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All-Pay Auctions with Budget Constraints: The Two Bidder Case

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Abstract

Consider an all-pay auction with interdependent, affiliated valuations and private budget constraints. We characterize a symmetric equilibrium for the case of two players. In contrast with the second-price auction, making budgets more severe can depress the bids of unconstrained bidders.

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Suppose firms are lobbying for a lucrative government contract to supply some product. Clearly, the contract's net value has an idiosyncratic component as each firm's production costs may differ. On the other hand, each firm also has a privately known limit on how much it is able to spend on lobbying. Perhaps the management approves of small restaurant meals but large expenditures or bribes are morally too much to stomach. A competitor, on the other hand, may be less hampered in its lobbying strategy despite valuing the contract similarly. How does the lobbying game unfold when competitors differ in their valuation for the prize and in their ability or capacity to compete for it?

Interpreting the above example as an all-pay auction, we build on Krishna & Morgan (1997)'s analysis of the all-pay auction with interdependent valuations by introducing *private* budget constraints. Restricting attention to a symmetric, two-bidder setting we characterize a symmetric equilibrium strategy, which is piecewise differentiable. The model admits affiliated and interdependent valuations and therefore is an extension of the general symmetric model of Milgrom & Weber (1982). As will be clear in the discussion below, the restriction to two bidders is not substantive for our equilibrium characterization, but greatly simplifies exposition. A key conclusion of the equilibrium characterization is that the presence of private budget constraints may encourage more aggressive bidding by relatively wealthy participants. The ability to exploit the financial misfortune of others is a tempting strategic option.

Although the model below is phrased as an all-pay auction with players being called "bidders," it applies to many contest situations where resources are irreversibly expended in pursuit of a goal. For example, firms may be engaged in a patent race with the firm devoting more resources to the research effort claiming victory. Suppose the resulting invention will generate similar, but not necessarily identical, profits for any of its potential developers. Any given firm may be constrained in how many resources it can allocate to the specific project for exogenous reasons. In the short-term it may have many competing demands on its finances, potential investors may be reluctant,[1] or it may lack the physical space and talent necessary to mount a large-scale effort. In many research projects such private constraints are certainly paramount in guiding decision making.

Given the economic salience of contests, there exists a substantial literature on all-pay auctions with both complete-information (Baye *et al.* , 1996) and incomplete-information models (Krishna & Morgan, 1997) thoroughly explored. This paper builds on the latter environment and allows for affiliated, interdependent valuations. As the examples above demonstrate, we feel this is a natural benchmark for the model's information structure. Private- and common-values, are special cases of the model.

This study also complements a growing literature on auctions with budget constraints. The baseline environment is analogous to those seen in Fang & Parreiras (2002) and Kotowski (2010) who study the second-price and first-price auctions respectively. Both of those studies build on Che & Gale (1998), which is a seminal paper in this strand of literature.

The remainder of the paper is organized as follows. Section 1 introduces the model and section 2 characterizes a symmetric equilibrium strategy. Section 3 considers the equilibrium's comparative static properties. Special emphasis is placed on contrasting its behavior with the behavior of the second-price auction equilibrium strategy. A discussion of open questions and the prospects of extending the analysis to the war of attrition concludes.

1 The Model

Suppose there are two risk-neutral bidders $\{i, j\}$. The generalization of the model to N bidders is conceptually straightforward, but cumbrous. As should be clear from the arguments below, no characteristic feature of the equilibrium depends on the two-bidder assumption provided that any generalization is within the confines of "general symmetric model" from

[1]Financiers may disagree with the firm regarding the invention's actual value and therefore may provide only partial financing.

Milgrom & Weber (1982).

Each bidder receives a two-dimensional private signal $\theta_i = (s_i, w_i) \in [0,1] \times [\underline{w}, \bar{w}]$. s_i is a bidder's *value-signal* which is her private information about the item for purchase. For example, if bidders are bidding for oil-drilling rights, s_i would be bidder i's estimate of the quantity of oil in a given tract. w_i is a bidder's *budget* above which she cannot bid. Budgets are therefore understood as hard limits and may (literally) correspond to a bidder's cash holdings or credit limit.

Conditional on $\mathbf{s} = (s_i, s_j)$, the value of the item to bidder i is $v_i(s_i, s_j) = v(s_i, s_j)$; thus, valuations are interdependent and pure common values and private values are special cases of the environment.[2] Suppose $v\colon [0,1]^2 \to [0,1]$ is continuous, strictly increasing in the first argument, nondecreasing in the second, and is normalized such that $v(0,0) = 0$ and $v(1,1) = 1$. A maintained assumption is that $\bar{w} > 1$. As valuations are bounded, this assumption implies that there are bidders who do not expend their entire wealth in (any) equilibrium. Ex post we will observe that this assumption is too conservative given that bidders will shade their bid relative to their valuation; however, we maintain it to simplify the derivation. Fang & Parreiras (2002) show that this assumption is important in deriving the equilibrium strategy in the second-price auction.

As standard, bidders are assumed to be risk-neutral. In the all-pay auction, if bidder i wins with a bid of $b_i \leq w_i$, her utility is $v(s_i, s_j) - b_i$; otherwise it is $-b_i$.[3] The introduction of risk aversion into this model would introduce several complications analogous to those encounter in the first-price auction. As shown by Kotowski (2010), the interaction of a bidder's private budget with her risk preferences often introduces countervailing incentives rendering the existence of "monotone" equilibria a subtle question.

[2] $v_i(s_i, s_j)$ is bidder bidder i's best estimate of the item's actual value conditional on (s_i, s_j). If the true value of the item described by some random variable V_i, then $v_i(s_i, s_j) \equiv \mathbb{E}[V_i | S_i = s_i, S_j = s_j]$.

[3] Formally, ties can be resolved by a coin flip. Ties are zero probability events in the equilibria which we construct.

The following assumption summarizes the information structure.

Assumption 1. *The distribution of types satisfies the following:*

1. *Value-signals are affiliated and their joint-distribution admits a density* $h(s_i, s_j)$. *Moreover,* $\forall x, y \in [0,1]$, $0 < h(x,y) = h(y,x) < \infty$.

