Classification of competitiveness types using copula

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Abstract

In this paper we classify competitive markets using a new form of normalized Herfindahl index and the degree of dominance of the leader. For this purpose we use the notion of copula, which connects two or more random variables with given marginals.

The parameters of the two marginals (which are supposed to be normal) are estimated by the moments' method, and the parameter of the copula is computed using the value $\tau$ of Kendall.

1. Introduction

In [7] there is defined the market share of the company $i$ by the formula

$$Cp_i = \frac{CA_i}{\sum_{j=1}^{n} CA_j} = \frac{CA_i}{CA_r},$$

where $CA_i$ is the benefit of the company $i$. We denote next by $p_i = Cp_i$, and we reorder the companies such that $p_1 \geq \ldots \geq p_n$. In this case $p_1$ is the weight of the leader (see [7]).

The Herfindahl index, or the informational energy of Onicescu is (see [9,10,7,11])

$$H = \sum_{i=1}^{n} p_i^2.$$  

In [7] there are considered 553 clustered markets, 235 in 2004 and 318 in 2008 as follows

1) In the year 2004:
   a) 174 markets clustered CAEN Rev. 1 at group level (three digits)
   b) 47 markets clustered CAEN Rev. 1 at division level (two digits)
   c) 13 markets clustered CAEN Rev. 1 at section level (one alphabetic character)
   d) one national system

2) In the year 2008:
   a) 218 markets clustered CAEN Rev. 2 at group level (three digits)
   b) 80 markets clustered CAEN Rev. 2 at division level (two digits)
   c) 19 markets clustered CAEN Rev. 2 at section level (one alphabetic character)
   d) one national system

If we denote by $n$ the number of companies and by $p_1$ the weight of the leader we obtain the regression

$$\log H = 1.239375 \log p_1 - 0.163945 \log n + 0.164457$$

$$\left[0.016777\right] \quad \left[0.008147\right] \quad \left[0.016634\right]$$

$$R^2 = 0.97060142$$ and the estimated standard deviation 0.10470.

Because the Herfindahl index has a high variation degree (in the above case the ratio between
maximum and minimum is 832.2702) Mereuță (see [7]) introduced the normalized Herfindahl index:

\[ M = \frac{\ln H + \ln n}{\ln n}. \]  

(4)

The above cohesion measure is the normalized quadratic Rényi entropy, where the quadratic Rényi entropy is \( R = -\ln H \).

Another parameter used to measure the cohesion of the market shares is the degree of dominance of the leader (see [7])

\[ Gdl = \frac{n^2 - \frac{1}{n}}{1 - \frac{1}{n}}. \]  

(5)

Noticing that \( 0 \leq M \leq 1 \) and \( 0 \leq Gdl \leq 1 \) Mereuță defines the matrix of cohesion degrees. There are obtained nine regions of the unit square using the lines \( Gdl = 0.4 \), \( Gdl = 0.6 \), \( M = 0.4 \) and \( M = 0.6 \).

**Definition 1 ([12,8,13])** A copula is a function \( C : [0,1]^n \to [0,1] \) such that

1) If there exists \( i \) such that \( x_i = 0 \) then \( C(x_1,...,x_n) = 0 \).

2) If \( x_j = 1 \) for all \( j \neq i \) then \( C(x_1,...,x_n) = x_i \).

3) \( C \) is increasing in each argument.

We have the following theorem (see [12,8,13]).

**Theorem 1 (Sklar)** Let \( X_1, X_2, ..., X_n \) be random variables with the cumulative distribution functions \( F_1, F_2, ..., F_n \), and the common cdf \( H(x_1,...,x_n) = P(X_1 \leq x_1,...,X_n \leq x_n) \). In this case there exists a copula \( C(u_1,...,u_n) \) such that \( H(x_1,...,x_n) = C(F_1(x_1),...,F_n(x_n)) \). The copula \( C \) is well defined on the chartesian product of the images of the marginals \( F_1, F_2, ..., F_n \).

**Definition 2 ([12,14,15])** If \( n = 2 \) the copula \( C \) is Archimedean if \( C(u,v) < u \) for any \( u \in (0,1) \) and \( C(C(u,v),w) = C(u,C(v,w)) \) for any \( u,v,w \in [0,1] \). If \( n > 2 \) the copula \( C \) is Archimedean if there exists a \( n-1 \) Archimedean copula \( C_1 \) and a \( 2- \)Archimedean copula \( C_2 \) such that \( C(u_1,...,u_n) = C_2(C_1(u_1,...,u_{n-1}),u_n) \).

