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On the Calculation of Price Sensitivities with Jump-Diffusion Structure

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Abstract

We provide an alternative approach for estimating the price sensitivities of a trading position with regard to underlying factors in jump-diffusion models using jump times Poisson noise. The proposition that results in a general solution is mathematically proved. The general solution that this paper offers can be applied to compute each price sensitivity. The suggested modeling approach deals with the shortcomings of the Black-Scholes formula such as the jumps that can occur at any time in the stock's price. Via the Malliavin calculus we show that differentiation can be transformed into integration, which makes the price sensitivities operational and more efficient. Thus, the solution that is provided in this paper is expected to make decision making under uncertainty more efficient.

Keywords: Malliavin Calculus, Asset Pricing, Price sensitivity, Jump-diffusion models, Jump times Poisson noise, European options.
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JEL Classification: G12, G10, C60.

1 Introduction

It is widely agreed in the literature that the modeling of financial derivatives is more precise if the price of the underlying asset is treated as a stochastic process. One of the most applied models for option pricing is the Black and Scholes formula [BS73]. However, the Black and Scholes model suffers from the continuity of the Brownian motion and thus from the exclusion of jumps. The aim of this paper is to develop

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an approach that can be used to remedy the shortcomings of the Black and Scholes model. This is achieved by developing a method for the computation of the price sensitivities of a trading position with respect to four main factors when the stochastic process describing the stock's price includes jumps. There are five price sensitivities of a trading position that are called "Greeks" in the literature. The importance of a precise calculation of these price sensitivities is paramount in financial markets pertinent to risk management. The change of the trading position with regard to the price of the underlying asset is called Delta. The rate of the change of the delta of a portfolio of options with regard to the price of the underlying asset is known as Gamma. The other source of risk is denoted by Vega that represents the sensitivity of the trading position with regard to the volatility of the underlying asset. The change of the portfolio with regard to time under the ceteris paribus condition is known as Theta. Finally, the sensitivity of the trading position with regard to the interest rate is known as Rho in the literature. Each Greek measures a source of risk for the underlying trading position. Therefore, the importance of computing the price sensitivities accurately is paramount to the investors and financial institutions. Traders need to calculate their Greeks at the end of every trading day in order to take necessary action if the internal risk limits are exceeded, in the underlying financial institution that the trader is linked to, in order to avoid dismissal. We utilize the Malliavin calculus to provide an accurate and operational solution for four of these price sensitivities. This approach is particularly useful since the price of the option characterized by an stochastic structure cannot be given in closed form. Therefore, the study of price sensitivities is very important in this context. Via the Malliavin calculus we can transform the differentiation into integration and thereby make the price sensitivities operational and more efficient.

Most previous work on the price sensitivities make use of the finite difference method. However, the Malliavin method is more efficient in terms of convergence. There has been some work done on this issue using the Malliavin method. The main contribution of this paper is to extend the Malliavin approach to calculating the price sensitivities when the price of the underlying asset follows a jump-diffusion process. To ensure the market is arbitrage free one should find a probability equivalent to the historical one under which the discounted prices are martingale (see the first part of fundamental theorem of asset pricing).

The application of the Malliavin calculus to the computations of price sensitivities were introduced by [FLLLT99] for markets with Brownian information. Their approach rests on the Malliavin derivative on the Wiener space and consists in:

1. applying the chain rule,
2. using the fact that this derivative has an adjoint (Skorohod integral) which coincides with the Itô integral for adapted processes.

Many papers employing this method have been developed for markets with jumps. For pure jump markets, in [KP04] the Poisson noise coming from the jump times is used, while in [BBM07] the authors differentiate with respect to both the jump times and the amplitude of the jumps. For jump-diffusion models, in [DJ06] the Malliavin calculus w.r.t the Brownian motion is applied after conditioning w.r.t the Poisson component, on the other hand in [BM06] the Poisson noise acts on the amplitude of the jumps. More recently in [KK10], Greeks formulae are obtained for Lévy process models of time-changed Brownian motion type using Malliavin calculus on the Wiener space conditionally on the time-changing process. And in [KT10] they use a scaling property of gamma processes w.r.t the Esscher transform parameter to perform formulas for Greeks in the case of asset price dynamics driven by gamma processes.
Our aim is to generalize the work of [KP04] by including a Brownian part and by covering European options. The Greeks formulae will be performed then by using

both the Malliavin derivative on the Wiener space and the jump times Poisson noise. We concentrate in this work on showing that we can differentiate w.r.t the jump times for jump-diffusion markets which is a different approach from [BM06] where the differentiation is with respect to the jump amplitudes and from [DJ06] where only the Malliavin calculus on the Wiener space is applied. For this, we need a new version of the gradient used in [KP04][‡] which has the required properties : it is a derivative and it has an adjoint satisfying the fact 2 above; and to be able to deal with European-like payoffs, the new version must contain the Poisson process in its domain.