2. *For all* $s_j \in [0,1]$, $v(\cdot, s_j)h(s_j|\cdot) \colon [0,1] \to \mathbb{R}_+$ *is nondecreasing.*

3. *For all* i, *the cumulative distribution function of a player's budget is* $G(w_i)$. $G(\cdot)$ *has full support on* $[\underline{w}, \bar{w}]$ *and is* $\mathcal{C}^2$. *Let* $g \equiv G'$ *denote the associated density function.*

Affiliated value-signals are a standard assumption introduced to the auction literature by Milgrom & Weber (1982). Assumption 1.2 is also fruitfully employed by Krishna & Morgan (1997). Intuitively, it limits the degree of correlation between s_i and s_j relative to, say, s_i's direct influence on player i's preferences. The assumption always holds if signals are independent. It is also satisfied when $v(s_i, s_j) = (s_i + s_j)/2$ and $h(s_i, s_j) \propto 1 + s_i s_j$. Additionally, it is implied by another often encountered assumption in the auction literature, namely that for each $s_j \in [0,1]$, $v(\cdot, s_j)\frac{h(s_j|\cdot)}{1-H(s_j|\cdot)} \colon [0,1] \to \mathbb{R}_+$ is nondecreasing.[4] This latter assumption proves central in characterizing equilibrium bidding in the first- and second-price auctions with budget constraints. It is interesting to note, therefore, that we require a weaker assumption on the information structure in this regard.

Players' budgets are mutually independent and both share the same cumulative distribution function. This assumption enhances tractability and is standard in studies of auctions with budget constraints. The differentiability assumption on G is for expositional convenience and can be relaxed. Finally, it is necessary to stress that $\underline{w}$ need not be zero. Indeed, in many situation it is common knowledge that participants in a contest or auction satisfy a certain threshold level of resources. For example, bidders may be required to post a bond to be allowed to bid or there may be (unmodeled) participant selection into the auction.

[4] A consequence of affiliation is that $\frac{h(s_j|\cdot)}{1-H(s_j|\cdot)}$ is non-increasing.

2 A Symmetric Monotone Equilibrium

Following similar analysis of the first- and second-price auctions, we wish to identify a symmetric equilibrium of the form

$$\beta(s,w) = \min\left\{\bar{b}(s), w\right\} \tag{1}$$

where $\bar{b}\colon [0,1] \to \mathbb{R}_+$ is strictly increasing, continuous and (piecewise) differentiable. One can interpret a bid of $\bar{b}(s)$ as an unconstrained bid—the bidder has the ability to bid more, but chooses not to do so. As we are considering symmetric equilibria, where both players follow the same equilibrium strategy, we will generally suppress player subscripts.

As customary, we begin with an heuristic derivation of $\bar{b}(s)$. Suppose for the moment that there is a symmetric equilibrium of the form (1) and consider a bidder of type $(s,\bar{w})$. By assumption, such a bidder would be unconstrained at all of her equilibrium bids as $\bar{b}(s) \leq v(1,1) = 1 \leq \underline{w}$. Treating the auction as a revelation mechanism, suppose this player bids as if she were a type $(x,\bar{w})$, i.e. $\beta(x,\bar{w}) = \bar{b}(x)$. Following this bid, her expected utility is

$$U(\beta(x,\bar{w})|s,\bar{w}) = G(\bar{b}(x))\int_0^1 v(s,y)h(y|s)dy + (1-G(\bar{b}(x)))\int_0^x v(s,y)h(y|s)dy - \bar{b}(x). \tag{2}$$

The first term is the expected gain from winning the auction and defeating a budget constrained opponent, i.e. $w_j < \bar{b}(x)$. The second term accounts for the expected gain from winning the auction versus an opponent with a budget in excess of $\bar{b}(x)$ but who chooses to bid less than $\bar{b}(x)$, i.e. $s_j < x$ but $w_j \geq \bar{b}(x)$. The final term is the bidder's payment, which she makes regardless of the auction's outcome.

If $\bar{b}(s)$ is this player's equilibrium best response, an "announcement" of $x = s$ must satisfy a local first-order optimality condition. Specifically, because $\bar{b}(s)$ is assumed to be sufficiently smooth then for almost every s, $\left.\frac{\partial U(\beta(x,\bar{w})|s,\bar{w})}{\partial x}\right|_{x=s} = 0$. Shuffling terms in the resultant expression gives the following differential equation describing the behavior of $\bar{b}(s)$ at points of differentiability:

$$\bar{b}'(s) = \frac{[1 - G(\bar{b}(s))]v(s,s)h(s|s)}{1 - g(\bar{b}(s))\int_s^1 v(s,y)h(y|s)dy} \tag{3}$$

Prior to identifying the appropriate boundary conditions, two observations are worthwhile. First, if $\bar{b}(s) < \underline{w}$ is an optimal bid, then (3) reduces to

$$\bar{b}'(s) = v(s,s)h(s|s). \tag{4}$$

(4) is the same differential equation identified by Krishna & Morgan (1997) as characterizing bidding behavior in the all-pay auction absent budget constraints. Second, when $\bar{b}(s) > \underline{w}$, (3) accounts for the change in marginal incentives faced by unconstrained bidders: slight bid increases not only defeat opponents with slightly higher valuations but they also defeat all opponents with sufficiently low budgets regardless of their valuation. This second effect ameliorates the well-known winner's curse phenomenon in interdependent-value settings.

Building on these two observations, we will argue that under appropriate regularity conditions made precise below, there will be an equilibrium of the form in (1) and $\bar{b}(s)$ will be composed of at most two differentiable components characterized by (3) if $\bar{b}(s) > \underline{w}$ and (4) with $\bar{b}(s) < \underline{w}$.

The following lemma, essentially a house-keeping statement, identifies when the presence of budget constraints is strategically irrelevant and the usual non-budget-constraints equilibrium strategy obtains.

Lemma 1. *Suppose* $\underline{w} \geq \int_0^1 v(y,y)h(y|y)dy$. *Then* $\beta(s,w) = \int_0^s v(y,y)h(y|y)dy$ *is a symmetric equilibrium strategy profile.*

Proof. This follows immediately from Krishna & Morgan (1997, Theorem 2). $\square$

Henceforth, suppose

$$\underline{w} < \int_0^1 v(y,y)h(y|y)dy. \tag{5}$$

Given condition (5), there exists a unique $\tilde{s} \in [0,1)$ such that $\underline{w} = \int_0^{\tilde{s}} v(y,y)h(y|y)dy$. Going forward, we offer two propositions characterizing equilibrium bidding; each depends of different assumptions concerning $G(\cdot)$, the distribution of a bidder's budget.

