# One numerical procedure for two risk factors modeling

Cocozza, Rosa and De Simone, Antonio

University of Naples "Federico II"

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# ONE NUMERICAL PROCEDURE FOR TWO RISK FACTORS MODELING

*Rosa Cocozza and Antonio De Simone,*
*University of Napoli "Federico II", Italy*
*Email: rosa.cocozza@unina.it, a.desimone@unina.it,*
*www.docenti.unina.it/rosa.cocozza*

**Abstract**. We propose a numerical procedure for the pricing of financial contracts whose contingent claims are exposed to two sources of risk: the stock price and the short interest rate. More precisely, in our pricing framework we assume that the stock price dynamics is described by the Cox, Ross Rubinstein (CRR, 1979) binomial model under a stochastic risk free rate, whose dynamics evolves over time accordingly to the Black, Derman and Toy (BDT, 1990) one-factor model. To this aim, we set the hypothesis that the instantaneous correlation between the trajectories of the future stock price (conditional on the current value of the short rate) and of the future short rate is zero. We then apply the resulting stock price dynamics to evaluate the price of a simple contract, i.e. of a stock option. Finally, we compare the derived price to the price of the same option under different pricing models, as the traditional Black and Scholes (1973) model. We expect that, the difference in the two prices is not sensibly large. We conclude showing in which cases it should be helpful to adopt the described model for pricing purposes.

**Keywords: option pricing, stochastic short rate model, binomial tree**

**JEL classification: C63, C65, G12, G13**
**MSC classification: 91G20, 91G30, 91G60**

## 1 Introduction

In the modern option pricing theory many attempts have been accomplished in order to release some of the traditional assumptions of the Black and Scholes (1973) model and, in general, to develop a pricing framework depending not only on the solely underlying dynamics . Very distinguished in this field are the models allowing for stochastic interest rate, as suggested for the first time by Merton (1973). Afterward, Brennan and Schwartz (1980) proposed a stochastic interest rate model to evaluate the price of convertible bonds. Within this context, if the conversion date coincides with the bond maturity, the future value of the bond is certain and no assumption on the interest rate dynamics is necessary. In this case the issues faced in evaluating the convertible are not sensibly different from those arising in determining the market value of an equity-linked endowment policy. On the contrary, if the conversion occurs before the maturity, some assumptions about the future dynamics of the term structure are necessary because, in this case, the future value of the bond is a random variable depending on the unknown level of the interest rates at the conversion date.

Even in the insurance segment the specification of the interest rate dynamics is very helpful. Recently, life insurance companies issued policies that typically combine a guaranteed return with a contingent spread on a reference asset return paid out to the policyholder under particular conditions. This is the case of participating life insurance policies allowing policyholders to gain a defined amount and at the same time to participate to the eventual additional profit margin of the insurance company according to a predefined participation rate. As showed elsewhere (Cocozza and Orlando, 2007; Cocozza et al. 2011), these contracts embed an option that exposes the issuer to two sources of financial risk: the return on the reference asset and the appropriate discount factor. On the contrary, if we consider an equity-linked endowment policy, the equilibrium price, as Brennan and Schwartz (1976) showed in their seminal paper, is equal to the sum of the present value of a zero coupon bond and those of an immediately exercisable call option on the reference asset or, alternatively, to the present value of the reference asset plus that of an immediately exercisable put option on the same reference asset. As a matter of fact, such a kind of contracts is not an insurance policy but a proper financial product issued not only by the life insurance companies but also by other financial institutions.

In this article we propose an innovative numerical procedure for the pricing of financial contracts whose contingent claims are exposed to two risk sources: the stock price and the interest rate. More precisely, in our pricing framework we assume that the stock price dynamics is described by the Cox, Ross and Rubinstein (1979) binomial model (CRR) under a stochastic risk free rate, whose dynamics evolves over time accordingly to the Black, Derman and Toy (1990) one-factor model (BDT). The BDT model avoids some drawbacks that typically affect equilibrium model of the term structure, such as negative spot interest rates. At the same time it offers the relevant opportunity to efficiently calibrate the risk factor trajectories, preventing from the adoption of parameters not allowing the endogenous reproduction the observed term structure. Last but not least, the conjoint adoption of CRR and BDT models may sensibly reduce the computational efforts in the estimation of the parameters by adopting implied volatility measures.