Consider a standard Poisson process $N = (N_t)_{t\in\mathbb{R}_+}$ with jump times $(T_i)_{i\in\mathbb{N}}$ and let H denote the Cameron-Martin space

$$H = \left\{ u = \int_0^{\cdot} \dot{u}_t dt \ : \ \dot{u} \in L^2(\mathbb{R}_+) \right\}.$$

For $u \in H$ and a smooth functional $F_n = f(T_1, \dots, T_n),\ f \in \mathcal{C}_b^1(\mathbb{R}^n), n \geq 1$ of the Poisson process, we let

$$D_u^N F_n := -\sum_{k=1}^{k=n} u_{T_k} \partial_k f(T_1, \dots, T_n).$$

Unfortunately, N_t does not belong to Dom (D^N) the domain of D^N, so an underlying asset price $(S_t)_{t\in\mathbb{R}_+}$ given by

$$dS_t = \mu_t S_t dt + \sigma_t S_{t^-}(dN_t - dt), \quad t \in \mathbb{R}_+, \quad S_0 = x > 0,$$

does not belong to Dom (D^N). Nevertheless, for $T \in \mathbb{R}_+$, $\int_0^T S_t dt \in$ Dom (D^N) since it can be written as

$$\int_0^T S_t dt = \sum_{k\geq 0} \int_{T_k\wedge T}^{T_{k+1}\wedge T} x e^{\int_0^t (\mu_t - \sigma_t)dt} \prod_{i=0}^{i=k} (1 + \sigma_{T_i}) dt.$$

For this reason, in [KP04], only options with payoff of the form $f(\int_0^T S_t dt)$ are considered and those with payoff $f(S_T)$ are excluded.

[‡]Since the Malliavin gradient on the Poisson space is not a derivative (cf. for example [AOPU00]), another version of the gradient introduced in [CP90] and in [ET93] is used in [KP04].

Consider a smooth functional $F = \sum_{n=1}^{n=m} 1_{\{N_T=n\}} F_n,\ m \geq 1$, where $F_n := f(T_1, \ldots, T_n)$ and $f \in \mathcal{C}_b^1(\mathbb{R}^n)$. Let

$$\widetilde{D}_u^N F := \sum_{n=1}^{n=m} 1_{\{N_T=n\}} D_u^N F_n.$$

$\widetilde{D}^N$ is a derivative (See Prop.3) and it has and adjoint and satisfies the fact 2 (see Prop.4). Moreover N_t belong to Dom $(\widetilde{D}^N)$.

In this paper we apply the Malliavin calculus to compute Greeks for options with pay-off $f(S_T)$ for discontinuous models. The market is incomplete and there are infinitely many of Equivalent Martingale Measures (E.M.M). An E.M.M is a probability equivalent to the historical one, under which the discounted prices are martingales. Let the dynamic of the underlying asset price under a fixed E.M.M satisfy the stochastic differential equation

$$\frac{dS_t}{S_t} = r_t dt + \sigma_t[dW_t + (dN_t - dt)], \quad t \in [0,T], \quad S_0 = x > 0, \tag{1.1}$$

where $W = (W_t)_{t\in[0,T]}$ is a Brownian motion and $(r_t)_{t\in[0,T]}$ and $(\sigma_t)_{t\in[0,T]}$ are deterministic processes such that $\sigma > -1$ and it is not a constant[§]. We compute the Greeks by using the gradient gradient $\widetilde{D}^N + D^W$ (D^W denotes the Malliavin derivative on Wiener space).

After this introduction the remaining part of the paper is organized as follows: Section two is devoted to the Brownian and Poisson Malliavin derivatives. In Section three we apply the Malliavin calculus to derive the formula for computing the Greeks. The last Section concludes the paper.

2 Malliavin derivatives

In this section we give a brief presentation of the Malliavin derivative on the Wiener space and its adjoint. The new version of the Poisson gradient introduced in [Pr09]

[§]The derivative of σ must not vanish to avoid the division by zero, see the computation of the Delta in Section. 3.

is also presented. The Possonian operator is a derivative and it admits an adjoint which coincides with the Poissanian Itô integral for adapted processes. For more details about the Malliavin calculus we refer¶ to [Øks96] and [Nu95] on the Wiener space and to [Bi83], [CP90], [D00], [ET93], [NV90], [Pr94] and [Pr09] on the Poisson space.

2.1 Malliavin derivative on the Wiener space

From now on, we fix a terminal time $T > 0$ and consider the Wiener space $C_0([0,T])$, the set of continuous functions on $[0,T]$ vanishing in 0.
Let $(D_t^W)_{t\in[0,T]}$ be the Malliavin derivative on the Wiener space. We denote by $\mathbb{P}$ the set of random variables $F : \Omega \to \mathbb{R}$, such that F has the representation

$$F(\omega) = f\left(\int_0^T f_1(t)dW_t, \ldots, \int_0^T f_n(t)dW_t\right),$$

where $f(x_1,\ldots,x_n) = \sum_\alpha a_\alpha x^\alpha$ is a polynomial in n variables $x_1,\ldots,x_n$ and deterministic functions $f_i \in L^2([0,T])$. Let $\|.\|_{1,2}$ be the norm

$$\|F\|_{1,2} := \|F\|_{L^2(\Omega)} + \|D_\cdot^W F\|_{L^2([0,T]\times\Omega)}, \quad F \in \text{Dom } (D^W).$$

We have $\mathbb{P} \subset \text{Dom } (D^W)$ and the following Proposition holds:

Proposition 1 *Given* $F = f\left(\int_0^T f_1(t)dW_t, \ldots, \int_0^T f_n(t)dW_t\right) \in \mathbb{P}$. *We have*

$$D_t^W F \;=\; \sum_{k=1}^{k=n} \frac{\partial f}{\partial x_k}\left(\int_0^T f_1(t)dW_t, \ldots, \int_0^T f_n(t)dW_t\right) f_k(t).$$

From now on, for any stochastic process u and for $F \in \text{Dom } (D^W)$ such that $u_\cdot D_\cdot^W F \in L^2([0,T])$ we let

$$D_u^W F := \langle D^W F, u\rangle_{L^2([0,T])} := \int_0^T u_t D_t^W F dt.$$

¶The list is not exhaustive.