Proposition 1. *Suppose that* $g(w) < 1$*. Then* $\beta(s,w) = \min\{\bar{b}(s), w\}$ *where*

$$\bar{b}(s) = \int_0^s v(y,y)h(y|y)dy \tag{6}$$

for $s < \tilde{s}$ *and* $\bar{b}(s)$ *is the unique solution to the differential equation* (3) *with the boundary condition* $\bar{b}(\tilde{s}) = \underline{w}$*, is a symmetric equilibrium strategy.*

The economic interpretation of the bound on $g(\cdot)$ is straightforward. Essentially, Proposition 1 relies on the assumption that the probability that a bidder is budget constrained is relatively low. This assumption may be reasonable in many situations of interest, especially where the contest's stakes are relatively low and most competitors have sufficient resources to mount a capable challenge.

Proposition 2. *Suppose* $\underline{w} = 0$ *and* $G''(w) \equiv g'(w) < 0$*. Let*

$$\psi(s) \equiv \inf\left[\left\{b \colon 1 - g(b)\int_s^1 v(s,y)h(y|s)dy > 0\right\} \cap [\underline{w}, \bar{w}]\right] \tag{7}$$

and suppose $\psi(0) \in (0,1)$*. Then*

$$\beta(s,w) = \min\{\bar{b}(s), w\}, \tag{8}$$

where $\bar{b}(s)$ is the unique strictly increasing solution to the differential equation (3) *with the boundary condition $\bar{b}(0) = \psi(0)$, is an equilibrium of the all-pay auction.*

The bound on $g(\cdot)$ becomes more tenuous once stakes are increased or the environment's primitives suggest a salient constraint on a participant's liquidity. For instance, if $\underline{w} = 0$ and $G(w) = \sqrt{\frac{w}{\bar{w}}}$ then the local bound on $g(\cdot)$ is violated near zero. In such circumstances, an alternative regularity condition allows for the existence of an equilibrium of the form (1). This is formalized in the next proposition.

It is clear that the assumptions of Propositions 1 and 2 are independent. The concavity condition in Proposition 2 admits a natural economic interpretation: there are fewer participants with higher wealth levels.

Proof. Here we supply the proof of Proposition 1. The proof of Proposition 2 is technically similar with the requisite additional arguments relegated to the appendix.

First, due to the assumption following Lemma 1, $\tilde{s} \in [0, 1)$ exists and is uniquely defined. Second, note that (3) satisfies the Lipschitz condition within its domain in $(s, b) \in [0, 1] \times [\underline{w}, 1]$. To verify this fact, observe that for all s, $1 - g(b) \int_s^1 v(s, y)h(y|s)dy$ is a continuous function of b and is by hypothesis bounded away from zero for all s. Therefore, there exists ϵ sufficiently small such that $1 - g(b) \int_s^1 v(s, y)h(y|s)dy \geq \epsilon > 0$. Then, note that if

$$b'(b, s) = \frac{[1 - G(b)]v(s, s)h(s|s)}{1 - g(b) \int_s^1 v(s, y)h(y|s)dy},$$

we can calculate

$$\begin{aligned}\left|\frac{d}{db}b'(b, s)\right| &= \left|\frac{(-g(b))\left(1 - g(b(s)) \int_s^1 v(s, y)h(y|s)dy\right) - (1 - G(b))\left(-g'(b) \int_s^1 v(s, y)h(y|s)dy\right)}{\left(1 - g(b(s)) \int_s^1 v(s, y)h(y|s)dy\right)^2}\right| v(s, s)h(s|s) \\ &\leq \frac{(1) + |g'(b)|}{\epsilon^2} \sup_{s\in[0,1]} v(s, s)h(s|s) \equiv K < \infty\end{aligned}$$

Hence, for $b, \hat{b}$ we have $|b'(b,s) - b'(\hat{b},s)| \leq K|b - \hat{b}|$. Consequently, the differential equation admits a unique solution on $[\tilde{s}, 1]$ with the boundary condition $\bar{b}(\tilde{s}) = \underline{w}$. Moreover, as $\int_s^1 v(s,y)h(y|s)dy \leq 1$ and $g(b) < 1$ for $b < 1$, for bids in the range $[\underline{w}, 1]$ $\bar{b}'(s) > 0$.

Given that player j is following the strategy $\beta(s,w)$, we need only verify that player i does not wish to deviate to any bid in the range of $\beta(s,w)$, or more specifically, in the range of $\bar{b}(s)$. The argument is organized into several steps:

Suppose first a player of type (s,w) chooses to bid $\bar{b}(x) \leq w$. This bidder's payoff at such a bid is

$$U(\bar{b}(x)|s,w) = G(\bar{b}(x)) \int_0^1 v(s,y)h(y|s)dy + (1 - G(\bar{b}(x))) \int_0^x v(s,y)h(y|s)dy - \bar{b}(x).$$

$U(\bar{b}(x)|s,w)$ is differentiable in x at all $x \neq \tilde{s}$. At $x = \tilde{s}$ is both right and left differentiable although these values may generally differ from one another.

Consider first a bidder of type $s \leq \tilde{s}$. Deviations to bids in the range $[0, \underline{w}]$ are not profitable. To see this observe that for $x < \tilde{s}$,

$$\frac{\partial U(\bar{b}(x)|s,w)}{\partial x} = v(s,x)h(x|s) - b'(x).$$

Then for $x < s$ we have that

$$\begin{aligned} \frac{\partial U(\bar{b}(x)|s,w)}{\partial x} &= v(s,x)h(x|s) - b'(x) \\ &\geq v(x,x)h(x|x) - b'(x) = 0 \end{aligned}$$

and for $x > s$, $\frac{\partial U(\bar{b}(x)|s,w)}{\partial x} \leq 0$. Therefore, expected utility is maximized at the bid $\bar{b}(s)$ and such a bidder has no profitable deviation in the range $[0, \underline{w}]$.