The paper is structured as follows. In section 2 main contributions to option pricing theory in a stochastic interest rate framework are reported together with a brief illustration of the BDT and the CRR models. Section 3 reports the main assumptions and the mechanics of the numerical procedure adopted to determine the price of a plain vanilla call option. Section 4 shows some numerical examples while section 5 concludes the paper with main final remarks.

## 2 Option pricing models with stochastic interest rate

As stated, a stochastic interest rate model for option pricing was proposed for the first time by Merton (1973), where the Gaussian process was adopted to describe the continuous-time short rate dynamics. The adoption of a Gaussian process was very common in the '80s and in the early '90s before the advent of the lognormal term structure models. A discrete time dynamics for short rate process equivalent to those adopted by Merton was subsequently discussed by Ho and Lee (1986), while other option pricing formulae under Gaussian interest rate were introduced by Rabinovitch (1989) and Amin and Jarrow (1992).

The success of the Gaussian models of the term structure relies on the mathematical tractability and thus on the possibility of obtaining closed formulas and solutions for the price of stock and bond options. In fact, the Gaussian process was for the first time adopted to derive the price of bond options by Vasicek (1977). Furthermore, the calibration of the Gaussian models does not require particularly demanding computational effort.

Although the Gaussian models have been very successful for research purposes, some relevant inner drawbacks prevented their diffusion among the practitioners, as for example the possibility for the interest rate trajectories to assume negative values. In response, other equilibrium models for the interest rate term structure have been developed. One of this is the well-known Cox, Ingersoll and Ross (1985) model (CIR), where the interest rate dynamics is described by a square root mean reversion process that, under the Feller condition (Feller, 1951), does not allow the interest rate to become negative. CIR dynamics has subsequently been adopted also to describe the stochastic short rate framework for pricing stock option (Kunitomo and Kim, 1999 and 2001) and for pricing endowment policies, with an asset value guarantee, where the benefit is linked to fixed income securities (Bacinello and Ortu, 1996).

However, as showed elsewhere (De Simone, 2010), equilibrium models are in general not able to ensure an efficient calibration of the interest rate dynamics, because they are based on a limited number of parameters, in general unable to guarantee an acceptable fitting of the model prices to market prices. Moreover, a satisfactory calibration of the model is sometimes a hard task, because many equilibrium models rely on an instantaneous interest rate (spot or forward) that, in general, is not directly observable on the market. The relevance of this problem increased over time, especially after the diffusion of standard market practices of pricing derivatives within the Black and Scholes environment (Black and Scholes, 1973; Black, 1976).

As mentioned, limitations of the equilibrium models may be overcome by *market models*, as for example the BDT model and the Libor Market Model (Brace et al. 1997). A particular characteristic of both models is the assumption of lognormal interest rate dynamics, even if this hypothesis applies only asymptotically for the BDT model. Such feature allows for a satisfactory calibration by adopting implied volatility measures according to the standard market practices. However, opposite to many equilibrium models, market models do not in general allow to obtain closed price formulas so that the price of interest rates derivatives has to be evaluated numerically. Between the two mentioned models, we choose the BDT because of its simplified approach in pricing derivatives. The BDT model allows to obtain a binomial tree for the dynamics of the Libor rate by adopting, as input data, the term structure of interest rates and of the corresponding volatilities. An exhaustive explanation of the procedure adopted for the construction of the tree is reported in Neftci (2008). Figure 1 shows an hypothetical BDT tree for the 12-monts spot Libor rate $L(t, s)$, where $s - t = 12$ months for each $t, s$.

At time $t = 0$ the 12-months spot Libor rate $L(0,1)$ is directly observable and therefore not stochastic. After one year, at time $t = 1$, the following 12-months Libor rate $L(1,2)$ can go up to the level $L(1,2)u$ or down to the level $L(1,2)d$. Similarly, at time $t = 2$ the Libor rate $L(1,2)j$, in the state of the world $j=u, d,$ may go up or down with equal risk neutral probability. We finally notice that since $L(t, s)ud=L(t, s)du$ the BDT tree is *recombining*.

In the next section we show how the information from the BDT tree is employed to describe the dynamics of the interest rate adopted as risk free rate in the CRR model. We choose such models for two main reasons: (1) both the models are based on a binomial tree and (2) both the risk factors are lognormal in the limit.