2.1.1 Skorohod integral

Let δ^W be the Skorohod integral on the Wiener space. The next Proposition is well known, it says that δ^W is the adjoint of D^W and is an extension of the Itô integral (see for example [Øks96]).

Proposition 2 *a) Let* $u \in \text{Dom }(\delta^W)$ *and* $F \in \text{Dom }(D^W)$*, we have*

$$E[F\delta^W(u)] = E[D_u^W F], \quad \text{for evry } F \in \text{Dom }(D^W).$$

b) Consider a $L^2(\Omega \times [0,T])$*-adapted stochastic process* $u = (u_t)_{t\in[0,T]}$*. We have*

$$\delta^W(u) = \int_0^T u_t dW_t.$$

c) Let $F \in \text{Dom }(D^W)$ *and* $u \in \text{Dom }(\delta^W)$ *such that* $uF \in \text{Dom }(\delta^W)$ *thus*

$$\delta^W(uF) = F\delta^W(u) - D_u^W F.$$

2.2 Poisson derivative

Let $\mathcal{S}$ denote the set of smooth functionals

$$F = \sum_{n=1}^{n=m} 1_{\{N_T=n\}} F_n, \quad \text{where} \quad F_n = f_n(T_1, \cdots, T_n) \in \text{Dom }(D^N), \quad m \in \mathbb{N}^* = \{1, 2, \dots\},$$

and for $1 \leq n \leq m$, $f_n \in \mathcal{C}_b^1(\mathbb{R}^n)$.

Definition 1 *Given an element* u *of the Cameron-Martin space* H *and* $F \in \mathcal{S}$ *as in the above, we define the gradient*‖

$$\widetilde{D}_u^N F := \sum_{n=1}^{n=m} 1_{\{N_T=n\}} D_u^N F_n = \sum_{n=1}^{n=m} 1_{\{N_T=n\}} \left(-\sum_{k=1}^{k=n} u_{T_k} \partial_k f_n(T_1, \cdots, T_n) \right). \quad (2.1)$$

The next proposition shows that the gradient $\widetilde{D}^N$ is a derivative.

Proposition 3 *Consider* $F = \sum_{n=1}^{n=m} 1_{\{N_T=n\}} F_n$ *and* $G = \sum_{n=1}^{n=m} 1_{\{N_T=n\}} G_n$ *two smooth functionals in* $\mathcal{S}$*, where* $F_n = f_n(T_1, \cdots, T_n) \in \text{Dom }(D^N)$ *and* $G_n = G_n(T_1, \cdots, T_n) \in \text{Dom }(D^N)$*. We have*

$$\widetilde{D}_u^N (FG) = F\widetilde{D}_u^N G + G\widetilde{D}_u^N F.$$

‖see [Pr09], section 7.3.

Proof. We have

$$FG = (\sum_{n=1}^{n=m} 1_{\{N_T=n\}} F_n)(\sum_{l=1}^{l=m} 1_{\{N_T=l\}} G_l) = \sum_{n=1}^{n=m} 1_{\{N_T=n\}} F_n G_n.$$

Thanks to the chain rule of the gradient D^N, we have

$$\begin{aligned} \widetilde{D}_u^N(FG) &= \sum_{n=1}^{n=m} 1_{\{N_T=n\}} F_n D_u^N G_n + \sum_{n=1}^{n=m} 1_{\{N_T=n\}} G_n D_u^N F_n \\ &= F\widetilde{D}_u^N G + G\widetilde{D}_u^N F. \end{aligned}$$

□

Remark 1 *Let* Dom $(\widetilde{D}^N)$ *be the domain of* $\widetilde{D}^N$.

1. Dom $(D^N) \subset$ Dom $(\widetilde{D}^N)$. *In fact any* $F \in$ Dom (D^N) *can be written as* $F = \sum_{n>0} 1_{\{N_T=n\}} F$. *We have* $\widetilde{D}_u^N F = D_u^N F$.

2. Dom $(\widetilde{D}^N)$ *contains* N_T *and* $\widetilde{D}_u^N N_T = 0$, *since* $N_T = \sum_{n\geq 0} 1_{\{N_T=n\}} n$.

2.2.1 Adjoint

The following proposition gives the adjoint gradient for D^N, it is well-known, cf. e.g. [CP90], [Pr94], [Pr02].

Proposition 4 *Consider* $F \in$ Dom (D^N) *and* $u \in H$, *we have*
a) The gradient D^N *is closable and admits an adjoint* δ^N *such that*

$$E[D_u^N F] = E[F\delta^N(u)].$$

b) For $u \in$ Dom (δ^N) *such that* $uF \in$ Dom (δ^N) *we have*

$$\delta^N(uF) = F\delta^N(u) - D_u^N F.$$

c) Moreover, δ^N *coincides with the compensated Poisson stochastic integral on the adapted processes in* $L^2(\Omega; H)$:

$$\delta^N(u) = \int_0^\infty \dot{u}_t(dN_t - dt).$$

To be able to use the Malliavin method for the computations of Greeks we need to show first the existence of an adjoint for $\widetilde{D}^N$ satisfying the properties of δ^N listed in Prop. 4. The relationship between $\widetilde{D}^N$ and D^N will be very helpful. In fact, we have δ^N is the adjoint of $\widetilde{D}^N$ as it is shown in the following proposition **.