Consider a function \( \phi : (0,1) \to R \) decreasing and convex with \( \phi(1) = 0 \) and its pseudo-inverse \( g \) (\( g(y) \) has the value \( x \) if there exists \( x \) such that \( \phi(x) = y \) and \( 0 \) in the contrary case). We know (see [5,12]) that a copula \( C \) is Archimedean if and only if there exists a function \( \phi \) as above such that for any \( x, y \in [0,1] \) we have

\[ C(x,y) = g(\phi(x) + \phi(y)). \]  

(6)

In [14,15] there are presented methods to simulate Archimedean copulas, and in [2] there are presented algorithms to simulate queueing systems with one channel with arrivals and services depending through copulas.

In [4] there are found analytical formulae for the copulas that connect the number of customers in a Gordon and Newell queueing network, and their corresponding Spearman \( \rho \) and Kendall \( \tau \). This value is (see [8]):

\[ \tau = P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0) = 4 \int_0^1 \int C(u,v) \frac{\partial^2 C}{\partial u \partial v} \, du \, dv - 1 = 4 \int_0^1 \int \frac{\partial C}{\partial u} \frac{\partial C}{\partial v} \, du \, dv. \]

Sometimes we need the overlay probabilities, and we need in this case the notion of co-copula (see [14])

\[ C^*(u_1,...,u_n) = C(1-u_1,...,1-u_n) + \sum_{i=1}^n u_i - n + 1. \]  

(8)

The probabilistic interpretation of the co-copula is that if \( X_1, ..., X_n \) are random variables
having the marginals $F_1,\ldots, F_n$ and they are connected by the copula $C$, we have

$$H(x_1,\ldots,x_n) = P(X_1 \geq x_1,\ldots,X_n \geq x_n) = C(F_1(x_1),\ldots,F_n(x_n)),$$

where $F_i(x_i) = 1 - F_i(x_i)$.

### 2. The new matrix of cohesion degrees using isolines

In the matrix of cohesion degrees defined by Mereuță (see [7]) the regions are separated by $X = \alpha$, or by $Y = \beta$. The regions created by these lines are

$$
\begin{cases}
P(X \leq \alpha) = F(\alpha) \\
P(X \geq \alpha) = 1 - F(\alpha)
\end{cases}, \text{ respectively}
$$

$$
\begin{cases}
P(Y \leq \beta) = G(\beta) \\
P(Y \geq \beta) = 1 - G(\beta)
\end{cases},
$$

where $F$ and $G$ are the marginals of the $Gdl$ (first axis) and $M$ (second axis). But they do not take into account on the relation between the random variables $Gdl$ and $M$.

Suppose that the above random variables have normal marginals, as in [7], but they are connected by the copula $C$. The marginal parameters are estimated using the moments’ method, and the parameter $\theta$ of the copula $C$ is estimated as follows.

First we estimate $\tau$ using the empirical probabilities in the above formula, and next we compute the last term: we find $\tau$ in function of $\theta$. For instance, in the case of Farlie-Gumbel-Morgestern copula (see [8,12]) we find

$$\tau = \frac{2\theta}{9}, \text{ and from here}$$

$$\theta = \frac{9\tau}{2}.$$  

For the Fréchet family the copula is a mixture between the upper Fréchet bound, min and the copula product (the independence case) with the weights $\theta$, respectively $1-\theta$. Due to the fact that in the min case we have $\tau = 1$, and in the product case we have $\tau = 0$ we obtain

$$\theta = \tau. $$

When the copula is Archimedean and we know the function $\varphi$ in (6) we use the variables change $x = \varphi(u)$ and $y = \varphi(v)$, and finally we obtain

$$\tau = 1 - 4 \cdot \int_0^\varphi(0)(\varphi(u))^2 du dv. $$

In the case of Clayton family we have

$$C(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{\frac{1}{\theta}}.$$  

From $\frac{\partial C}{\partial u} = \frac{\varphi(u)}{\varphi(u)\varphi(v)}$ we obtain first $\varphi'(u) = -u^{-\theta-1}$, and from here

$$\varphi(u) = \frac{u^{-\theta} - 1}{\theta}, \text{ and}$$

$$g(w) = (\theta w + 1)^{\frac{1}{\theta}}.$$  

Using (7') we obtain

$$\tau = \frac{\theta}{\theta + 2}, \text{ and from here}$$

$$\theta = \frac{2 \cdot \tau}{1 - \tau}. $$

Other family of Archimedean copulas presented in [5,6,8] and simulated in [2] is the Frank family. In this case for $\theta \in \mathbb{R}^+$ we have
\[ C(u, v) = -\frac{1}{\theta} \ln \left( \frac{e^{-\theta(u+v)} - e^{-\theta u} - e^{-\theta v} + e^{-\theta}}{e^{-\theta} - 1} \right). \]  

