New indices of labour productivity growth: Baumol's disease revisited

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New Indices of Labour Productivity Growth: Baumol's Disease Revisited

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Abstract

We introduce two new indexes of labour productivity growth. Both indexes are intended to capture the shift in the short-run production frontier, which can be attributed to technological progress or growth in capital inputs. The two indexes adopt distinct approaches to measuring the distance between the production frontiers. One is based on the distance function and the other is based on the profit function. In the end, we show that these two theoretical measures coincide with the index number formulae that are computable from the observable prices and quantities of output and input. By applying these formulae to the U.S. industry data of the years 1970–2005, we compare newly proposed index of labour productivity growth with the growth of average labour productivity over periods and across industries. We revisit the hypothesis of Baumol's disease throughout our observations on the trend of industry labour productivities in the service sector.

Key Words: Labour productivity, index numbers, Malmquist index, Törnqvist index, output distance function, input distance function, Baumol's disease, service sector

JEL classification: C14, D24, O47, O51

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1. Introduction

Productivity measure is defined as the ratio of an index of outputs to an index of inputs. Economists think of productivity as measuring the current state of the technology used in producing the goods and services of a firm, which is a technical constraint on the firm's profit maximizing behaviour. The production frontier, consists of inputs and the maximum output attainable from such inputs, characterizes technology. Hence, the productivity growth index is interpreted as the shift in the production frontier, reflecting technological change.[1 2] There are multiple index number formulae for a productivity growth index. The idea that the productivity growth index should capture the shift in the production frontier helps to decide between different index number formulae. This approach to the choice of index numbers is called the economic approach.

Productivity measures can be classified into two types: total factor productivity (TFP) and partial factor productivity (PFP). The former index relates a bundle of total inputs to output, while the latter index relates a part of total inputs to output. Caves, Christensen and Diewert (1982) use the economic approach to justifying the choice of index number formula for the TFP growth index. They define the Malmquist productivity index, which measures the shift in the production frontier. Since it is a theoretical productivity index that is defined by the distance functions, one cannot compute it without knowing its functional form and its parameters. They show that the Malmquist productivity index coincides with the Törnqvist productivity index under the general assumption on the distance function. The Törnqvist productivity index is computable from the observed prices and quantities of outputs and inputs. Hence, they provide a good justification for the use of the Törnqvist productivity index. Diewert and Morrison (1986) also adopt the economic approach but use the profit function to define the theoretical TFP growth index. They show this index coincides with another index number formula of prices and quantities of outputs and inputs.

The present paper deals with the PFP growth index. Our focus is on the labour productivity growth index in particular.[3] Following Caves, Christensen and Diewert (1982), and Diewert and Morrison (1986), we apply the economic approach to the index number problem of labour productivity growth. We start from the idea that labour productivity should represent the technical constraint that a firm faces when it decides the optimum level of labour input. To put it differently, labour productivity measures the current state of the production technology that transforms labour inputs into output, holding fixed capital services. Hence, the production technology associated with the use of labour can be characterized by the short-run production frontier, which consists of labour inputs and the maximum output attainable from such

[1] See Griliches (1987). The same interpretation is also found at Chambers (1988).

[2] In principle, productivity improvement can take place through technological progress and technical efficiency gain. Technical efficiency is the distance between production plan and production frontier. The present paper assumes a firm's profit maximizing behaviour, and in our model the current production plan is always on the current production frontier. The assumption of profit maximization is a common practice in the economic approach to index numbers. See Caves, Christensen and Diewert (1982), and Diewert and Morrison (1986).

[3] We deal with the general model consisting of multiple labour inputs. Hence, our reasoning can be applied to any partial productivity growth measure that is associated with any combination of inputs in total.

labour inputs, holding fixed capital services. We propose theoretical labour productivity growth indexes that measure the shift in the short-run production frontier, using the distance function as well as the profit function. Two indexes are purely theoretical indexes. Under the assumption on the particular functional forms to represent the underlying technology, we derive the index number formulae, which coincide with distinct theoretical indexes.

The most standard labour productivity measure is average labour productivity, which is defined as output per unit of labour input. The shift in the short-run production frontier is not the only source of the growth of average labour productivity. The decrease in labour inputs could also raise average productivity, exploiting scale economies. This is the reason why average labour productivity steers us to the wrong conclusion about underlying technology constraint for the firm profit maximizing behaviour in some cases. New indexes of labour productivity growth can be considered extracting scale economies effect from the growth in average labour productivity.

Triplett and Bosworth (2004), (2006) and Bosworth and Triplett (2007) discussed the phenomenon that the service sector has a much lower growth of labour productivity than other industries and it drag down the growth of the aggregate labour productivity from the early 1970s until the middle 1990s. They call it Baumol's disease, since it was firstly pointed out by Baumol (1968).[4] However, their analysis is based on average labour productivity. We compare labour productivity in the service sector and other sectors applying new labour productivity growth index, which the present paper introduces, the U.S. industry data.