For deviations to bids in the range $[\underline{w}, \bar{b}(1)]$ we can similarly calculate that

$$\frac{\partial U(\bar{b}(x)|s,w)}{\partial x} = g(\bar{b}(x))\bar{b}'(x)\int_x^1 v(s,y)h(y|s)dy + (1 - G(\bar{b}(x))v(s,x)h(x|s) - \bar{b}'(x).$$

As $x > s$, then

$$\begin{aligned}
&\frac{\partial U(\bar{b}(x)|s,w)}{\partial x} \\
&= g(\bar{b}(x))\bar{b}'(x)\int_x^1 v(s,y)h(y|s)dy + (1 - G(\bar{b}(x))v(s,x)h(x|s) - \bar{b}'(x) \\
&\leq g(\bar{b}(x))\bar{b}'(x)\int_x^1 v(x,y)h(y|x)dy + (1 - G(\bar{b}(x))v(x,x)h(x|x) - \bar{b}'(x) = 0.
\end{aligned}$$

Therefore, expected payoffs are decreasing further for all competitive bids in excess of $\underline{w}$.

Consideration of a bidder with a value signal of $s \geq \tilde{s}$ leads to an analogous argument and is omitted for brevity. We conclude therefore that no (unconstrained) bidder has an incentive to deviate to another bid in the range of $\bar{b}(s)$.

Suppose instead that $\beta(s,w) = w$. We must confirm that this is a constrained optimum bid. Such a bidder can only deviate to bids in the range $[0,w]$; equivalently, they can only choose to bid like a type $x \in [0,\hat{s}]$ where $\hat{s} = \bar{b}^{-1}(w)$. $\hat{s}$ is well defined because $\bar{b}(s)$ is strictly increasing. As $\bar{b}(s) > w$, it must be that $\hat{s} < s$. Therefore, from the arguments above, for all $x \in [0,\hat{s}]$, $\frac{\partial U(\bar{b}(x)|s,w)}{\partial x} \geq 0$ making reductions in bid unprofitable.

Finally, we verify that all bidders receive nonnegative equilibrium expected payoffs. As $\bar{b}(0) = 0$, the above arguments show that all bidders prefer the bid of $\beta(s,w)$ to the bid of 0. As a bid of zero guarantees a payoff of zero, it follows that $U(\beta(s,w)|s,w) \geq 0$ for all (s,w). This completes the proof. $\square$

Remark 1. Because expected payoffs are non negative for all bidders, $\bar{b}(s) < 1$. The fact that the cumulative distribution function of budgets, $G(w)$, is well-behaved for $w > 1$ is not necessary for the equilibrium to obtain. How budgets are distributed beyond the range of relevant bids is not important for equilibrium behavior.

Remark 2. As constructed above, $\bar{b}(s)$ is differentiable at $s \in (0,1)\backslash\{\tilde{s}\}$. At $\tilde{s}$, $\lim_{s\to\tilde{s}^-}\bar{b}(s) \neq \lim_{s\to\tilde{s}^+}\bar{b}(s)$. The discrete change in the slope of this function accounts for the change in marginal incentives once bids cross the $\underline{w}$ threshold. In addition to bidders with value-signals less than s, a bid of $\bar{b}(s) > \underline{w}$ defeats a mass of bidders who have higher signals but have lower budgets.

Example 1. Suppose that value-signals $S_i \overset{i.i.d.}{\sim} U[0,1]$ while budgets $W_i \overset{i.i.d.}{\sim} U[0.08, 2.08]$. Let $v(s_i, s_j) = (s_i + s_j)/2$. It is readily verified that $\bar{b}(s) = s^2/2$ for $s < \tilde{s} = 0.4$. Of course, this function is also the equilibrium strategy in this model absent budget constraints.

For $s > \tilde{s}$, $\bar{b}(s)$ is the solution to the differential equations

$$\bar{b}'(s) = \frac{4(2-\bar{b}(s))s}{7+s(3s-2)} \tag{9}$$

with the boundary condition $\bar{b}(.4) = 0.08$ and the resulting equilibrium strategy is

$$\beta(s,w) = \begin{cases} \frac{s^2}{2} & s \leq 0.4 \\ \min\{\bar{b}(s), w\} & s > 0.4 \end{cases}$$

The functions $\bar{b}(s)$ and $s^2/2$ are plotted in Figure 1. It is worth noting that the presence of budget constraints encourages some bidders to bid more aggressively. Locally, to the right of $\tilde{s} = 0.4$, $\bar{b}(s) > s^2/2$. This fact is easily verified analytically and holds more generally as shown by the following lemma.

Lemma 2. *Suppose the conditions of Proposition 1 are satisfied and* $\underline{w} > 0$. *Then* $\lim_{s\to\tilde{s}^+}\bar{b}'(s) > \lim_{s\to\tilde{s}^-}\bar{b}'(s)$.

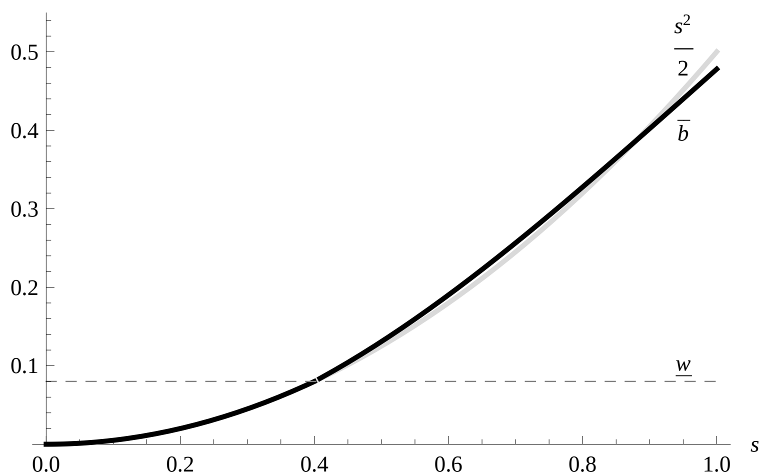

Figure 1: The functions $\bar{b}(s)$ and $s^2/2$ in the characterization of equilibrium bidding in Example 1.