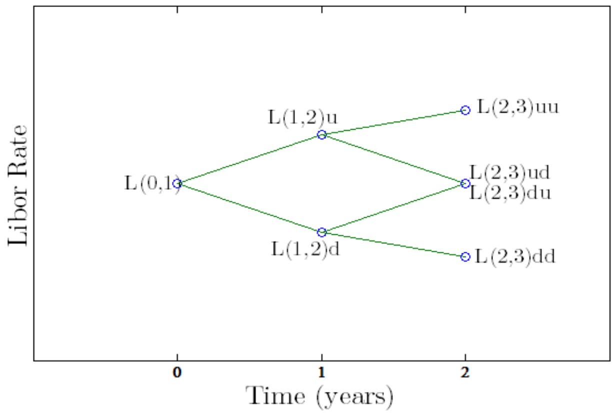


**Figure 1**. The BDT tree of the 12-months Libor rate *L(t, s)*.

An important property of the implied stock tree is that the rate at which the stock price increases/decreases *(u/d)* is constant over time, so that $S_T^j = S_t j$ , *j=u,d*. As shown by Cox et al. (1979), the risk neutral probability *p* is:

$$p = \frac{m-d}{u-d}$$

where *m=1+L(t, T)*, and where *u·d=1* because also the CRR tree is recombining. Moreover, we remark that the discrete time dynamics approximates the Black and Scholes dynamics of the stock price, according to the following stochastic differential equation (SDE):

$$\Delta S_t = \mu S_t \Delta + \sigma_{CRR} S_t \sqrt{\Delta} \varepsilon_t \qquad (1)$$

where $S_t$ is the stock price at time *t*, $\mu$ is the drift of the process Δ is the time distance between two observations, $\sigma_{CRR}$ is the instantaneous volatility coefficient and $\varepsilon_t$ can be interpreted as the outcome of a binomial random variable under the *natural* probability. Notice that as Δ→0, $\varepsilon_t$ tends in distribution to a standard normal random variable so that $\sqrt{\Delta}\varepsilon_t$ can be interpreted as a Wiener increment. As a result, the final stock price tends in distribution to a lognormal random variable. This property entails that the adoption of implied volatility measures for pricing equity linked contingent claims ensures a satisfactory calibration of the model to the market data. Of course, as the drift change, the expected value of the stock price changes too because of the subsequently modification of the probability space. On the contrary, the levels of the future stock prices in each state of the world only depend on the current stock price and on the volatility parameter $\sigma_{CRR}$ . In the next section we adopt the Libor rate as risk free rate in evaluating equity linked contingent claims. Despite the Libor rate refers to banks AA or AA- rated, we consider it as a proxy of the risk free rate. In section 5 we study how to release this assumption and a possible solution to this issue is proposed.

## 3 The procedure

In the majority of cases, when there is a two risk factor pricing model we have to ascertain:

1. the dynamics according to which the two factors evolve;
2. the measure of the correlation between the two dynamics;
3. the estimate of relevant parameters under risk neutral environment.

The three tasks can be extremely difficult to perform properly. As a consequence, a certain level of approximation is often required. Even the simulation based on copula function, not always available with reference to the distribution under observation, can be very questioning with reference to the choice of an appropriate correlation measure. In this order of ideas, we developed a numerical procedure to get the arbitrage free price of a European call option in a stochastic short rate framework. To begin with, we notice that the numerical procedure adopts the results from the BDT and from the CRR models, whose assumptions hold also for our model.

*3.1 Main assumptions*

According to the stated environment (section 2), the arbitrage free dynamics of the stock price is described by equation 1. Therefore the risk neutral dynamics of the stock price is defined as follows:

$$\Delta S_t = r(t)S_t\Delta + \sigma_{CRR} S_t \sqrt{\Delta}\varepsilon_t^p \qquad (2)$$

where *r(t)* is the instantaneous risk free interest rate at time t under the corresponding appropriate risk neutral probability *p*. The crucial point in our approach is that the instantaneous risk free rate is *piecewise constant* and evolves over time according to the BDT dynamics, since the spot rate *r*(*t*) is not a random variable unless the tenor *(δ)* of the rate matures.