Proposition 5 *With previous notations:*

a) $\widetilde{D}^N$ *is closable and admits* δ^N *as adjoint. Moreover, if* $F = \sum_{n=1}^{n=m} 1_{\{N_T=n\}} F_n$ *in* $\mathcal{S}$ *with* $F_n = f_n(T_1, \cdots, T_n) \in \mathrm{Dom}\ (D^N)$ *and* $u \in H$ *such that*†† $\int_0^T \dot{u}_t dt = 0$ *then*

$$E[\widetilde{D}_u^N F] = E[F\delta^N(u)].$$

b) For $F, G \in \mathrm{Dom}\ (\widetilde{D}^N)$ *and* $u \in \mathrm{Dom}\ (\delta^N)$ *with* $\int_0^T \dot{u}_t dt = 0$ *:*

$$E\left[G\widetilde{D}_u^N F\right] = E\left[F(G\delta^N(u) - \widetilde{D}_u^N G)\right].$$

3 Computations of Greeks

In this section we compute the Greeks for European options with maturity T and payoff $f(S_T)$, where $(S_t)_{t\in[0,T]}$ denotes the underlying asset price driven by the sum of a Brownian motion and a compensated standard Poisson process. Let $B = (B_t)_{t\in[0,T]}$ be a standard Brownian motion and $N = (N_t)_{t\in[0,T]}$ denote a standard Poisson process. The market is incomplete, since there are infinitely many of P-E.M.M. A P-E.M.M. Q is characterized by its Radon-Nikodym density with respect to P given by

$$\begin{aligned}\rho_T &= \exp\left(\int_0^T \alpha_s dB_s - \frac{1}{2}\int_0^T \alpha_s^2 ds + \int_0^T \ln(1+\beta_s)(dN_s - ds)\right.\\ &\quad \left. + \int_0^T (\ln(1+\beta_s) - \beta_s) ds\right),\end{aligned}$$

**The proof of this proposition can be found in [Pr09]: Section 7.3. However another proof is provided in the appendix with the condition $\int_0^T \dot{u}_t dt = 0$.

††The condition $\int_0^T \dot{u}_t dt = 0$ is necessary to prove the existence of the adjoint for the new version of the gradient introduced in [CP90] and in [ET93], see the proof of the Lemma. 1 in the appendix.

where $\beta > -1$ and by the equation

$$\mu_t - r_t + \alpha_t \sigma_t + \beta_t \sigma_t = 0.$$

Consider the two processes $W = (W_t)_{t\in[0,T]}$ and $M = (M_t)_{t\in[0,T]}$ where for $t \in [0,T]$

$$W_t = B_t - \int_0^t \alpha_s ds \quad \text{and} \quad M_t = N_t - \int_0^t (1+\beta_s) ds.$$

By Girsanov theorem [J79] W is a Q-Brownian motion and M is a Q-compensated Poisson process. The dynamic of $(S_t)_{t\in[0,T]}$ under Q is

$$\frac{dS_t}{S_t} = r_t dt + \sigma_t [dW_t + dM_t], \quad t \in [0,T], \quad S_0 = x > 0.$$

We have

$$S_T \;=\; x \exp\left(\int_0^T \sigma_t dW_t + \int_0^T (r_t - \sigma_t(1+\beta_t) - \frac{1}{2}\sigma_t^2) ds \right) \times \prod_{k=1}^{k=N_T} (1+\sigma_{T_k}),$$

where $(T_k)_{k\geq 1}$ denotes the jump times of $(N_t)_{t\in[0,T]}$. Let ζ be a parameter taking the values: $S_0 = x$, the volatility σ, or the interest rate r. Let $C = E[f(S_T^\zeta)]$ be the price of the option. The computations of Greeks by the Malliavin approach rest on the integration by parts formula -cf. [FLLLT99] for the Brownian case and [KP04] for the Poisson case- given in the following proposition.

Proposition 6 *Let I be an open interval of $\mathbb{R}$, $(F^\zeta)_{\zeta\in I}$ and $(G^\zeta)_{\zeta \in I}$ be two families of random functionals in* Dom $(\widetilde{D}^N) \bigcap$ Dom (D^W)*, continuously differentiable with respect to the parameter $\zeta \in I$. Let $(u_t)_{t\in[0,T]}$ be a process satisfying*

$$(\widetilde{D}_u^N + D_u^W) F^\zeta \neq 0, \quad a.s. \text{ on } \{\partial_\zeta F^\zeta \neq 0\}, \quad \zeta \in I,$$

such that $uG^\zeta \partial_\zeta F^\zeta / (\widetilde{D}_u^N + D_u^W) F^\zeta$ is continuous in ζ in Dom $(\delta^N) \bigcap$ Dom (δ^W) *and $\int_0^T \dot{u}_t dt = 0$. We have*

$$\frac{\partial}{\partial \zeta} E\left[G^\zeta f\left(F^\zeta\right)\right] = E\left[f\left(F^\zeta\right) \left(\frac{G^\zeta \partial_\zeta F^\zeta}{(\widetilde{D}_u^N + D_u^W)F^\zeta} \delta^N(u) - \widetilde{D}_u^N \left(\frac{G^\zeta \partial_\zeta F^\zeta}{(\widetilde{D}_u^N + D_u^W)F^\zeta} \right)\right)\right]$$
$$+ E\left[f\left(F^\zeta\right) \left(\frac{G^\zeta \partial_\zeta F^\zeta}{(\widetilde{D}_u^N + D_u^W)F^\zeta} \delta^W(u) - D_u^W \left(\frac{G^\zeta \partial_\zeta F^\zeta}{(\widetilde{D}_u^N + D_u^W)F^\zeta} \right)\right)\right] + E\left[f\left(F^\zeta\right) \partial_\zeta G^\zeta \right],$$

for any function f such that $f\left(F^\zeta\right) \in L^2(\Omega)$, $\zeta \in I$.