Proof. A direct calculation gives

$$\lim_{s \to \tilde{s}^+} \bar{b}'(s) = \left(\frac{1}{1 - g(\underline{w}) \int_{\tilde{s}}^{1} v(\tilde{s}, y) h(y|s) dy} \right) v(\tilde{s}, \tilde{s}) h(\tilde{s}|\tilde{s})$$
$$> v(\tilde{s}, \tilde{s}) h(\tilde{s}|\tilde{s}) = \lim_{s \to \tilde{s}^-} \bar{b}'(s)$$

The bracketed term is positive and strictly greater than 1 because of the local bound $g(\underline{w}) < 1$. □

The conclusion that the introduction of budget constraints encourages some auction participants to place higher bids in an important conclusion. It also holds for the first-price and the second-price auction.

3 Comparative Statics

To gain some intuition of the equilibrium's behavior, we offer several comparative static exercises. The first exercise considers changes in the budget distribution $G(\cdot)$. Changes in the distribution of budgets may reflect broader economic fluctuations that may constrain bidder's behavior. The second exercise examines the non-strategic release of value-relevant information by a third party, such as the seller. The release of positive information encourages more aggressive bidding by unconstrained bidders. Analogous comparative statics are offered by Fang & Parreiras (2002) for the second-price auction and by Kotowski (2010) for the first-price auction.

3.1 Changes in the Budget Distribution

First consider the comparative static exercise of making budgets more likely to be low. In particular fix an auction environment where $\underline{w} = 0$, but consider two distributions of

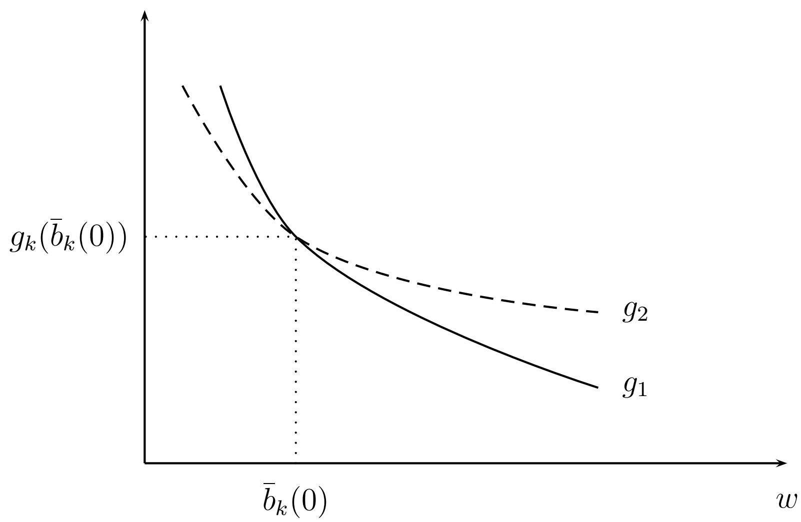

Figure 2: The densities $g_2(w)$ and $g_1(w)$.

budgets which are strictly concave: $G_1(w)$ and $G_2(w)$. Let $\beta_k(s,w) = \min\{\bar{b}_k(s), w\}$ denote the corresponding equilibrium strategies and suppose:

(A) G_2 likelihood ratio dominates G_1; i.e., $\frac{g_2(w)}{g_1(w)}$ is increasing. And,

(B) $g_1(\bar{b}_1(0)) = g_2(\bar{b}_1(0))$.

Likelihood ratio dominance implies hazard rate dominance, $\frac{g_1(w)}{1-G_1(w)} > \frac{g_2(w)}{1-G_2(w)}$, and first-order stochastic dominance, $G_1(w) \geq G_2(w)$. Budget levels are more likely to be low when they are distributed according to G_1. Point (B), which can be weakened, places additional structure on the relationship between the distributions. Along with strict concavity, it implies that $\bar{b}_1(0) = \bar{b}_2(0)$. Figure 2 summarizes the implied situation. Changing the distribution of budgets in this manner allows us to conclude:

Proposition 3. *For all (s,w), $\beta_2(s,w) \geq \beta_1(s,w)$ and the inequality is strict for bidders with a sufficiently high budget.*

Proof. It is sufficient to establish $\bar{b}_2(s) > \bar{b}_1(s)$ for $s > 0$. Let $b > \bar{b}_2(0) = \bar{b}_1(0)$. Then

$G_1(b) > G_2(b)$ and $g_1(b) < g_2(b)$. Hence, for all b where (3) is positive

$$0 < \frac{[1 - G_1(b)]v(s,s)h(s|s)}{1 - g_1(b)\int_s^1 v(s,y)h(y|s)dy} < \frac{[1 - G_2(b)]v(s,s)h(s|s)}{1 - g_2(b)\int_s^1 v(s,y)h(y|s)dy}.$$

This inequality implies that if $\bar{b}_2(s)$ and $\bar{b}_1(s)$ intersect, $\bar{b}_1(s)$ must cross $\bar{b}_2(s)$ from above. Hence, if $\bar{b}_1(s) > \bar{b}_2(s)$ for any $s > 0$, then $\bar{b}_1(s) > \bar{b}_2(s)$, $s \in (0, \hat{s})$, for $\hat{s}$ sufficiently small. Suppose $0 < s < \hat{s}$, then

$$g_2(\bar{b}_2(s)) \geq g_1(\bar{b}_2(s)) \geq g_1(\bar{b}_1(s)), \text{ and}$$
$$G_2(\bar{b}_2(s)) \leq G_1(\bar{b}_2(s)) \leq G_1(\bar{b}_1(s)).$$

Therefore, $\bar{b}_2'(s) \geq \bar{b}_1'(s)$. Finally, because $\bar{b}_k(s)$ can be expressed as an integral equation,

$$\bar{b}_1(0) = \bar{b}_1(\hat{s}) - \int_0^{\hat{s}} \bar{b}_1'(x)dx > \bar{b}_2(\hat{s}) - \int_0^{\hat{s}} \bar{b}_2'(x)dx = \bar{b}_2(0)$$

which contradicts $\bar{b}_1(0) = \bar{b}_2(0)$. Therefore, $\bar{b}_2(s) > \bar{b}_1(s)$ for $s > 0$ as desired. $\square$

Although the comparative static result is consistent with a prima facie intuition that more stringent budgets should depress bids, it contrasts with the analogous result in the second-price auction. In the second-price auction, making budgets more likely to be low in the sense of (A) increases the bid of high-budget bidders (Fang & Parreiras, 2002). This difference highlights the competing effects of budget constraints. Binding budgets ameliorate the winner's curse and therefore encourage more aggressive bidding. A bidder is more optimistic regarding the item's value because in equilibrium she may defeat an opponent with a high value-signal who has a low budget. However, budgets also stratify competition. With fewer bidders capable of competing at higher bid levels the marginal incentive to bid higher declines. This serves to depress bidding. In the second-price auction the former effect dominates the latter. In the all-pay auction, however, the net effect can go the other way.