In other words, once we choose a term structure (in our examples the Libor rate term structure), the variability of the rate can be observed in practice according to the 'nodes' of the term structure. Therefore, if the rate is, as in our case, the Libor, the variability time interval goes from 1 week up to 12 months and coincides with the tenor the chosen rate. Therefore, once we adopted the 12 months rate, the dynamics over time on a discrete time interval is equal to 12 months. This accounts for a piecewise constant dynamics. The length of the time interval of the process describing the rate dynamics is therefore set according to the relevant node of the term structure. Theoretically, any node could be used, but if we decide to go for a BDT application we need also volatility data. Since not all the nodes have the same liquidity, the significance of the corresponding implied volatility is not the same across the term structure. We are therefore forced towards those nodes showing the maximum liquidity, since this guarantees the more efficient measure of implied volatility. If we assume (or better observe) that the most liquid node is 12 months, we will adopt a BDT model on a year basis. As a consequence on a certain time horizon (longer than one year in the case under estimation), we will have an array of one year rates defining a corresponding set of probability spaces which can be used for evaluating the stock price dynamics. As a consequence, the evaluating numeraire is piecewise constant and in a sense "rolls over time" according to the term structure tenor.Therefore what can be regarded as a random variable at the evaluation date is the *δ*-rate. As a consequence, the instantaneous risk free rate *r*(*t*) and the δ-Libor rate *L(t, T)*, with δ = *T* – *t*, are connected as follows:

$$\exp\{-r(t)n\Delta\} = \frac{1}{1+L(t,T)\delta}, \quad t<T \qquad (3)$$

where *n* is the number of steps and is such that nΔ=δ. As the Libor rate changes, the drift and the up probability in equation 2 also change. We notice that the instantaneous risk free rate is constant from *t* to *T*, so that *r(j)=r(j+Δ)* for each *j=t,…T – Δ*. Afterwards, the current Libor rate changes and the instantaneous risk free rate will change too.

Another important assumption of our procedure is that, at each time, the correlation between the future stock price and the future spot rate, conditional on the *current value* of the spot rate, is zero. This assumption is clearly not realistic, but it allows us to obtain the joint Probability Density Function (PDF) of the two considered random variables (the stock price and the short rate) just by multiplying their marginal PDFs. The following example may explain the implications of this assumption.

According to the BDT model, the future stock price at time *t* may assume the value of *L(t, T)u* with (risk neutral) probability *q=1/2* or the value of *L(t, T)d* with probability *1 – q=1/2*. At the same time, according to the CRR model, the future stock price may assume the value of $S_t^u$ with (risk neutral) probability *p* or of $S_t^d$ with probability *1 – p*. In other word, we refer to *p* and *q* as the marginal up probability respectively of the stock price and of the Libor rate. If we set the hypothesis of null correlation between *L(t, T)* and $S_t$, conditional on the current value of the short rate and of the stock price, the joint conditional PDF of stock price and short rate is shown in figure 2:

*3.2 The mechanics of the procedure*

To show how the procedure can be effectively applied, we consider a two step scheme. The length of each period is equal to the tenor *δ* of the Libor rate that is, for simplicity, set equal to 1. The number of steps *n* of the stock tree is thus equal to 2 and, consequently, the step size Δ is equal to 1.

Let *L(0,1)* be the current 12 month Libor rate, whose value may increase to *L(1,2)u* or decrease to *L(1,2)d* with equal (marginal) probability (*q=1/2*), according to the BDT model. Analogously, let $S_0$ be the current stock price whose value, according to the implied stock tree model, may increase to $S_1^u = S_0 u$ with risk neutral marginal probability $p_1$ or decrease to $S_1^d = S_0 d$ with probability $1 - p_1$, where $p_1 = \frac{m_1 - d}{u - d}$ and $m_1 = 1 + L(0,1)$. We can therefore identify four "states of the world" whose probabilities are reported in figure 2.