We obtain also the copula \textit{Prod} for \( \theta = 0 \) and the copula \textit{min} for \( \theta \to \infty \). For \( \theta \to -\infty \) we obtain the lower Fréchet bound \( W \).

From \( \frac{\phi(u)}{\phi(v)} = \frac{\phi'(u)}{\phi'(v)} \) we obtain first \( \phi'(u) = \frac{4e^{-\theta u}}{e^{-\phi u} - 1} \), and from here

\[ \phi(u) = \ln \left( \frac{1 - e^{-\theta}}{1 - e^{-\theta u}} \right), \quad \text{and} \]

\[ g(w) = -\frac{1}{\theta} \ln(e^{-\theta} + 1), \quad \text{where} \quad e^{-\theta} = 1 - \gamma. \]  

For this family we obtain \[ \tau = 1 - 4 \cdot I, \quad \text{where} \]

\[ I = \frac{1}{\ln^2(1 + \gamma)} \int_0^1 \frac{\ln(1 + x)}{x} + \frac{1}{1 + x} \, dx. \]

In the case \( \theta \neq 0 \) we multiply the relation (18') by \( \ln^2(1 + \gamma) \), and in the case \( \theta = \gamma = \tau = 0 \) and \( I = \frac{1}{4} \) we compute \( I'(0) = \frac{1}{4\tau} \) using the Taylor series for \( \ln(1 + x) \) and \( \frac{1}{1 + x} \). We obtain the Cauchy problem

\[ \begin{cases} 
\gamma'(I) = \frac{\ln^2(1 + \gamma(1))}{4 - \ln^2(1 + \gamma(1))} & \text{for } I \neq \frac{1}{4} \\
\gamma'(\frac{1}{4}) = 36 \\
\gamma(\frac{1}{4}) = 0
\end{cases} \]

Because \( I = \frac{1}{4} \) we obtain the Cauchy problem

\[ \begin{cases} 
\gamma'(\tau) = \frac{\ln^2(1 + \gamma(\tau))}{4 - \ln^2(1 + \gamma(\tau))} & \text{for } \tau \neq 0 \\
\gamma'(0) = -9 \\
\gamma(0) = 0
\end{cases} \]

Finally we take into account that \( \gamma(\theta) = e^{-\theta} - 1 \) and \( \gamma'(\tau) = -e^{-\theta} \cdot \theta'(\tau) \). We obtain

\[ \begin{cases} 
\theta'(\tau) = \frac{\theta^2}{2(1 - \tau) + 4 \theta^2} & \text{for } \tau \neq 0 \\
\theta'(0) = 9 \\
\theta(0) = 0
\end{cases} \]

The above Cauchy problem is solved using the Runge-Kutta method.

In the case of the Gumbel-Hougaard family (see [5,8,13]) we have for \( \theta \geq 1 \) and \( \beta = \frac{1}{\theta} \)

\[ C(u, v) = e^{-[(\ln u)^\beta + (\ln v)^\beta]^\beta}. \]

For \( \theta = 1 \) we obtain the copula \textit{Prod} and for \( \theta \to \infty \) we obtain the copula \textit{min}.

