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Counterparty credit Risk

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Abstract

Credit valuation adjustment in Basel III is studied from the perspective of the mathematics involved. A bank covers mark-to-market losses for expected counterparty risk with a CVA capital charge. The CVA is known as credit valuation adjustments. In this paper it will be argued that CVA and conditioned value at risk (CVaR) have a common mathematical ancestor. The question is raised why the Basel committee, from the perspective of CVaR, has selected a specific parameterization. It is argued that a fine-tuned supervision, on the longer run, will be beneficial for counterparties with a better control over their spread.

Keywords: CVA, CVaR, statistical methodology

1 Introduction

1.1 Measures of counterparty risk

The reservation of capital is based on what to expect in the future. In Basel III the Credit Valuation Adjustment (CVA) capital charge takes a prominent role [1] in the treatment of mark-to-market counterparty risk. Banks with IMM approval for counterparty credit risk need to calculate additional capital charge by modeling the impact of changes in the counterparties' credit spreads. The capital charge calculation for each counterparty is based on the CVA formula of page 31 of the framework document and is represented below in equation (1).

We may ask if a presently sufficient decrease in counterparty spread can be trusted for the future such that it will, rightfully, not add to the required capital charge. Our method enables a comparison of counterparties and a 'trusworthiness' factor to be incorporated in the CVA. The method on which 'trust' can be made numerical is e.g. to compare historical series of spread and to use the variance of the spread. An alternative method to compare is to employ Fourier analysis on the fluctuations of different spread. If principal wavenumber components of two series are more or less equal, both counterparties suffer from the same economic processes. If one counterparty has a greater mean amplitude than the other, one can conclude that this counterparty is less in control over his spread. Therefore it could be wise to reserve capital despite of a sufficient decrease in spread in the present period. The proposed parametrization of fine-tuning on counterparty risk is based on the method of conditional value at risk [2]. Of course other assesments of trust, like a panel of experts, can be employed too to classify trust in counterparty spread.

The CVA is defined in the framework [1] as:

$$CVA = L_{MKT} \sum_{i=1}^{T} \max\left\{0, e^{-\hat{s}_{i-1}t_{i-1}} - e^{-\hat{s}_i t_i}\right\} \left(\frac{EE_{i-1}D_{i-1} + EE_i D_i}{2}\right) \tag{1}$$

Here, EE_i is the expected exposure [4] to the counterparty at time (bucket) t_i, with $t_0 = 0$ and $i = 1, 2, ...T$. The EE_i is obtained in the new capital conservation mechanism of the committee and refers to a required countercyclical capital buffer [3]. Moreover, D_i is a default risk-free discount factor at t_i. It is somewhat unclear what is intended here. The L_{MKT} is the so-called loss-given-default of the counterparty and must be based on the spread of a market instrument of the couterparty. It is a market assesment of the loss and, hence, does not depend on time buckets t_i. The filer $\max(0, y)$ in equation (1) equals y when $y > 0$ and vanishes if $y \leq 0$. Note that L_{MKT} does not change for the measurement period from t_0 to t_T. But that does not imply that this factor will be constant for a possible next period. The $\hat{s}_i$ will be discussed below. The difference with Basel II [4] appears to be substantial because in Basel II supervision the committee only subscribed banks to employ the following maturity measure, M.

$$M = \frac{\sum_{k=1}^{t_k \leq 1\, year} E_{ff}\,(EE_k)\,\Delta\, t_k\, D_k + \sum_{k \in \{t_k > 1\, year\}}^{maturity} EE_k\,\Delta\, t_k\, D_k}{\sum_{k=1}^{t_k \leq 1\, year} E_{ff}\,(EE_k)\,\Delta\, t_k\, D_k} \tag{2}$$

Conceptually, M is the (effective) credit duration of counterparty exposure [4].