Proof. For function $f \in \mathcal{C}_b^\infty(\mathbb{R})$, we have

$$\begin{aligned}\frac{\partial}{\partial \zeta} E\left[G^\zeta f\left(F^\zeta\right)\right] &= E\left[G^\zeta \partial_\zeta f\left(F^\zeta\right)\right] + E\left[f\left(F^\zeta\right)\partial_\zeta G^\zeta\right] \\ &= E\left[G^\zeta \partial_\zeta F^\zeta \frac{(\widetilde{D}_u^N + D_u^W) f\left(F^\zeta\right)}{(\widetilde{D}_u^N + D_u^W) F^\zeta}\right] + E\left[f\left(F^\zeta\right)\partial_\zeta G^\zeta\right].\end{aligned}$$

Then we conclude using Propositions 2 and 5. The extension to $f\left(F^\zeta\right) \in L^2(\Omega)$ with $\zeta \in I$, can be obtained from the same argument as in p. 400 of [FLLLT99] and in [KP04] p. 167 for the Poisson case, using the bound

$$\begin{aligned}&\left|\frac{\partial}{\partial \zeta} E\left[G^\zeta f_n(F^\zeta)\right] - E\left[f\left(F^\zeta\right)\left(V^\zeta(\delta^N(u) + \delta^W(u)) - (\widetilde{D}_u^N + D_u^W)V^\zeta + \partial_\zeta G^\zeta\right)\right]\right| \\ &\quad\leq \|f(F^\zeta) - f_n(F^\zeta)\|_{L^2(\Omega)} \left\|V^\zeta(\delta^N(u) + \delta^W(u)) - (\widetilde{D}_u^N + D_u^W)V^\zeta + \partial_\zeta G^\zeta\right\|_{L^2(\Omega)},\end{aligned}$$

and an approximating sequence $(f_n)_{n\in\mathbb{N}}$ of smooth functions, where $V^\zeta := G^\zeta \partial_\zeta F^\zeta / (\widetilde{D}_u^N + D_u^W) F^\zeta$. □

Consider an option with payoff $f\left(F^\zeta\right)$.

Delta, Rho, Vega

The Greeks Delta $:= \frac{\partial C}{\partial x}$, Rho $= \frac{\partial C}{\partial r}$ and Vega $= \frac{\partial C}{\partial \sigma}$ can be computed from Proposition 6

$$\frac{\partial}{\partial \zeta} E\left[f\left(F^\zeta\right)\right] = E\left[f\left(F^\zeta\right) L^\zeta(\delta^N(u) + \delta^W(u)) - (\widetilde{D}_u^N + D_u^W) L^\zeta\right], \tag{3.1}$$

where we let $G^\zeta = 1$ and $L^\zeta := \frac{\partial_\zeta F^\zeta}{(\widetilde{D}_u^N + D_u^W) F^\zeta}$. As an example we compute the delta[‡‡] of an European option using (3.1) with $\zeta = x$, $f(F^\zeta) = f(S_T)$, and $\partial_\zeta F^\zeta = \partial_x S_T = \frac{1}{x} S_T$. We have

$$\text{Delta} = \partial_x E\left[e^{-\int_t^T r_s ds} f\left(S_T\right)\right] = e^{-\int_t^T r_s ds} E\left[f\left(S_T\right)\left(L^x(\delta^N + \delta^W)(u) - (\widetilde{D}_u^N + D_u^W) L^x\right)\right],$$

where

$$L^x = \frac{1}{x} \frac{S_T}{(\widetilde{D}_u^N + D_u^W) S_T} = \frac{1}{x} \frac{1}{\int_0^T u_t \sigma_t dt - \int_0^T \frac{u_t \sigma_t'}{1+\sigma_t} dN_t}.$$

[‡‡]we can use the same techniques for Rho and Vega.

And

$$
\begin{aligned}
D_u^W L^x &= 0 \\
\widetilde{D}_u^N L^x &= (L^x)^2 \widetilde{D}_u^N \left(\int_0^T \frac{u_t \sigma_t^{'}}{1+\sigma_t} dN_t \right) = -(L^x)^2 \left(\int_0^T u_t \partial_t \frac{u_t \sigma_t^{'}}{1+\sigma_t} dN_t \right) \\
&= -(L^x)^2 \int_0^T \frac{u_t}{1+\sigma_t} \left((\sigma_t^{'} u_t^{'} + u_t \sigma_t^{''}) - \frac{u_t (\sigma_t^{'})^2}{1+\sigma_t} \right) dN_t,
\end{aligned}
$$

here we supposed that $\sigma^{'} \neq 0$.

We can use

$$
\begin{aligned}
\delta^N(u) &= \int_0^T \dot{u}_t dN_t = \int_0^T \dot{u}_t (dN_t - dt) = \sum_{k \geq 0} \dot{u}_{T_k}, \\
\delta^W(v) &= \int_0^T v_t dW_t = \sum_{j \geq 1} v_{t_{j-1}} (W_{tj} - W_{t_{j-1}}),
\end{aligned}
$$

for $u \in H$ such that $\int_0^T \dot{u}_t dt = 0$ and v adapted.