From \( \frac{\phi(u)}{\phi(v)} = \frac{\phi'(u)}{\phi'(v)} \) we obtain first \( \phi'(u) = -\frac{\theta(\ln u)^{\theta-1}}{u} \), and from here

\[ \phi(u) = (\ln u)^\theta, \quad \text{and} \]

\[ g(x) = e^{-x^\theta}. \]  

For this family we obtain \[ \tau = 1 - \frac{1}{\theta}, \quad \text{and from here} \]

\[ \theta = 1 - \frac{1}{\tau}. \]
\[ \theta = \frac{1}{1 - \tau}. \]  

The Gumbel-Barnett copula is
\[ C(u,v) = u \cdot v \cdot e^{-(\theta \ln u)(\ln v)}, \] with \( 0 < \theta \leq 1. \)  

We notice that we have also the copula product (independence) for \( \theta \to 0. \)

From \( \frac{\partial}{\partial u} = \phi'(u) \phi'(v) \) we obtain first \( \phi'(u) = -\frac{1}{u(1-\theta \ln u)^2} \), and from here
\[ \phi(u) = \frac{\ln(1-\theta \ln u)}{\theta}, \] and
\[ g(x) = e^{\frac{1}{\theta^2}}. \]

Using (7') we obtain
\[ \tau = -\beta \cdot \int_{-\infty}^{x} e^{\frac{-x}{\beta}} dx < 0. \]

where \( \beta = \frac{\alpha}{\theta}. \)

The Ali-Mikhail-Haq copula is
\[ C(u,v) = \frac{u \cdot v}{1 - \theta(1-u)(1-v)}, \] with \( -1 \leq \theta \leq 1. \)

We notice that we have the copula Prod (independence) for \( \theta = 0. \)

From \( \frac{\partial}{\partial u} = \phi'(u) \phi'(v) \) we obtain first \( \phi'(u) = -\frac{1}{u(1-\theta \ln (1-u))} \), and from here
\[ \phi(u) = \frac{1}{1-\theta} \ln \left( \frac{\theta + 1-\theta}{u} \right), \] and
\[ g(x) = e^{\frac{1-\theta}{(1-\theta) x - \theta}}. \]

Using (7') we obtain
\[ \tau = 1 - \frac{2(1-\theta)^2 \ln(1-\theta)}{3\theta^2} - \frac{2}{3\theta^2}. \]

In the above formula \( \tau \) is increasing on \( \theta \), and we have \( \tau(-1) = \frac{5-8\ln 2}{3} \) and \( \tau(1) = \frac{1}{3}. \) If we know \( \tau \) we obtain \( \theta \) using the bisection method.

**Definition 3** Let \((X,Y)\) be a bi-variate random variable such that the random variables \(X\) and \(Y\) are connected by the copula \(C\).

The copula of non-overlay, non-overlay for \((X,Y)\) is \(C_{11}(u,v) = C(u,v).\)

The copula of overlay, overlay for \((X,Y)\) is \(C_{00}(u,v) = C^*(1-u,1-v).\)

The copula of non-overlay, overlay for \((X,Y)\) is \(C_{10}(u,v) = u - C(u,v).\)

The copula of overlay, non-overlay for \((X,Y)\) is \(C_{01}(u,v) = v - C(u,v).\)

**Remark 1** If the marginal distributions are uniform on \([0,1]\) then, if we denote by \(H\) the common cumulative distribution function, we have: \(C_{11}(u,v) = H(u,v) = P(X \leq u,Y \leq v),\) \(C_{00}(u,v) = \overline{H}(u,v) = P(X \geq u,Y \geq v),\) \(C_{10}(u,v) = P(X \leq u,Y \geq v)\) and \(C_{01}(u,v) = P(X \geq u,Y \leq v).\) For other marginal distributions, \(F\) and respectively \(G\), we have \(P(X \leq x,Y \leq y) = C_{11}(F(x),G(y)),\) \(P(X \geq x,Y \geq y) = C_{00}(F(x),G(y)),\) \(P(X \leq x,Y \geq y) = C_{10}(F(x),G(y))\) and \(P(X \geq x,Y \leq y) = C_{01}(F(x),G(y)).\)

We will find the isolines in \(u,v\) given by \(\alpha_1, \ldots, \alpha_k\) with \(0 < \alpha_1 < \ldots < \alpha_k\)
\[
\begin{align*}
C_{11}(u,v) &= \alpha_j \\
C_{00}(u,v) &= \alpha_j \\
C_{10}(u,v) &= \alpha_j \\
C_{01}(u,v) &= \alpha_j 
\end{align*}
\quad \text{where } j = 1, k \tag{23}
\]

The corresponding isolines in \( x, y \) are built from the isolines in \( u, v \) such that \( x = F^{-1}(u) \) and \( y = F^{-1}(v) \). These are the separators of the regions bordered by \( x_{\min} = \min_{i=1,n} X_i, \ x_{\max} = \max_{i=1,n} X_i, \ y_{\min} = \min_{i=1,n} Y_i, \ y_{\max} = \max_{i=1,n} Y_i \). The above regions have corresponding regions in the plane \( Ouv \) in the box bordered by \( u_{\min} = F(x_{\min}), \ u_{\max} = F(x_{\max}), \ v_{\min} = G(y_{\min}) \) and \( v_{\max} = G(y_{\max}) \).