3.2 Public Signals

Suppose prior to bidding in the auction, both players observe a public signal $s_0 \in [0,1]$. In many auction environments public signals are relevant along two dimensions. First, a public signal may be directly relevant for a player's own payoffs. All else equal, higher realizations of this signal are good news concerning the item's ultimate value: $\frac{\partial v_i(s_i,s_j|s_0)}{\partial s_0} > 0$. Second, a public signal may be irrelevant to a player's own payoffs, $\frac{\partial v_i(s_i,s_j|s_0)}{\partial s_0} = 0$, but may be correlated with the private information of the opposing bidder. While various combinations of these two scenarios are quite natural, we will focus on the extreme cases to highlight the competing effect of public information on equilibrium bidding in this environment.

Formally, suppose player's valuations are now given by the function $v_i(s_i, s_j|s_0) = v(s_i, s_j|s_0)$ which is strictly increasing in s_i and non-decreasing in s_j and s_0. As before, normalize the payoff function such that $v(0,0|0) = 0$ and $v(1,1|1) = 1$.

Proposition 4. *Consider the environment of Proposition 1 and suppose the public signal is independent of (s_i, s_j) but is strictly payoff relevant. Then,*

(a) There exists a symmetric equilibrium of the form $\beta_i(s_i, w_i|s_0) = \min\{\bar{b}(s_i, s_0), w_i\}$.

(b) If $s_0^h > s_0^l$, then $\bar{b}(s_i, s_0^h) \geq \bar{b}(s_i, s_0^l)$.

Proof. The existence of equilibrium of the form $\beta_i(s_i, w_i|s_0) = \min\{\bar{b}(s_i, s_0), w_i\}$ follows immediately from Proposition 1 with statements conditional on s_0 replacing the unconditional statements. In particular, $\bar{b}(s_i, s_0)$ inherits the analogous properties of $\bar{b}(s_i)$ from the original model.

Let $s_0^h > s_0^l$ be two realizations for the public signal. As it is only payoff relevant, we have that $h(s_j|s_i, s_0^h) = h(s_j|s_i, s_0^l) \equiv h(s_j|s_i)$ for all s_i and s_j.

Define $\tilde{s}_k$ as the value-signal at which $\int_0^{\tilde{s}_k} v(y,y|s_0^k)h(y|y)dy = \underline{w}$. Then, for any $s < \min\{\tilde{s}_h, \tilde{s}_l\}$ we have that

$$\begin{aligned}\bar{b}(s,s_0^h) - \bar{b}(s,s_0^l) &= \int_0^s v(s,y|s_0^h)h(y|s)dy - \int_0^s v(s,y|s_0^l)h(y|s)dy \\ &= \int_0^s \left[v(s,y|s_0^h) - v(s,y|s_0^l)\right] h(y|s)dy > 0.\end{aligned}$$

Therefore, for $s < \min\{\tilde{s}_h, \tilde{s}_l\}$ we have $\bar{b}(s,s_0^h) > \bar{b}(s,s_0^l)$. Consequently, $s_0^h < s_0^l$. Moreover, we know that for $s \in [\tilde{s}_h, \tilde{s}_l]$, $\bar{b}(s,s_0^h) \geq \underline{w} \geq \bar{b}(s,s_0^l)$.

It remains to be verified that for all $s \in [\tilde{s}_l, 1]$ the inequality $\bar{b}(s,s_0^h) \geq \bar{b}(s,s_0^l)$ is maintained.

Suppose the contrary. Then there exists a $\hat{s} \in [\tilde{s}_l, 1]$ such that $\bar{b}(\hat{s},s_0^h) = \bar{b}(\hat{s},s_0^l) = \hat{b} > \underline{w}$. Then,

$$\begin{aligned}\bar{b}'(\hat{s},s_0^h) &= \frac{[1-G(\hat{b})]v(\hat{s},\hat{s}|s_0^h)h(\hat{s}|\hat{s})}{1 - g(\hat{b})\int_{\hat{s}}^1 v(\hat{s},y|s_0^h)h(y|\hat{s})dy} \\ &> \frac{[1-G(\hat{b})]v(\hat{s},\hat{s}|s_0^l)h(\hat{s}|\hat{s})}{1 - g(\hat{b})\int_{\hat{s}}^1 v(\hat{s},y|s_0^l)h(y|\hat{s})dy} = \bar{b}'(\hat{s},s_0^l)\end{aligned}$$

Therefore, at any such $\hat{s}$, $\bar{b}(s,s_0^h)$ must cross $\bar{b}(s,s_0^l)$ from below. This however is a contradiction as at $\tilde{s}^l$, $\bar{b}(\tilde{s}^l,s_0^h) > \bar{b}(\tilde{s}^l,s_0^l) = \underline{w}$. □

The following example shows that when the public signal s_0 has no direct effect on a bidder's valuation but is affiliated with (s_i, s_j) she may bid less in equilibrium—at least over a range of value-signals. This conclusion is orthogonal to the presence of budget constraints and complements earlier analysis of the all-pay auction by Krishna & Morgan (1997).

Example 2. Suppose $v(s_i,s_j) = (s_i+s_j)/2$, $\underline{w} > 0$ and $h(s_i,s_j,s_0) \propto 1 + s_i s_j s_0$. For s sufficiently low, the bidder's equilibrium bid is $\beta(s,w|s_0) = \int_0^s v(y,y)h(y|y,s_0)dy$. A direct calculation gives that

$$v(y,y)h(y|y,s_0) = \frac{2y\,(s_0y^2+1)}{s_0y+2} \tag{10}$$

Simple algebra shows shows that for $y < 1/2$, $\frac{d}{ds_0}v(y,y)h(y|y,s_0) < 0$. Therefore for s sufficiently small, $\beta(s,w|s_0^h) < \beta(s,w|s_0^l)$.