Gamma

To compute the Gamma $= \frac{\partial^2 C}{\partial x^2}$, let $H^x := L^x(\delta^N(u) + \delta^W(u)) - (\widetilde{D}_u^N + D_u^W) L^x$. We have using (3.1) and Prop. 6

$$
\begin{aligned}
\text{Gamma} &= e^{-\int_t^T r_s ds} \frac{\partial^2}{\partial x^2} E\left[f\left(F^x\right)\right] = e^{-\int_t^T r_s ds} \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} E\left[f\left(F^x\right)\right] \right) = e^{-\int_t^T r_s ds} \frac{\partial}{\partial x} E\left[f\left(F^x\right) H^x\right] \\
&= e^{-\int_t^T r_s ds} \Bigg\{ E\left[f\left(F^x\right) \left(\frac{H^x \partial_x F^x}{(\widetilde{D}_u^N + D_u^W) F^x} \delta^N(u) - \widetilde{D}_u^N \left(\frac{H^x \partial_x F^x}{(\widetilde{D}_u^N + D_u^W) F^x} \right) \right) \right] \\
&+ E\left[f\left(F^x\right) \left(\frac{H^x \partial_x F^x}{(\widetilde{D}_u^N + D_u^W) F^x} \delta^W(u) - D_u^W \left(\frac{H^x \partial_x F^x}{(\widetilde{D}_u^N + D_u^W) F^x} \right) \right) \right] + E\left[f\left(F^x\right) \partial_x H^x\right] \Bigg\}.
\end{aligned}
$$

4 Conclusions

Making use of options is a common practice in financial markets by investors and other financial agents in order to neutralize or reduce the price risk of the underlying asset. Thus, option pricing is an integral part of modern financial risk management. One of the most utilized tools for this purpose is the Black and Scholes (1973) formula. However, this formula suffers from the continuity of the Brownian motion and consequently from the exclusion of jumps. The current paper aims at developing an approach that can be used to remedy the shortcomings of the Black and Scholes (1973) model by taking into account the impact of the potential jumps. This is operationalized by developing an alternative method for the computation of the price sensitivities of a trading position with respect to the main underlying factors when the stochastic process describing the asset price is characterized by jumps. It is shown how the Malliavin derivative on the Wiener space and the jump times Poisson noise can be utilized to calculate the much needed price sensitivities more accurately. Thus, we propose an alternative approach for calculating the Delta, Gamma, Vega, and the Rho more accurately. These four price sensitivities of a trading position have important repercussions in financial risk management. Hence, the more precise approach developed in this paper for calculating these price sensitivities is expected to be enormously valuable to investors as well as financial institutions in their constant pursue of finding and constructing financial risk management strategies that are more successful in hedging against the potential sources of the underlying price risk. The solution that is provided in this paper can therefore become an essential tool for good decision making under uncertainty.

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Appendix

Proof of the Prop. 5

First, we need the following lemma

Lemma 1 *Consider $u \in H$ such that $\int_0^T \dot{u}_t dt = 0$ and a smooth functional $f(T_1, \cdots, T_n) \in$ Dom (D^N), we have*

$$E\left[D_u^N f(T_1, \cdots, T_n) \mid N_T = n\right] = E\left[f(T_1, \cdots, T_n)\delta^N(u) \mid N_T = n\right].$$

Proof of the Lemma 1.

Proof. Let $u \in H$ such that $\int_0^T \dot{u}_t dt = 0$. We follow [Pr02], Lemma 1. We consider the simplex $\Delta_n = \{(t_1, \ldots, t_n) \in [0, T]^n : 0 \leq t_1 < \cdots < t_n\}$. We have for $f \in L^2(\Delta_n, dt_1, \ldots, dt_n)$,

$$E[f(T_1, \cdots, T_n) \mid N_T = n] = \frac{n!}{T^n} \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f(t_1, \cdots, t_n) dt_1 \ldots dt_n.$$

And

$$E[D_u^N f(T_1, \cdots, T_n) \mid N_T = n] = -\sum_{k=1}^{k=n} I_k,$$

where

$$I_k := \frac{n!}{T^n} \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} u_{t_k} \partial_k f(t_1, \cdots, t_n) dt_1 \ldots dt_n.$$

<u>k=1</u> We have by integration by parts

$$\int_0^{t_2} u_{t_1} \partial_1 f(t_1, \cdots, t_n) dt_1 = -\int_0^{t_2} \dot{u}_{t_1} f(t_1, \cdots, t_n) dt_1 + u_{t_2} f(t_2, t_2, \cdots, t_n).$$

Thus

$$I_1 = A_1 + B_2,$$

where

$$\begin{aligned} A_1 : &= -\frac{n!}{T^n} \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} \dot{u}_{t_1} f(t_1, \cdots, t_n) dt_1 \cdots dt_n \\ B_2 : &= \frac{n!}{T^n} \int_0^T \int_0^{t_n} \cdots \int_0^{t_3} u_{t_2} f(t_2, t_2, \cdots, t_n) dt_2 \cdots dt_n. \end{aligned}$$