1.2 Mathematical resemblance with CVaR

The first thing that catches the eye in the Basel III CVA formulation is the use of the $\max(0, y)$ operator. Its systematics resembles the way in which the β-CVaR [5] is obtained. The basis for β-CVaR is the expression

$$F_\beta(\vec{x}, \alpha) = \alpha + \frac{1}{(1-\beta)T} \sum_{i=1}^{T} [f(\vec{x}, \vec{y}_i) - \alpha]^+ \tag{3}$$

with $[t]^+ = t$ when $t > 0$ and $[t]^+ = 0$ when $t \leq 0$. Furthermore, $f(\vec{x}, \vec{y})$ is a general measure of loss associated to a decision vector $\vec{x} \in X \subset R^n$. The $\vec{x}$ refer to the 'portfolio' while $\vec{y}_i \in Y \subset R^m$ refer to the uncertainties that arise from market parameters. Note that the CVaR is a general scheme.

As can be seen in equation (3) the $\vec{y}$ are supposed to be uniformly distributed with $1/T$, for bucket numbering $i = 1, 2, .., T$. In the definition, the index β is the probability level of the loss, i.e. there is a probability β that the loss will *not* exceed α. Later on we will make this more explicit.

In order to establish a relation between the CVA defined in (1) and the CVaR, we can make the following comparison: the EE_i together with the t_i can be assigned to the $\vec{x}$ from (3), the $\hat{s}_i = \frac{s_i}{L_{MKT}}$ in (1) naturally are assigned to the $\vec{y}_i$ from (3). The s_i refer to the credit spread of the counterparty at, what the committee calls, 'tenor' t_i. When the α is selected zero we acknowledge a form $[g_i]^+$ employed in (1). Or,

$$f(\vec{x}, \vec{y}_i) \sim \left(e^{-\hat{s}_{i-1}t_{i-1}} - e^{-\hat{s}_i t_i}\right)\left(\frac{EE_{i-1}D_{i-1} + EE_i D_i}{2}\right) = g_i \qquad (4)$$

On the point of using a maximum operator $\max(0, g_i)$ one can see the resemblance beween CVA and CVaR because $[g_i]^+ = \max(0, g_i)$. Hence, the methods to optimize CVaR can, in principle, be applied to CVA as well. The mathematical statistics apparently allows an optimizing approach. For the ease of comparing the CVA in equation (1) with CVaR we may write

$$CVA = L_{MKT} \sum_{i=1}^{T} \max\left(0, g_i\right) \qquad (5)$$

with g_i as defined in (4).

2 Factor dynamics in the CVA

Here we study the dynamics of CVA. Recall that the factor $\hat{s}_i$ is defined by $\hat{s}_i = \left(\frac{s_i}{L_{MKT}}\right)$ and s_i refers to the credit spread of the counterparty. Now because, $t_i > t_{i-1} \geq t_0 = 0$, it follows that, when the (weighted) credit spread of the counterparty, or the, $\hat{s}_i$ at t_i is greater than $\hat{s}_{i-1}$, then, $e^{-\hat{s}_{i-1}t_{i-1}} - e^{-\hat{s}_i t_i} > 0$ and the mean $\left(\frac{EE_{i-1}D_{i-1}+EE_i D_i}{2}\right) > 0$ times the $e^{-\hat{s}_{i-1}t_{i-1}} - e^{-\hat{s}_i t_i}$ adds to the CVA. The weight factor is the difference between the exponentials.

Furthermore, if $\hat{s}_i < \hat{s}_{i-1}$ then $\left(\frac{EE_{i-1}D_{i-1}+EE_i D_i}{2}\right)$ still may add to the CVA provided

$$\delta t_{i-1} > \left(\frac{\hat{s}_{i-1}}{\hat{s}_i} - 1\right) t_{i-1} \qquad (6)$$

given $t_i = t_{i-1} + \delta t_{i-1}$. This means that for relative large time steps a certain amount of diminishing of credit spread for the counterparty appears to still add to the capital charge CVA. This is due to the fact that in the initial diminishing of credit spread over time, the spread of the counterparty can still increase in a, more uncertain, larger time period.

Only when $\hat{s}_i < \hat{s}_{i-1}$ and equation (6) is *not* valid does the term g_i in time period t_i not contribute to the CVA capital charge. Hence, a lower CVA capital charge turns out to depend on a sufficient decreasing series $\{\hat{s}_i\}_{i=I_0}^{I_1}$ with $I_0 < I_1$ in the set $\{1, 2, ..., T\}$, that is a consecutive series of decreasing spread periods. Moreover, when $e^{-\hat{s}_{i-1}t_{i-1}} - e^{-\hat{s}_i t_i} \approx 0$ there is litte contribution to the CVA. No doubt, when δt_i is relatively small a prudent control will then require a relatively large T.