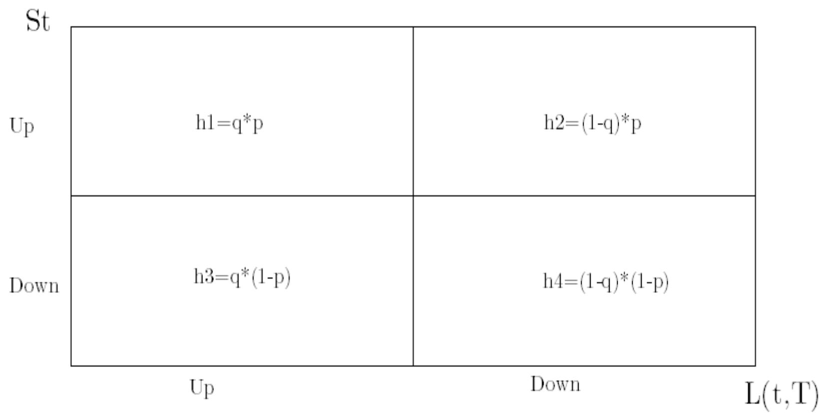


**Figure 2**. The joint PDF of the short rate *L(t, T)* and of the stock price $S_t$ in the case of zero correlation.

Now, let us consider the case in which the short rate increases to *L(1,2)u* and also the stock price increases to $S_1^u$. Since the two events are independent, the probability that they occur contemporaneously is: $\Pr ob\left(S_1^u \cap L(1,2)u \mid S_0, L(0,1)\right) = \Pr ob\left(S_1^u \mid S_0, L(0,1)\right) * \Pr ob\left(L(1,2)u \mid S_0, L(0,1)\right) = p_1 * \frac{1}{2}$. This procedure may be repeated for each possible couple of the short rate and stock price at time *t=1*.

Now, let us consider the state of the world where stock price is $S_1^u$ and the Libor rate is L(1,2)u. At the successive time step, *t=2*, the stock price may again increase to $S_2^{uu} = S_0 u^2$ with risk neutral marginal probability $p_2$ or decrease to $S_2^{ud} = S_0 ud = S_0$ with probability $1 - p_2$, with $p_2 = \frac{m_2^u - d}{u - d}$ and $m_2^u = 1 + L(1,2)u$. At the same time, the Libor rate may increase to *L(2,3)uu* or decrease to *L(2,3)ud* with equal probability. The probability that both the stock price and the Libor rate increase for two consecutive times is therefore: $\Pr ob\left(S_2^{uu} \cap L(2,3)uu \mid S_1^u, L(1,2)u\right) = p_1 * p_2 * \frac{1}{2} * \frac{1}{2}$. We repeat this procedure until, at the end of the second time interval, all the 16 final states of the world and their respective probabilities are available, as described in table 1.

| State of the world | Libor rate | Stock price | Probability |
|---|---|---|---|
| 1 | *L(2,3)uu* | $S_2^{uu}$ | $h_1$ |
| 2 | *L(2,3)ud* | $S_2^{uu}$ | $h_2$ |
| 3 | *L(2,3)du* | $S_2^{uu}$ | $h_3$ |
| 4 | *L(2,3)dd* | $S_2^{uu}$ | $h_4$ |
| 5 | *L(2,3)uu* | $S_2^{ud}$ | $h_5$ |
| 6 | *L(2,3)ud* | $S_2^{ud}$ | $h_6$ |
| 7 | *L(2,3)du* | $S_2^{ud}$ | $h_7$ |
| 8 | *L(2,3)dd* | $S_2^{ud}$ | $h_8$ |
| 9 | *L(2,3)uu* | $S_2^{du}$ | $h_9$ |
| 10 | *L(2,3)ud* | $S_2^{du}$ | $h_{10}$ |
| 11 | *L(2,3)du* | $S_2^{du}$ | $h_{11}$ |

| | | | |
|---|---|---|---|
| 12 | *L(2,3)dd* | $S_2^{du}$ | $h_{12}$ |
| 13 | *L(2,3)uu* | $S_2^{dd}$ | $h_{13}$ |
| 14 | *L(2,3)ud* | $S_2^{dd}$ | $h_{14}$ |
| 15 | *L(2,3)du* | $S_2^{dd}$ | $h_{15}$ |
| 16 | *L(2,3)dd* | $S_2^{dd}$ | $h_{16}$ |

**Table 1:** An example of the discrete joint probability density function of the stock price and of the Libor rate

The current stock price $S_0$ and the future stock price $S_2$ are thus linked as follows:

$$S_0 = E_0^h\left[\frac{S_2}{1+L(1,2)}\right]\frac{1}{1+L(0,1)} \qquad (4)$$

where *h* is the joint probability measure associated to the future values of the stock price and of the Libor rate and $E_0^h$ denotes the expected value under the probability measure *p* conditional on the information available at the valuation date *t=0*.