Recently, Nin, Arndt, Hertel and Preckel (2003) also defined the PFP by the shift in the production frontier. However, their productivity measure is the firm's productivity of producing a particular type of output amongst a comprehensive set of outputs using all the inputs. They also propose a procedure to calculate their measure of PFP. However, our study is based on data envelopment analysis (DEA) rather than index number technique. Thus, our result is independent of their result in all respects.

Section 2 proposes two measures of labour productivity growth. We also show how they can be calculated from observable prices and quantities. Section 3 explains the good aggregation property of these two measures, which we cannot find in average labour productivity growth. Section 4 applies these two measures to the analysis of labour productivity in U.S. industries. We compare these two measures with standard average labour productivity growth. Section 5 concludes.

2. Measuring the Shift in Production Frontier

We consider the labour productivity (LP) growth index that measures the shift in the short-run production frontier. The short-run production frontier represents the maximum output attainable from each bundle of labour inputs, holding fixed the level of technology and capital services. Let us consider the problem of measuring the

[4] However, they also showed that the labour productivity of service sectors has been even higher than other industries since 1995. Thus Baumol's disease has long since been cured. All these papers discuss the difference in productivity growth across industries through the industries' average labour productivity.

labour productivity growth of a firm from period 0 to period 1. Our approach to measuring the shift in the short-run production frontier can be illustrated in a simple model of one output y and two inputs, capital service x_K and labour input x_L. Suppose that a firm produces output y^0 and y^1, using inputs (x_K^0, x_L^0) and (x_K^1, x_L^1). Period t production technology is described by a production frontier (= function) $y = f^t(x_K, x_L)$ for t = 0 and 1. The technical constraint that a firm faces when it chooses the optimum level of labour input is characterized by the period t short-run production frontier $y = f^t(x_K^t, x_L)$, which indicates the output attainable from labour input x_L, holding fixed the period t technology and the period t capital service x_K^t.

We consider a preferable case for the use of labour: the situation when the production possibility frontier uniformly expands between periods 0 and 1 (Figure 1). Any level of labour input can produce more output in period 1 than in period 0 in this case. Thus, we can say that the productivity of labour improves in all respects between these two periods. The points A and B indicate the production plans for period 0 and 1.

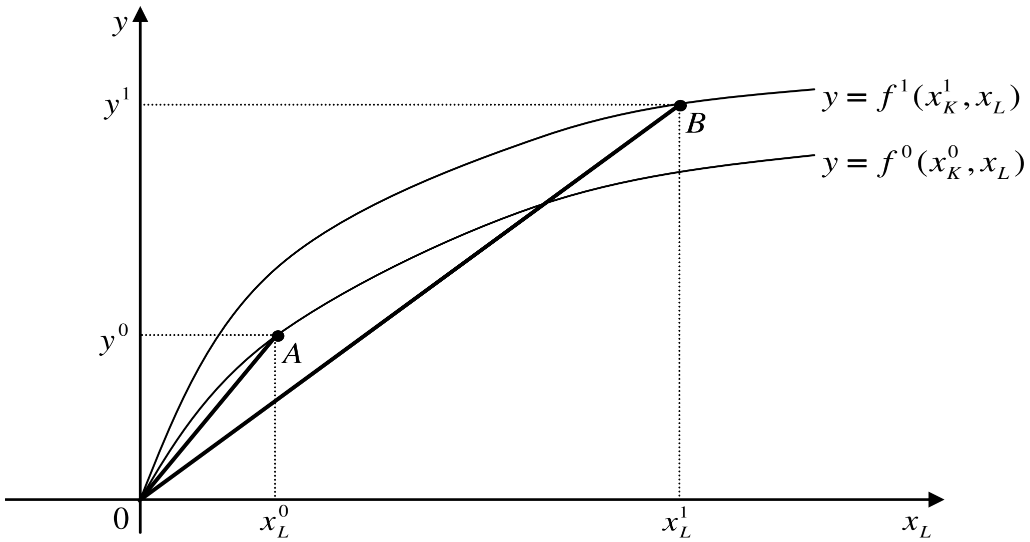

Figure 1: Average Labour Productivity and the Shift in the Short-run Production Frontier

Average labour productivity is the most popular measure of labour productivity and is interpreted as the units of output that one unit of labour can produce. We investigate how average labour productivity changes under expansion of the short-run production frontier, as illustrated by Figure 1. Average labour productivity, however, deteriorates from points A to B, reflecting the large increase in labour input. Thus, average labour productivity leads us to draw a counterintuitive conclusion in this case.

The problem of the misevaluation of the average labour productivity results from it not being associated with the shift in the short-run production frontier. We introduce the LP growth indexes to measure its shift. Given the quantity of labour input x_L, the shift in the short-run production frontier can be calculated as the ratio of the output being attainable from the period 1 capital service x_K^1 using the period 1 technology to the output being attainable from the period 0 capital service x_K^0 using the period 0 technology. If this ratio is larger (smaller) than one, the same quantity of labour input can produce more (less) output in period 1 compared with the reference period 0. In Figure 2, the maximum attainable level of output from labour input x_L^* changes from y_a to y_b between periods 0 and 1. Hence, the labour productivity growth is calculated as y_b/y_a. Note that its value depends on the reference quantity of labour input. Labour inputs of the periods under consideration, x_L^0 and x_L^1 are often adopted for defining the productivity measures.