3 CVA and CVaR

For comparison with CVaR the CVA is made dependent on α. Hence,

$$CVA(\alpha) = \alpha + L_{MKT} \sum_{i=1}^{T} \max\left(0, g_i - \alpha\right) \tag{7}$$

g_i defined as in (4). As can be seen from equation (1) the Basel III CVA has, $\alpha = 0$.

Subsequently we note that the CVaR is defined by

$$\phi_\beta(\vec{x}) = (1-\beta)^{-1} \int_{f(\vec{x},\vec{y}) \geq \alpha_\beta} f(\vec{x}, \vec{y}) P(d\vec{y}) \tag{8}$$

with α_β a parameter depending on α and β. The CVaR can be obtained as a minimalization problem: $\phi_\beta(\vec{x}) = \min\{F_\beta(\vec{x}, \alpha) |\, \alpha \in R\}$, with $F_\beta(\vec{x}, \alpha)$ as in (3), or more generally

$$F_\beta(\vec{x}, \alpha) = \alpha + (1-\beta)^{-1} \int_{\vec{y} \in R^m} [f(\vec{x}, \vec{y}) - \alpha]^+ P\left(d\vec{y}\right). \tag{9}$$

Here, $P(\vec{y})$ is a, Lebesgue, probability measure over the vector $\vec{y}$, similar as used by Komogorov [6]. In case of equation (3), the Lebesgue measure 'collapses' to a discrete space with uniform density, $\rho(\vec{y}_i) = 1/T$ for $\vec{y}_i \in R^m, i = 1, 2, 3, , , T$. Hence, according to [7] the minimum can be found through the expression

$$\frac{\partial}{\partial \alpha} F_\beta(\vec{x}, \alpha) = 1 + (1-\beta)^{-1} \left(\Psi(\vec{x}, \alpha) - 1\right) \tag{10}$$

with

$$\Psi(\vec{x}, \alpha) = \int_{f(\vec{x},\vec{y}) \leq \alpha} P(d\vec{y}) \tag{11}$$

This is equal to the expression that the random variable $f(\vec{x}, \vec{y})$ does not exceed a treshold α. Hence, $\Psi(\vec{x}, \alpha) = \beta$, or in words, the probability that $f(\vec{x}, \vec{y})$ does not exceed a treshold α is equal to $\beta \in (0, 1)$. Having, $\frac{\partial}{\partial\alpha} F_\beta(\vec{x}, \alpha) = 0$, the minimum of $F_\beta(\vec{x}, \alpha)$ is found and hence, the minimum CVaR is determined as $\phi_\beta(\vec{x})$ with the probability that a test random variable does not exceed a value α with a probability β.

Interestingly, introducing a Heaviside θ function as correction term $\alpha L_{MKT} \sum_{i=1}^{T} \theta(g_i - \alpha)$ to equation (7) does not lead to a minimum for terms similar to $F_\beta(\vec{x}, \alpha)$. This is true when we reformulate $CVA'(\alpha) = CVA(\alpha) + \alpha L_{MKT} \sum_{i=1}^{T} \theta(g_i - \alpha)$ in terms of $G_\beta(\vec{x}, \alpha)$ in CVaR terms like

$$G_\beta(\vec{x}, \alpha) = F_\beta(\vec{x}, \alpha) + \alpha(1 - \beta)^{-1} \int_{\vec{y} \in R^m} \theta(g(\vec{x}, \vec{y}) - \alpha) P(d\vec{y}) \tag{12}$$

with $F_\beta(\vec{x}, \alpha)$ from equation (9). Because

$$\int_{\vec{y} \in R^m} \theta(g(\vec{x}, \vec{y}) - \alpha) P(d\vec{y}) = 1 - \Psi(\vec{x}, \alpha) \tag{13}$$

with, $f(\vec{x}, \vec{y})$ in equation (11) replaced by $g(\vec{x}, \vec{y})$, it follows that

$$\frac{\partial}{\partial\alpha} G_\beta(\vec{x}, \alpha) = 1 - \alpha(1 - \beta)^{-1} \int_{\vec{y} \in R^m} \delta(g(\vec{x}, \vec{y}) - \alpha) P(d\vec{y}). \tag{14}$$