Equation (4) states that the current stock price $S_0$ is the expected value of the future stock price $S_2$ discounted at an appropriate stochastic interest rate. We notice that the discount factor is only partly included in the expectation operator because, at time *t=0*, the current Libor rate *L(0,1)* is known while the Libor rate that will run during the period from *1* to *2* is a random variable. Since the discount factor and the stock price are independent random variables, it can be shown that:

$$S_0 = E_0^p[S_2]E_0^q\left[\frac{1}{1+L(1,2)}\right]\frac{1}{1+L(0,1)} = \frac{E_0^p[S_2]}{[1+L(0,2)]^2} \qquad (5)$$

Equation (5) implies a very important property: the probability measure *h*, defined as the joint probability associated to the future values of the stock price and of the Libor rate, is such that the stock price is a martingale. Once the joint PDF of the final stock price and of the Libor rate is obtained, it is quite a simple task to determine the price $C_0$ of a European call option at time *t=0*, with strike price *X*:

$$C_0 = E_0^h\left[\frac{(S_2 - X)^+}{1+L(1,2)}\right]\frac{1}{1+L(0,1)} \qquad (6)$$

## 4 Numerical examples

Equation 6 implies that the price of a call option is approximately equal to the price of the same option calculated by means of the CRR model. In other words, the model here developed can be regarded as an extension of the implied stock tree model on a roll-over basis. However, this could not be the case if we release the hypothesis of zero correlation between interest rate and stock price, because in the original CRR model the interest rate is not stochastic and therefore there is no reason for any relationship between the rate and the stock price. On the contrary, a change of the term structure of interest rates produces a difference in the price of the call option, since that change can be regarded as the adoption of a different parameter. Therefore, there is a certain interest in evaluating pricing differences emerging from the implementation of one model against the other. To this aim it is necessary to get the joint PDF of the stock price and of the interest rate.

Let us consider the simple framework of the previous section where *n=2*, *δ=1*. We set the stock price equal to 100 and its implied annual volatility to 20%. Finally, we assume that the spot Libor rate is 1% for the first year and 2% for the second year and 3% for the third year, while the term structure of the volatility is equal to 20% and 30% for the second and the third year respectively. All the parameters are reported in table 2.

| Parameters | Time (years) | | |
|---|---|---|---|
| | **0** | **1** | **2** |
| $S_0$ | 100 | - | - |
| $L(0,T)$ | 1% | 2% | 3% |
| $\sigma_{CRR}$ | - | 20% | 20% |
| $\sigma_{BDT}$ | - | 20% | 30% |

**Table 2:** Parameters adopted for the calibration of the model.

Two remarks are necessary. Firstly, we notice that the Libor rate for maturities over 12 months is not directly observable on the market. Those values can however be obtained from the swap curve on the Libor rate by means of bootstrap technique (details in Hull, 2009). Secondly, we notice that to the aim of pricing a plain vanilla call option it is not necessary to specify the 3-year interest rate. We decide however to consider it to show the final joint PDF of the interest rate and stock price. Such PDF may be adopted to compute the price of financial products whose value depends contemporaneously on the level of stock price and interest rate at the maturity, as for example convertible bonds.

Figure 3a and 3b show respectively the BDT interest rate tree and the binomial stock tree (with marginal probabilities) calculated adopting the parameters shown in table 2.

