz curve of distribution  $Y$

$W(Y)$  = social welfare in  $Y$

$\text{var}(Y)$  = variance of  $Y$

$R$  means preferred by the more Rawlsian SWF.

It is worth noting again three important aspects:

- GL curves are required when either Lorenz curves cross or the dominating distribution has the lower mean (cases **3** and **4**).
- When GL curves cross once, additional restrictions on the form of the SWF  $W$  (i.e. its third derivative  $W'''' > 0$ ) are required in any case (cases **8** and **9**).
- When GL curves cross, welfare rankings are possible only if either the variance or the mean-variance conditions are satisfied, depending on the relation between mean incomes.

## 7 READERS' NOTES

### 7.1 Time requirements

The delivery of this module and related discussion may take two to three hours to an audience already familiar with concepts of policy, policy impact simulations, income and income distributions, Lorenz curves, social welfare and Social Welfare Functions.

### 7.2 Frequently asked questions

Frequently asked questions are, for example, the following:

- ✓ **What is the meaning and role of the preferences of the decision-maker?** i.e., what does it mean that the decision-maker is “inequality-averse” and an income-seeker? It is important in these cases to refer to the shape of the welfare function imposed by the restrictions on its first and second derivatives.
- ✓ **How is the “with policy” income distribution generated?** Selected trainees, not familiar with how to build policy scenarios may not understand how, in practical cases, the “with policy” income distribution is generated, i.e., how to logically link the policy proposal to the new income distribution. In addition, slightly more complex exercises than the examples provided in the module with real data should be prepared and carried out.

### 7.3 Complementary capacity building materials

The trainer may also consider the opportunity to present the relevant segment of the country case study based on real data: Inequality and poverty impacts of selected agricultural policies: the case of Armenia.

### 7.4 EASYPol links

This module belongs to a set of modules which discuss how to provide normative prescriptions when confronting alternative income distributions, i.e. how to identify the best income distribution in terms of social welfare, in a set of alternative income distributions. It is part of the modules composing a training path addressing [Analysis and monitoring of socio-economic impacts of policies](#).

The following EASYPOL modules form a set of materials logically preceding the current module, which can be used to strengthen the user's background:

- EASYPol Module 000: [Charting Income Inequality: The Lorenz Curve](#).
- EASYPol Module 001: [Social Welfare Analysis of Income Distributions: Ranking Income Distributions with Lorenz Curves](#)
- EASYPol Module 002: [Social Welfare Analysis of Income Distributions: Ranking Income Distribution with Generalised Lorenz Curves](#).

A case study presenting the use of crossing Generalised Lorenz curves to rank income distributions in the context an agricultural policy impact simulation exercise with real data is reported in the EASYPol Module 042: [\*Inequality and Poverty Impacts of Selected Agricultural Policies: The Case of Paraguay.\*](#)

## 8 FURTHER READINGS

- Anand S., 1983. *Inequality and Poverty in Malaysia*, Oxford University Press, London, UK.
- Lambert P. J., Aronson J. R., 1993. Inequality Decomposition Analysis and the Gini Coefficient Revisited, *Economic Journal*, **103**, 1221-1227.
- Lambert P., 1993. *The Distribution and Redistribution of Income – A Mathematical Analysis*, Manchester University Press, UK, 2<sup>nd</sup> edition.
- Lerman, Yitzhaki S., 1995. Changing Ranks and the Inequality Impacts of Taxes and Transfers, *National Tax Journal*, **48**, pp. 45-59
- Sen A., 1997. *On Economic Inequality*, Oxford University Press, Oxford, UK, 2<sup>nd</sup> edition.