Intuition behind this example is straightforward. Conditional on observing a high public signal s_0^h bidder i can infer that her opponent likely has a high signal and will in consequence bid high. A high bid by the opponent decreases the probability with which bidder i wins the auction, discouraging her from bidding aggressively (recall, in an al-pay auction she must pay her bid irrespective of auction outcome). In contrast, when the public signal has a direct effect on a bidder's value for the item, the resulting boost in expected payoff may be enough to counteract the discouragement effect stemming from affiliation alone.

4 Conclusion and Open Questions

In this paper, we establish the existence of a monotone, symmetric pure-strategy equilibrium in an affiliated value all-pay auction with private financial constraints. There are several exciting avenues of research within this benchmark model. For instance, in the preceding analysis we make no general statements about the equilibrium's uniqueness. Many standard techniques in establishing equilibrium uniqueness involve characterizing behavior of the bidding strategy's inverse; in our environment, the bidding strategy's inverse is multi-valued complicating the exercise.

A second avenue of open research concerns revenue rankings. In the environment considered, the first-price, second-price, and all-pay auctions share a similar functional form for an equilibrium: $\min\{b(s), w\}$. However, precise revenue comparisons are illusive. For instance, with independent private-values and financial constraints alone Che & Gale (1998) establish the following ranking of expected revenues:

$$\text{All-Pay} \geq \text{First-Price} \geq \text{Second-Price}$$

In contrast, with interdependent affiliated values and no budget constraints, Krishna & Morgan (1997) confirm the following revenue comparisons:

$$\text{All-Pay} \geq \text{First-Price}$$
$$\text{Second-Price} \geq \text{First-Price}$$

There exists no general ranking between the all-pay and the second-price auction. Whereas these limiting environments suggest the superiority of the all-pay auction above the first-price auction, it is an open question whether such rankings are preserved in generic instances of the current model. Moreover, it may be possible to identify (non-trivial) conditions on the distribution of budgets allowing for revenue rankings between the all-pay and second-price auctions in interdependent-value settings.

Given that the first-price, second-price, and all-pay auctions have symmetric equilibria of the general form $\min\{b(s), w\}$, a natural conjecture is that the war of attrition—the second-price, all-pay auction—also has an equilibrium in this class as well. The answer to this conjecture is negative. To see this conclusion, we can follow a similar derivation to that employed in (2) and express the expected utility from a bid of $b(x) > \underline{w}$ as

$$\begin{aligned}U(b(x)|s,w) = G(b(x)) &\int_0^1 v(s,y)h(y|s)dy \\ &+ (1 - G(b(x))) \int_0^x v(s,y)h(y|s)dy \\ &- (1 - G(b(x)))(1 - H(y|s))b(x) \\ &- \int_0^x \left[\int_0^{b(y)} zg(z)dz + \int_{b(y)}^1 b(y)g(z)dz \right] h(y|s)dy \\ &- \int_x^1 \left[\int_0^{b(x)} zg(z)dz \right] h(y|s)dy\end{aligned}$$

The resulting differential equation characterizing $b(\cdot)$ in the range of bids above $\underline{w}$ is

$$\bar{b}'(s) = \frac{(1 - G(\bar{b}(s)))h(s|s)v(s,s)}{(1 - G(\bar{b}(s)))(1 - H(s|s)) - g(\bar{b}(s)) \int_s^1 v(s,y)h(y|s)dy}$$

It is straightforward to verify that in many examples meeting the above assumptions (all) solutions to this equation are (strictly) decreasing, which is a contradiction. Indeed, based on Krishna & Morgan (1997) we conjecture that equilibria in the war of attrition with private budget constraints will be markedly different than those of the other auction formats considered. Even with bounded valuations, absent budget constraints the equilibrium strategy in the war of attrition is unbounded—a feature that cannot be readily reconciled with a world of budget constraints.

A Proof of Proposition 2

The main qualification necessary to accommodate Proposition 2 centers on identifying admissible solutions to the differential equation (3), reproduced here

$$b'(s) = \frac{[1 - G(b(s))]v(s,s)h(s|s)}{1 - g(b(s)) \int_s^1 v(s,y)h(y|s)dy} \tag{11}$$

with the requisite boundary condition $b(0) = \psi(0)$. Recall that a maintained hypothesis in deriving (11) was that its solution is strictly increasing for a.e. $s \in [0,1]$. Here we show that under the conditions of Proposition 2, maintained in this appendix, we can find an increasing solution that will be a constituent argument of the desired equilibrium strategy.

Lemma 3. *Suppose the hypotheses of Proposition 2 hold. Then there exists a strictly increasing function* $\bar{b}(s)\colon [0,1] \to \mathbb{R}_+$ *which is a solution to* (11) *such that* $\lim_{s\to 0^+} \bar{b}(s) = b_0 = \psi(0)$. *Moreover,* $\bar{b}(s)$ *is the unique function with these properties.*

Proof. Observe first that $\psi(s)\colon [0,1] \to \mathbb{R}_+$ is a well defined, continuous function. This conclusion follows immediately from the strict concavity of $G(\cdot)$ and the assumption $\psi(0) = b_0 \in (0,1)$.

Second, note that $\psi(s) < 1$ for all s. To see this conclusion, suppose the contrary. By the strict monotonicity of $g(\cdot)$, there exists a $\hat{s} \in (0,1)$ such that

$$1 - g(1) \int_{\hat{s}}^1 v(\hat{s},y)h(y|\hat{s})dy = 0.$$

As $\int_{\hat{s}}^1 v(\hat{s},y)h(y|\hat{s})dy \le 1$, then $g(1) > 1$ and for all $b \in [0,1]$, $g(b) > 1$ (by monotonicity). However, for $w \in (1, \underline{w}]$, we have

$$1 \ge G(w) = \int_0^w g(b)db = \int_0^1 g(b)db + \int_1^w g(b)db > 1 + \int_1^w g(b)db > 1$$