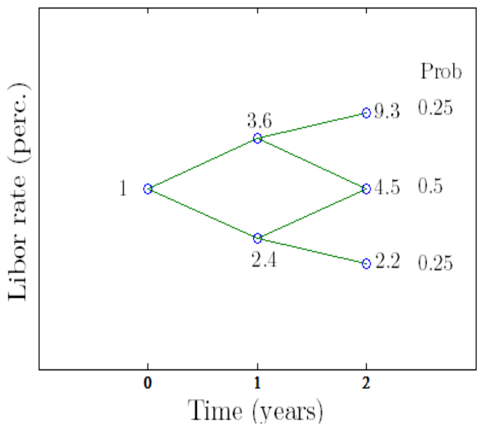


**Figure 3a.** Example of the BDT tree

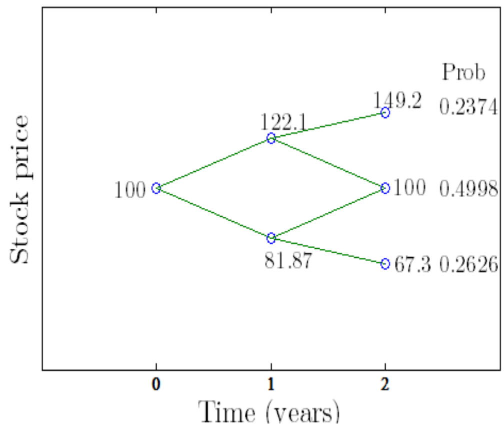


**Figure 3b.** Example of the binomial stock tree

Given the values in figure 3a and 3b, it is a simple task to determine the joint PDF of the interest rate and of the stock tree. For example, the probability that the stock price will be 100 at the end of the 2 year jointly with an interest rate level of 9.3% is 0.25*0.4998.

We remark that, if we set the hypothesis of zero correlation, the price of the call option calculated by means of the procedure exposed in section 3 coincide with the price of the same option calculated by means of the CRR model in the case where the term structure of the interest rate is not flat. More precisely, the risk free rate for the first year coincides with the corresponding *spot rate* while the risk free rate for the second year coincides with the corresponding *forward rate.* In this way we are able to incorporate market expectations (at the valuation date) on the future interest rates in the pricing of option.

To show how interest rate expectations affect the stock option pricing, we compare the price of a call option, derived in our framework, to the Black and Scholes price. We decide to adopt the Black and Scholes model for the comparison firstly, because the CRR price tends to the Black and Scholes price as $\Delta \rightarrow 0$, and secondly, because the Black and Scholes price is not able to capture the expectations on the future interest rates.

Table 3 reports the differences (in percentage) between the price of a two year ATM vanilla call option calculated by means of our procedure ($P$) and the price of the same option calculated by means of the Black and Scholes (1973) model ($Bls$), according to the following formula:

$$\frac{P - Bls}{Bls}*100 .$$

Such differences are calculated for different term structures of the interest rates. The other parameters adopted to evaluate the price differences are those reported in table 2 but this time we consider, for each year, a higher number of steps for the stock tree. More precisely, since the stock exchange is open about 254 days per year, we thus set $n=508$ ($\Delta=1/254$).

We thus notice that if the term structure is flat (see the numbers on the diagonal of table 3), the percentage difference with the Black and Scholes formula is quite negligible, from .03% to .05%. However, as expected, such differences tend to increase as the difference between the interest rates for the two maturities increases. If, on the contrary, $L(0,1)<L(0,2)$ ($L(0,1)>L(0,2)$) the term structure is upward (downward) sloping and the price differences are positive (negative).

| L(0,1) \ L(0,2) | 1% | 2% | 3% | 4% | 5% |
|---|---|---|---|---|---|
| **1%** | 0.05% | 6.96% | 13.10% | 18.58% | 23.48% |
| **2%** | -7.41% | 0.04% | 6.68% | 12.60% | 17.90% |
| **3%** | -15.12% | -7.10% | 0.04% | 6.41% | 12.11% |
| **4%** | -23.06% | -14.45% | -6.80% | 0.04% | 6.15% |
| **5%** | -31.22% | -22.01% | -13.82% | -6.51% | 0.03% |

**Table 3.** Price differences with respect to the Black and Scholes (1973) formula for different interest rate term structures.

## 5 Final remarks

This paper shows a procedure to determine the price of financial contracts that are exposed to two sources of risk: the stock price and the interest rate. In particular, we assume that each risk factor evolves over time according to a binomial tree so that the final distribution is, in the limit for both risk factors, lognormal. To this aim, we set some hypotheses and in particular we assume that the correlation between the interest rate and the stock price is zero and that the Libor rate proxies for the risk free rate. We showed that under this assumptions, the stock price is a martingale under a particular (joint) probability measure that results from the product of the risk neutral marginal probabilities of the two considered risk factors. Even if this assumptions appear to be clearly unrealistic, we therefore set them to simplify the pricing approach. In this section, some techniques are proposed in order to release such hypotheses.