### 3. Application

Consider the above 553 clustered markets (see [7]). We have \( Gdl_{\min} = 0.074, \ Gdl_{\max} = 0.9997, \ M_{\min} = 0.1954, \ M_{\max} = 0.9769 \). In [7] the marginal distributions are considered normal. Using the moments method we obtain \( \hat{\mu}_{Gdl} = 0.47432, \ \hat{\sigma}_{Gdl}^2 = 0.05801, \ \hat{\mu}_M = 0.51181 \) and \( \hat{\sigma}_M^2 = 0.01745 \). From these estimated parameters we obtain the box in the plane \( Ouv \) \( u_{\min} = 0.0067, \ u_{\max} = 0.9988, \ v_{\min} = 0.0002 \) and \( v_{\max} = 0.9989 \).

The Kendal \( \tau \) is \( 0.53986 \). The parameter \( \theta \) depending on the copula family is as in the following table.

**Table 1:** The value of the estimated parameter \( \theta \) depending on the copula family.

<table>
<thead>
<tr>
<th>Family</th>
<th>Constraints on ( \tau \in [-1,1] )</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clayton: ( \theta &gt; 0 )</td>
<td>( \tau &gt; 0 )</td>
<td>2.34646</td>
</tr>
<tr>
<td>Frank: ( \theta \neq 0 )</td>
<td>( \tau \neq 0 )</td>
<td>0.00948</td>
</tr>
<tr>
<td>Gumbel-Hougaard: ( \theta \geq 1 )</td>
<td>( \tau \geq 0 )</td>
<td>2.17323</td>
</tr>
<tr>
<td>Gumbel-Barnett: ( 0 &lt; \theta \leq 1 )</td>
<td>( \tau &lt; 0 )</td>
<td>not our case</td>
</tr>
<tr>
<td>Ali-Mikhail-Haq: ( -1 \leq \theta \leq 1 )</td>
<td>( \frac{5-8\ln 2}{3} \leq \tau \leq \frac{1}{3} )</td>
<td>not our case</td>
</tr>
<tr>
<td>FGM: ( -1 \leq \theta \leq 1 )</td>
<td>(</td>
<td>\tau</td>
</tr>
<tr>
<td>Fréchet: ( \theta \geq 0 )</td>
<td>( \tau &gt; 0 )</td>
<td>0.53986</td>
</tr>
</tbody>
</table>

In the following graphics there are represented first the above boxes in \( Ouv \) and \( Oxy \), and the data points with \( U_i = F(Gdl_i) \) and \( V_i = G(M_i) \), respectively \( X_i = Gdl_i \) and \( Y_i = M_i \). We represent also the isolines (23) in the plane \( Ouv \) with \( k = 2, \ \alpha_t = 0.4 \) and \( \alpha_s = 0.6 \), and the corresponding isolines for the plane \( Oxy \). These graphics are represented for each case of copula for which we have estimated the parameter \( \theta \) (the constraints for the above tables are fulfilled).

In these graphics each class has a code given by four integer numbers: first number is for \( C_{00} \), the second number is for \( C_{01} \), the third number is for \( C_{10} \) and the last number is for \( C_{11} \). The number corresponding to a copula type depends on the position of its value and \( \alpha_t \), tacking \( \alpha_t = 0 \) and \( \alpha_s = 1 \). For instance the code from the fourth position is \( k \) if \( \alpha_k \leq C_{11}(u,v) < \alpha_{k+1} \) for any \( u, v \) in the considered region. The region from the center (containing the middle of the square) has the code \((0,0,0,0)\), and the class from the corner \((1,0)\) has the code \((0,0,2,0)\). For each of the 553 points \((U_i, V_i)\) obtained by the application of the (normal) cumulative distribution function on \( Gdl_i \) and \( M_i \) for the clustered market \( i \) we find the code of its class. When the point is in an old class we count it
for this class, and we memorize its class’s code if the point is in a new class. In each of the obtained regions we write the class number, and, between parentheses, the number of clustered markets from the involved class.