<u>**k=2**</u> We have

$$\begin{aligned}
&\int_0^{t_3} u_{t_2} \int_0^{t_2} \partial_2 f(t_1, \cdots, t_n) dt_1 dt_2 \\
&= \int_0^{t_3} u_{t_2} \partial_2 \int_0^{t_2} f(t_1, \cdots, t_n) dt_1 dt_2 - \int_0^{t_3} u_{t_2} f(t_2, t_2, \cdots, t_n) dt_2 \\
&= -\int_0^{t_3} \dot{u}_{t_2} \int_0^{t_2} f(t_1, t_2, \cdots, t_n) dt_1 dt_2 + u_{t_3} \int_0^{t_3} f(t_1, t_3, t_3, t_4, \cdots, t_n) dt_1 \\
&\quad - \int_0^{t_3} u_{t_2} f(t_2, t_2, \cdots, t_n) dt_2.
\end{aligned}$$

Thus

$$I_2 = A_2 - B_2 + B_3,$$

where

$$\begin{aligned}
A_2 : &= -\frac{n!}{T^n} \int_0^T \int_0^{t_n} \cdots \int_0^{t_3} \dot{u}_{t_2} \int_0^{t_2} f(t_1, t_2, \cdots, t_n) dt_1 \cdots dt_n \\
B_3 : &= \frac{n!}{T^n} \int_0^T \int_0^{t_n} \cdots \int_0^{t_4} u_{t_3} \int_0^{t_3} f(t_1, t_3, t_3, t_4, \cdots, t_n) dt_1 dt_3 \cdots dt_n.
\end{aligned}$$

By using the same argument of the above, for any <u>$k \in \{3, \ldots, n-1\}$</u>, we have

$$\begin{aligned}
&\int_0^{t_{k+1}} u_{t_k} \left(\int_0^{t_k} \cdots \int_0^{t_2} \partial_k f(t_1, \cdots, t_n) dt_1 \cdots dt_{k-1} \right) dt_k \\
&= \int_0^{t_{k+1}} u_{t_k} \partial_k \left(\int_0^{t_k} \cdots \int_0^{t_2} f(t_1, \cdots, t_n) dt_1 \cdots dt_{k-1} \right) dt_k \\
&\quad - \int_0^{t_{k+1}} u_{t_k} \int_0^{t_k} \int_0^{t_{k-2}} \cdots \int_0^{t_2} f(t_1, \cdots, t_{k-2}, t_k, t_k, \cdots, t_n) dt_1 \cdots dt_{k-2} dt_k \\
&= -\int_0^{t_{k+1}} \dot{u}_{t_k} \int_0^{t_k} \cdots \int_0^{t_2} f(t_1, \cdots, t_n) dt_1 \cdots dt_k \\
&\quad + \int_0^{t_{k+2}} u_{t_{k+1}} \int_0^{t_{k+1}} \int_0^{t_{k-1}} \cdot \int_0^{t_2} f(t_1, \cdot, t_{k-1}, t_{k+1}, t_{k+1}, \cdot, t_n) dt_1 \cdot dt_{k-1} dt_{k+1} \cdot dt_n \\
&\quad - \int_0^{t_{k+1}} u_{t_k} \int_0^{t_k} \int_0^{t_{k-2}} \cdots \int_0^{t_2} f(t_1, \cdots, t_{k-2}, t_k, t_k, \cdots, t_n) dt_1 \cdots dt_{k-2} dt_k.
\end{aligned}$$

Thus for $k \in \{3, \cdots, n-1\}$, we have

$$I_k = A_k - B_k + B_{k+1},$$

where

$$\begin{aligned}
A_k : &= -\frac{n!}{T^n}\int_0^T\int_0^{t_n}\cdots\int_0^{t_{k+1}}\dot{u}_{t_k}\int_0^{t_k}\cdots\int_0^{t_2} f(t_1,\cdots,t_n)dt_1\cdots dt_n,\\
B_k : &= \frac{n!}{T^n}\int_0^T\int_0^{t_n}\cdot\int_0^{t_{k+1}}u_{t_k}\int_0^{t_k}\int_0^{t_{k-2}}\cdot\int_0^{t_2} f(t_1,\cdot,t_{k-2},t_k,t_k,\cdot,t_n)dt_1\cdot d\hat{t}_{k-1}\cdot dt_n,
\end{aligned}$$

$d\hat{t}_k$ denotes the absence of dt_k.

<u>k=n</u> Let

$$\begin{aligned}
A_n &:= -\frac{n!}{T^n}\int_0^T\dot{u}_{t_n}\int_0^{t_n}\cdots\int_0^{t_k}\cdots\int_0^{t_2} f(t_1,\cdots,t_n)dt_1\cdots dt_n,\\
B_n &:= \frac{n!}{T^n}\int_0^T u_{t_n}\int_0^{t_n}\int_0^{t_{n-2}}\cdots\int_0^{t_2} f(t_1,\cdots,t_{n-2},t_n,t_n)dt_1\cdots d\hat{t}_{n-1}dt_n,
\end{aligned}$$

we have

$$\begin{aligned}
I_n = &\frac{n!}{T^n}\int_0^T u_{t_n}\int_0^{t_n}\cdots\int_0^{t_2}\partial_n f(t_1,\cdots,t_n)dt_1\cdots dt_n\\
= &\frac{n!}{T^n}\int_0^T u_{t_n}\partial_n\int_0^{t_n}\int_0^{t_{n-1}}\cdots\int_0^{t_2} f(t_1,\cdots,t_n)dt_1\cdots dt_n - B_n\\
= &A_n + \frac{n!}{T^n}u_T\int_0^T\int_0^{t_{n-1}}\cdots\int_0^{t_2} f(t_1,\cdots,t_{n-1},T)dt_1\cdots dt_{n-1} - B_n\\
= &A_n - B_n,
\end{aligned}$$

since $\int_0^T \dot{u}_t dt = 0$.