In particular, the assumption of zero correlation between the interest rate and the stock price may be released by redistributing, in a different way, the joint probabilities calculated in the case of independency that are exposed in figure 2, among the possible states of the world. For example, we can set the hypothesis that the stock price and the interest rate show perfectly negative correlation by equally distributing the probabilities (in case of independence) of contemporaneously up movements and down movements of stock price and rate to the other two states of the world, according to figure 4. The terminal stock price dynamics will thus result in a binomial random variable and, in the limit, is lognormal.

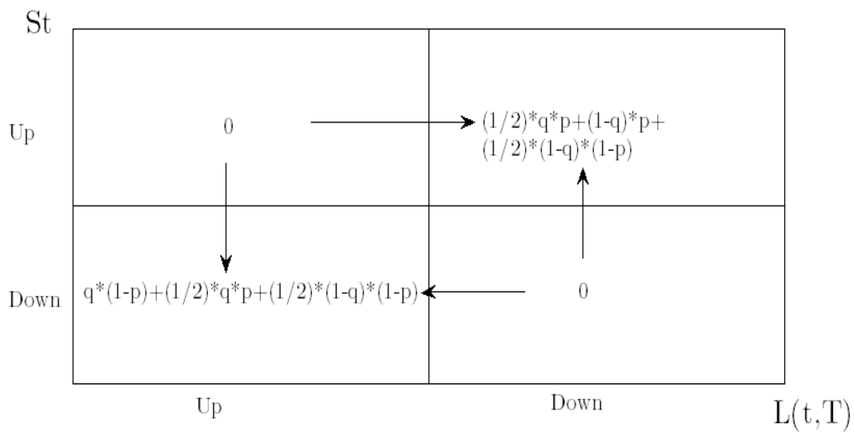


**Figure 4**. The joint PDF of the short rate *L(t, T)* and of the stock price $S_t$ in the case of perfectly negative correlation.

We notice that, if this is the case, the *instantaneous* correlation will be -1 but the *terminal* correlation may depend not only on the way the probabilities are redistributed, but also on the variances of the two risk factors. For these reasons, more sophisticated techniques must be applied in order to calibrate the model to the empirical correlation between interest rate and stock price. Finally we point out that, if a similar technique can be applied to impose perfectly positive correlation between the two risk factors, it is a harder task to fit the correlation parameter to the observed market data. In this case the problem is twofold. Firstly, the correlation between interest rate and stock price must be estimated. Secondly, we have to choose a redistribution rule (different from the above mentioned rule) such that the correlation of the model is equal to the estimated correlation. This second point can be solved in a very simple way. In the case of perfectly negative correlation, we set the probability of contemporaneous up and down movement of the two risk factor equal to zero. If on the contrary we decide to equally redistribute to the other two states of the world only a percentage $\gamma$ of the probability of contemporaneous up and down movement, it will result in a correlation equal to $-\gamma$. If we thus decide to equally redistribute only a 50% of the probability of contemporaneously up and down movement to the other states of the world, it will result in a correlation of -0.5. On the contrary, if we decide to equally distribute a percentage $\gamma$ of the probability of opposite movements of the considered risk factors to the other two states of the world, it will result in a correlation equal to $\gamma$.

We notice however that by relaxing the hypothesis of zero correlation, equation 5 does not hold and the stock price dynamics does not expose the martingale properties. However, even if the current stock price were still an unbiased estimate of the future stock price, it can be noticed that relaxing the hypothesis of zero correlation may produce a price for a call option different from those that can be obtained by adopting the Cox, Ross and Rubinstein (1979) model.

The second strong hypothesis of our model is in the adoption of the Libor rate as risk free rate. Since the Libor rate is in general higher than a AAA interest rate, its adoption will result in higher (lower) price for a call (put) option. To solve this problem, we can however consider the Libor rate as formed by the sum of a basic risk free rate *R(t, T)* and a spread *φ(t, T)*. In this case the risk free rate *R(t, T)* will be a random variable composed as follows:

$$R(t, T) = L(t, T) - \varphi(t, T)$$

Assuming that the spread is not a random variable and that it is constant over time at a certain level, it will only be function of the tenor of the interest rate $\delta$, and the risk free rate *R(t, T)* will be a random variable with the same distribution of *L(t, T)*, the same variance but a lower average.

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