**Fig. 1a**: The graphics in the coordinates $u, v$ in the case of Clayton copula

**Fig. 1b**: The graphics in the coordinates $x, y$ in the case of Clayton copula
Fig. 2a: The graphics in the coordinates $u, v$ in the case of Frank copula

Fig. 2b: The graphics in the coordinates $x, y$ in the case of Frank copula
Fig. 3a: The graphics in the coordinates \( u, v \) in the case of Gumbel-Hougaard copula

Fig. 3b: The graphics in the coordinates \( x, y \) in the case of Gumbel-Hougaard copula
We notice that the two national systems from 2004 (represented by a circle in the above graphics, with $Gdl = 0.2287$, $M = 0.4996$, $U = F(Gdl) = 0.1524$ and $V = G(M) = 0.4579$) and 2008 (represented by a square in the above graphics, with $Gdl = 0.2861$, $M = 0.4856$, $U = F(Gdl) = 0.2161$ and $V = G(M) = 0.4191$) are in the same class having the code $(1,0,0,0)$, i.e.
0.4 \leq C_{00}(U,V) < 0.6, C_{11}(U,V) < 0.4, C_{01}(U,V) < 0.4, C_{10}(U,V) < 0.4.

The numbers of clustered markets in each class depending on the year (2004, 2008 or both) and on the level (group, division, section or all three levels) are listed in Appendix A. The line of the class that contains the two national systems is bolded. The star at the exponent at “Total” means that the total does not contain the national system. Two stars in the last total means that we did not take into account the two national systems. For instance, in the case of Clayton copula the totals are 44* for the year 2004, 53* for the year 2008 and 97** for both year. It means that, if we take into account the national systems, these totals would be 45 for the year 2004, 54 for the year 2008 and 99 for both years.

4. Conclusions

Our classification has a probabilistic interpretation: each region obtained by isolines is such that the four probabilities resulting from the four copula types from definition 3 are in given intervals bordered by $\alpha_j$. It has also more possible classes than the nine regions from the case of Mereuţă (see [7]): in each case of copula family there are 13 possible classes. Even the effective number of classes is greater (11 in the cases of Clayton and Fréchet copula, respectively 12 in the cases of Frank and Gumbel-Hougaard copula).

There are also similitudes between the classification of Mereuţă and those from this paper. In the classification of Mereuţă the regions with small $Gdl$ and $M$, and with both values big (the heads of the main diagonal) there are relative big numbers of the contained clustered markets: both numbers are equal to 97. The same thing we can say about our case. For the smallest values of $Gdl$ and $M$ we obtain 180 clustered markets in the case of Clayton copula, 129 clustered markets in the case of Frank copula, 170 clustered markets in the case of Gumbel-Hougaard copula, respectively 168 clustered markets in the case of Fréchet copula. For the highest values of $Gdl$ and $M$ we obtain 114 clustered markets in the case of Clayton copula, 104 clustered markets in the case of Frank copula, 126 clustered markets in the case of Gumbel-Hougaard copula, respectively 117 clustered markets in the case of Fréchet copula.

On the secondary diagonal the above numbers are small. The number of clustered markets with high $Gdl$ and low $M$ (the class from bottom-right corner) is 2 in the case of Mereuţă, respectively 3 in the case of this paper, for each case of copula.

The number of clustered markets with low $Gdl$ and high $M$ (the class from top-left corner) is 5 in the case of Mereuţă, 2 in the case of Frank copula, and 0 (no clustered market in the class) in the other cases.

In the case of Frank copula there is also an interesting similitude between our classification and the classification of Mereuţă for the middle class (medium $Gdl$ and $M$): the number of clustered markets is 110 in the case of Mereuţă, and 111 in our case.

References


## Appendix A

The numbers of clustered markets depending on the year and on the level

### Table 2: The number of clustered markets in the case of Clayton copula

<table>
<thead>
<tr>
<th>Class number</th>
<th>2004</th>
<th></th>
<th></th>
<th>2008</th>
<th></th>
<th></th>
<th>2004 and 2008</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Group level</td>
<td>Division level</td>
<td>Section level</td>
<td>Total</td>
<td>Group level</td>
<td>Division level</td>
<td>Section level</td>
<td>Total</td>
<td>Group level</td>
</tr>
<tr>
<td>1</td>
<td>37</td>
<td>7</td>
<td>2</td>
<td>46</td>
<td>53</td>
<td>12</td>
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