Thus

$$\begin{aligned}
\sum_{k=1}^{k=n} I_k &= (A_1+B_2)+(A_2-B_2+B_3)+\sum_{k=3}^{k=n-1}(A_k-B_k+B_{k+1})+A_n-B_n\\
&= \sum_{k=1}^{k=n} A_k.
\end{aligned}$$

Then

$$\begin{aligned}
E[D_u^N f(T_1,\cdots,T_n)\mid N_T=n] &= -\sum_{k=1}^{k=n} I_k = -\sum_{k=1}^{k=n} A_k\\
&= \sum_{k=1}^{k=n}\frac{n!}{T^n}\int_0^T\int_0^{t_n}\ldots\int_0^{t_2}\dot{u}_{t_k}f(t_1,\cdots,t_n)dt_1\ldots dt_n
\end{aligned}$$

$$= \quad E\left[f(T_1,\cdots,T_n)\left(\sum_{k=1}^{k=n}\dot{u}_{T_k}\right) \mid N_T = n\right].$$

Now to show that

$$E\left[f(T_1,\cdots,T_n)\left(\sum_{k=1}^{k=n}\dot{u}_{T_k}\right) \mid N_T = n\right] = E\left[f(T_1,\cdots,T_n)\delta^N(u) \mid N_T = n\right].$$

it is sufficient to prove

$$E\left[f(T_1,\cdots,T_n)\left(\sum_{k>n}\dot{u}_{T_k} - \int_{T_n}^{\infty}\dot{u}_t dt\right) \mid N_T = n\right] = 0,$$

since $\int_0^T \dot{u}_t dt = 0$. Recall that for $k > n$ we have

$$\begin{aligned}
&E\left[f(T_1,\cdots,T_n,\cdots,T_k) \mid N_T = n\right] = \\
&\quad \frac{n!}{T^n}e^{-T}\int_0^{\infty} e^{-t_k}\int_0^{t_k}\cdots\int_0^{t_{n+1}}\int_0^{t_n}\cdots\int_0^{t_2} f(t_1,\cdots,t_n,\cdots,t_k)dt_1\ldots dt_k.
\end{aligned}$$

Therefore for $k > n$

$$\begin{aligned}
E[F\dot{u}_{T_k} \mid N_T = n] &= \frac{n!}{T^n}e^{-T}\int_0^{\infty}\dot{u}_{t_k}e^{-t_k}\int_0^{t_k}\cdots\int_0^{t_n}\cdots\int_0^{t_2} f(t_1,\cdots,t_n)dt_1\ldots dt_k \\
&= \frac{n!}{T^n}e^{-T}\int_0^{\infty}u_{t_k}e^{-t_k}\int_0^{t_k}\cdots\int_0^{t_n}\cdots\int_0^{t_2} f(t_1,\cdots,t_n)dt_1\ldots dt_k \\
&\quad -\frac{n!}{T^n}e^{-T}\int_0^{\infty}u_{t_{k-1}}e^{-t_{k-1}}\int_0^{t_{k-1}}\cdots\int_0^{t_n}\cdots\int_0^{t_2} f(t_1,\cdots,t_n)dt_1\ldots dt_{k-1} \\
&= E[F(u_{T_k} - u_{T_{k-1}}) \mid N_T = n] \\
&= E\left[F\int_{T_{k-1}}^{T_k}\dot{u}_t dt \mid N_T = n\right].
\end{aligned}$$

Then

$$E[D_u^N f(T_1,\cdots,T_n) \mid N_T = n] = E\left[f(T_1,\cdots,T_n)\left(\sum_{k=1}^{\infty}\dot{u}_{T_k} - \int_{T_n}^{\infty}\dot{u}_t dt\right) \mid N_T = n\right].$$

□

Now we can give the proof of Prop. 5.

Proof. a) We have using Lemma. 1 for any $u \in H$ such that $\int_0^T \dot{u}_t dt = 0$

$$E[1_{\{N_T=n\}}D_u^N F_n] \quad = \quad \sum_{i=1}^{\infty} E[1_{\{N_T=n\}}D_u^N F_n \mid \ N_T = i]P(N_T = i)$$

$$
\begin{aligned}
&= 1_{\{N_T=n\}} E[D_u^N F_n \mid N_T = n] P(N_T = n) \\
&= E\left[1_{\{N_T=n\}} F_n \delta^N(u) \mid N_T = n\right] P(N_T = n) \\
&= E\left[1_{\{N_T=n\}} F_n \delta^N(u)\right].
\end{aligned}
$$

Thus

$$
\begin{aligned}
E[\widetilde{D}_u^N F] &= E[\sum_{n=1}^{n=m} 1_{\{N_T=n\}} D_u^N F_n] \\
&= \sum_{n=1}^{n=m} E\left[1_{\{N_T=n\}} F_n \delta^N(u)\right] = E[F\delta^N(u)].
\end{aligned}
$$

b) Using the chain rule of $\widetilde{D}^N$ and a) we obtain

$$
E\left[G\widetilde{D}_u^N F\right] = E\left[\widetilde{D}_u^N(FG) - F\widetilde{D}_u^N G\right] = E\left[F(G\delta^N(u) - \widetilde{D}_u^N G)\right].
$$

□