It is highly likely that $g(\vec{x}, \vec{y})$ is unequal to α given $g(\vec{x}, \vec{y})$ and $\vec{x} \in X$, $\vec{y} \in Y$. Simply stated there appears not to exist trivial compensation terms for: $\Delta(\alpha) = -\alpha L_{MKT} \sum_{i=1}^{T} \theta(g_i - \alpha)$ in equation (7). This conclusion can be supported by noting the fact that, generally, $\max(0, z) = z\theta(z)$. Hence, for numerical purposes there appears no CVaR trivial way from $CVA(\alpha)$ to sums without $\Delta(\alpha)$.

4 Conclusion & discussion

4.1 CVar and CVA resemblance

As can be seen from comparing the two formula's the Basel III CVA methodology resembles the CVaR. Moreover, it was shown that a CVaR minimum exists for $\alpha \in R$. As can be seen from the definition (1), the Basel III CVA is connected to the CVaR with the use of $\alpha = 0$ and there is no CVaR trivial compensation term for $\Delta(\alpha)$ defined previously. If for proper choice of T and β we have $L_{KMT} = \frac{1}{(1-\beta)T}$, then $\max(0, g_i - \alpha) = [g_i - \alpha]^+$, with, $g_i = \left(e^{-\hat{s}_{i-1} t_{i-1}} - e^{-\hat{s}_i t_i}\right)\left(\frac{EE_{i-1} D_{i-1} + EE_i D_i}{2}\right)$ warrants a CVaR reformulation of CVA.

In this interpretation it makes sense to view the $\hat{s}_i$ as belonging to the $\vec{y}$ 'market' variables. Moreover, expected exposure, EE_i, can be seen as belonging to the state variables $\vec{x}$. For a certain period, L_{MKT} and D_i also belong to $\vec{y}$. Concerning the time-buckets they also can be assigned to the state variables in the sense that they can, directly, be influenced by the bank. More in particular, the δt_i appear state variables. Note that a bank can negotiate with the supervisor the δt_i of the reports and in this way the time-buckets certainly can be viewed as state variable.

4.2 Proof that $CVA(\alpha) > CVA(0)$ for $\alpha < 0$

In Figure 1 below the direction of the difference is already made plausible. Let us define the set I^+ as $I^+ = I(g_i > 0) = \{i|g_i > 0\}$ and $I^- = I(g_i \leq 0) = \{i|g_i \leq 0\}$ such that $I^+ \cup I^- = \{1, 2, 3, ...T\}$. This implies that one may write

$$CVA(0) = L_{MKT} \sum_{i \in I^+} g_i \tag{15}$$

If we note that $[g]^+ = g\theta(g)$, it follows for $\alpha < 0$, that, $CVA(\alpha) - CVA(0) =$

$$-|\alpha| + L_{MKT} \sum_{i \in I^-} (|\alpha| + g_i)\, \theta\, (|\alpha| + g_i) + L_{MKT}|\alpha| \sum_{i \in I^+} (|\alpha| + g_i) \tag{16}$$

This is true because, when $g_i > 0$ then $g_i + |\alpha| > 0$ too. Now because

$$L_{MKT}|\alpha| \sum_{i \in I^+} (|\alpha| + g_i) = card\left(I^+\right) L_{MKT}|\alpha|, \tag{17}$$

with *card* denoting 'cardinality'of the set, then if $card\,(I^+)\, L_{MKT} > 1$, it follows that $CVA(\alpha) - CVA(0) > 0$, because

$$L_{MKT} \sum_{i \in I^-} (|\alpha| + g_i)\, \theta\, (|\alpha| + g_i) \geq 0. \tag{18}$$