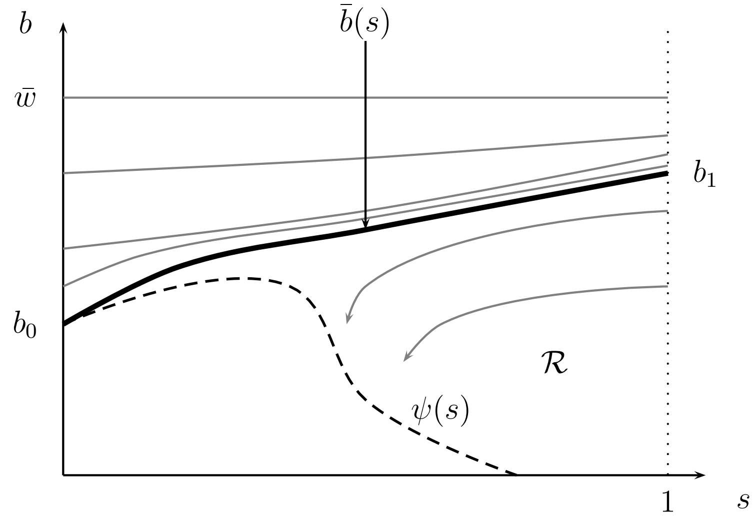

Figure 3: Construction of $\bar{b}(s)$ and representative solutions $b_x(s)$.

which is a contradiction. Therefore, $\psi(s) < 1$ for all s.

For the (hypothesized) solution $\bar{b}(s)$ to be strictly increasing, it is clear that it must satisfy $\bar{b}(s) > \psi(s)$ for all $s > 0$. It is easy to see that all solutions of (11) in the region $\mathcal{R} = \{(s, b)\colon s \in (0, 1), b \in (\psi(s), \bar{w})\}$ are increasing.

Let $b_x(s)$ be the (unique) solution to (11) with the boundary condition $(1, x)$. For instance, $b_{\bar{w}}(s) = \bar{w}$. If, however, $x < \sup \psi(s)$ then $b_x(s)$ does not reach the b-axis. As $b_x(0)$ (when it exists) varies continuously in x, consider the exercise of lowering x from $\bar{w}$ to b_1 such that $x \to b_1 \implies b_x(0) \to b_0$. (The appropriate solution to (11) can be computed in a region uniformly bounded away from $\psi(s)$ and then extended to $\mathcal{R}$'s boundary.) Let the resulting solution be $\bar{b}(s)$. Figure 3 pictures this construction. Fang & Parreiras (2002) employ a similar argument.

$\bar{b}(s)$ is the only solution of (11) in $\mathcal{R}$ such that $\lim_{s\to 0^+} \bar{b}(s) = b_0$. To see this, suppose there exists some other solution, call it $\hat{b}(s)$, with $\lim_{s\to 0^+} \hat{b}(s) = b_0$. Without loss of generality, suppose $\hat{b}(s) < \bar{b}(s)$ for all $s > 0$. (The solutions cannot intersect in the interior of $\mathcal{R}$.)

Noting $G(\cdot)$'s concavity, $\forall s > 0$

$$\bar{b}'(s) = \frac{[1 - G(\bar{b}(s))]v(s,s)h(s|s)}{1 - g(\bar{b}(s)) \int_s^1 v(s,y)h(y|s)dy} < \frac{[1 - G(\hat{b}(s))]v(s,s)h(s|s)}{1 - g(\hat{b}(s)) \int_s^1 v(s,y)h(y|s)dy} = \hat{b}'(s)$$

Then for $s > 0$,

$$b_0 = \bar{b}(s) - \int_0^s \bar{b}'(x)dx > \hat{b}(s) - \int_0^s \hat{b}'(x)dx = b_0$$

which is a contradiction.

□

The following lemma confirms that there is an equilibrium of the form $\beta(s,w) = \min\{\bar{b}, w\}$.

Lemma 4. *$\beta(s,w)$ defined in (8) is a symmetric equilibrium of the all-pay auction with private budget constraints.*

Proof. There are several cases to consider. First, for a bidder who bids $\beta(s,w) = \bar{b}(s) < w$, and therefore is *unconstrained*, identical arguments to those found in the proof of Proposition 1 rule out deviations in the range of $\bar{b}(s)$.

Consider now a bidder who ought to bid $\beta(s,w) = w \leq \psi(0)$. Given that the other bidder is following the strategy $\beta(s,w)$, a bid of $b \leq w$ gives a pay-off of

$$U(b|s,w) = G(b) \int_0^1 v(s,y)h(y|s)dy - b$$

A simple calculation gives that

$$\frac{\partial U}{\partial b} = g(b) \int_0^1 v(s,y)h(y|s)dy - 1.$$

Consider type $s = 0$, then for $b < \psi(0)$, we have that $\frac{\partial U}{\partial b} \geq 0$. Thus, a bid of $\beta(0,w) = w$ is a (constrained) optimum. It is also an optimum for bidders of type $s > 0$ as

$$\int_0^1 v(s,y)h(y|s)dy - 1 \geq \int_0^1 v(0,y)h(y|0)dy - 1 \geq 0 \tag{12}$$

where we have used Assumption 1.

As a bidder who bids zero receives a payoff of zero, it follows that all bidders who bid $\beta(s,w) \leq \psi(0)$ must receive a positive payoff in equilibrium. The concavity of expected utility in announced type ensures that a bidder's expected payoff is positive for all s.

Ruling out deviations by bidders bidding $\bar{b}(s)$ to bids below $\psi(0)$ is straightforward and for brevity is omitted here. □

References

Baye, Michael R., Kovenock, Dan, & de Vries, Casper G. 1996. The all-pay Auction with Complete Information. *Economic Theory*, **8**(2), 291–305.

Che, Yeon-Koo, & Gale, Ian. 1998. Standard Auctions with Financially Constrained Bidders. *Review of Economic Studies*, **65**(1), 1–21.

Fang, Hanming, & Parreiras, Sérgio. 2002. Equilibrium of Affiliated Value Second Price Auctions with Financially Constrained Bidders: The Two-Bidder Case. *Games and Economic Behavior*, **39**, 215–236.

Kotowski, Maciej H. 2010 (November). *First-Price Auctions with Budget Constraints*. Working Paper. University of California, Berkeley.

Krishna, Vijay, & Morgan, John. 1997. An Analysis of the War of Attrition and the All-Pay Auction. *Journal of Economic Theory*, **72**(2), 343–362.

Milgrom, Paul, & Weber, Robert J. 1982. A Theory of Auctions and Competitive Bidding. *Econometrica*, **50**(5), 1089–1122.