4.3 Financial consequence

The question now arises whether the selection $\alpha = 0$, implicitly present in the Basel III definition of risk capital required for counterparty risk (1), represents the best optimal value given the previous obvious subdivision in state and market variables. In other words, granted the structural resemblance between CVaR methodolgy and the required CVA, we may ask what the reasons for the Basel committee were to select the $\alpha = 0$ parameterization in their CVA required capital charge. The application of equation (7) enables to have $\alpha < 0$. Looking at the discussion of the dynamics of factors in section 2, one sees that

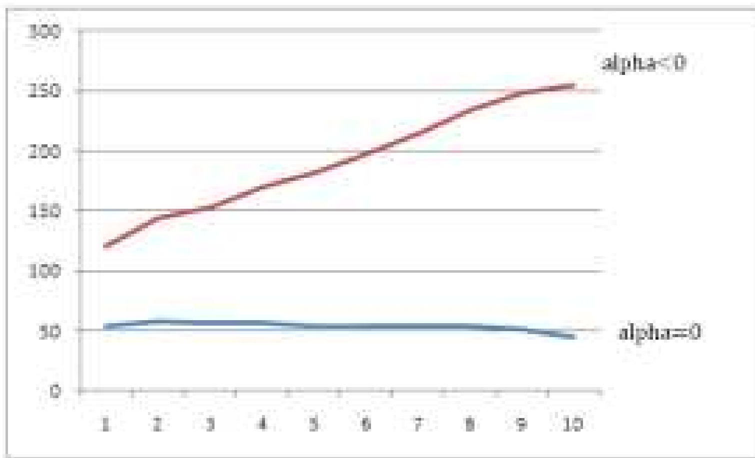

Figure 1: *The* $CVA(\alpha)$ *for* $\alpha < 0$ *and the Basel III* $CVA(0)$ *(horizontal line) are given where* $\alpha \in \{-0.01, -0.02, -0.03, -0.04, -0.05, -0.06, -0.07, -0.08, -0.09, -0.1\}$. *The* s_i, EE_i *and* D_i *are pairwise randomly generated from the interval (0,1) and* $L_{MKT} = 2$. *With the present parametrisation the gap in the lower* $|\alpha|$ *region requires to reserve 3 - 4* $\times$ *the amount of Basel III CVA*

$\alpha < 0$ is interesting for unstable counterparty spread. Despite the fact that e.g. in the t_i the spread of the counterparty is diminished related to t_{i-1}, one could still want to add $g_i + |\alpha|$ because of (expected) fluctuation in spread in some future. The latter selection enables a fine-tuning in discriminating stable and unstable counterparties basing onself on the variance of the couterparty spread, i.e. the $Var(s_i)$.

From figure 1 we present numerical proof of the fact that for $\alpha < 0$ the $CVA(\alpha)$ is larger than the $CVA(0)$ used in the Basel III framework. Use was made of parametrizations $L_{MKT} = 2$, $T = 10^3$ and s_i, EE_i together with D_i pairwise randomly generated (in a VBA program). The α are taken from the set $\{\alpha_n | n = 1, 2, ..., 10, \alpha_n = \alpha_{n-1} - \Delta\alpha, \alpha_0 = 0, \Delta\alpha = 0.01\}$. Hence, the introduction of $\alpha < 0$ enables a fine-tuning in the CVA in terms of more or less creditworthy counterparties.

4.4 Measure of trust

An alternative to the variance of the spread of the counterparty is a Fourier analysis on spread fluctuations. When characteristic frequencies for two given counterparties are approximately equal but there is a substantial difference in spread-amplitude a measure of trust in a counterparty is obtained that can be expressed in terms of $\alpha < 0$. Use of comparing characteristic frequencies will enable to judge whether counterparties are subjected to similar economic fluctuations.

4.5 Question of supervision

In view of possible negotiating processes on time-buckets and the creditworthiness of counterparties it can be questioned whether the present Basel III CVA form with $\alpha = 0$ is the best possible capital charge in a regulatory framework for more resilient banking. Differentiating on a numerical measure of trust will avoid having too low capital reservation for certain counterparties and will have the effect that, because of less required capital reserve, banks will prefer to do business with counterparties with a better control over their spread.

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