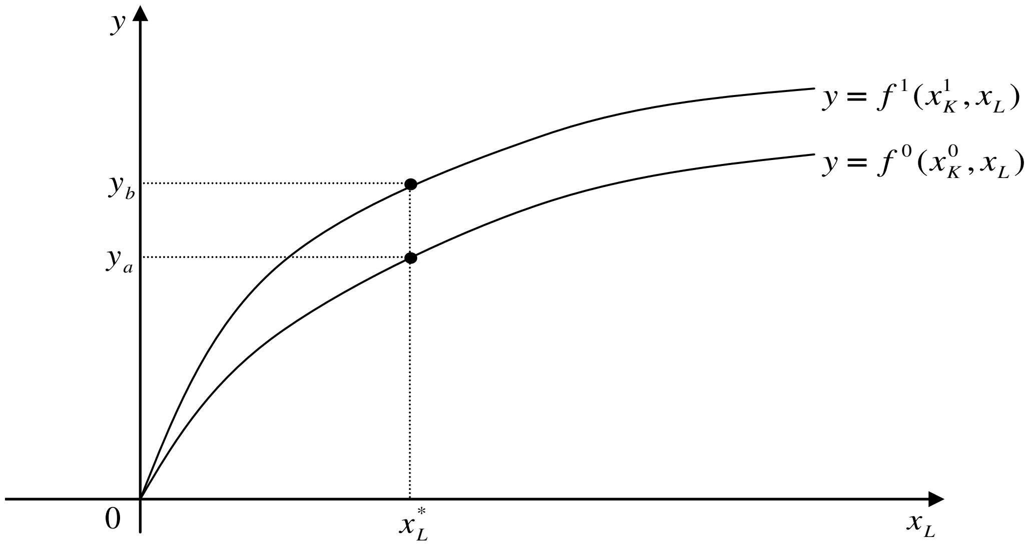

Figure 2: Measuring the Shift in the Short-run Production Frontier

The reason why average labour productivity declines, unrelated to the upward shifts in the short-run production frontier, is that the increase in labour input could decrease average labour productivity exploiting the scale economies. In figures, average labour productivity is the slope of the ray going from the origin to the point on the production frontier. Since the short-run production frontier is concave with respect to labour input, the slope of the ray decreases along the increase in labour input. Concavity is an indispensable property of the short-run production frontier. Even though the underlying production frontier exhibits constant returns to scale, the short-run production frontier, where capital inputs are fixed, is concave with respect to labour inputs and thus, it exhibits diminishing returns to scale.

3. Labour Productivity Growth Index

The explanation of measuring labour productivity growth based on a simple model of one output and two inputs can be generalized to allow for multiple outputs and multiple inputs. In this general model, the distance between the production plan and the production frontier can be measured by the output distance function as well as the profit function. We propose two theoretical indexes of labour productivity growth, both of which are defined as the ratio of the distance between a production plan and the short-run production frontier. Each index uses a distinct approach to measure its distance to the short-run production frontiers. One index adopts the primal approach and it is formulated by the output distance function. The other index adopts the dual approach and it is formulated by the profit function.

We discuss the labour productivity (LP) growth index of a firm between periods 0 and 1.[5] A firm is considered as a productive entity transforming inputs into outputs. We assume that there are M (net) outputs,[6] $\boldsymbol{y} = [y_1,\ldots, y_M]^T$ and $P + Q$ inputs consisting of

[5] Our theory can apply the comparisons of two distinct firms.

[6] Outputs include intermediate inputs. If output m is an intermediate input, then $y_m < 0$. Hence, the nominal value of (net) outputs $\boldsymbol{p}\cdot\boldsymbol{y}$ is the value-added that a firm generates.

P types of capital inputs $\boldsymbol{x}_K = [x_{K,1},\ldots,x_{K,P}]^{\mathrm{T}}$ and Q types of labour inputs $\boldsymbol{x}_L = [x_{L,1},\ldots, x_{L,Q}]^{\mathrm{T}}$. Outputs are sold at the positive producer prices $\boldsymbol{p} = [p_1,\ldots, p_M]^{\mathrm{T}}$, capital services are purchased at the positive rental prices $\boldsymbol{r} = [r_1,\ldots, r_P]^{\mathrm{T}}$, and labour inputs are purchased at the positive wage $\boldsymbol{w} = [w_1,\ldots, w_Q]^{\mathrm{T}}$. The period t production frontier is presented by the *period t input requirement function*, F^t, for $t = 0$ and 1:

(1) $x_{K,1} = F^t(\boldsymbol{y}, \boldsymbol{x}_{K,-1}, \boldsymbol{x}_L)$.

It represents the minimum amount of the first capital input that a firm can use at period t, producing the vector of output quantities $\boldsymbol{y}$, holding fixed other capital services $\boldsymbol{x}_{K-1} = [x_{K,2},\ldots,x_{K,P}]^{\mathrm{T}}$ and labour inputs $\boldsymbol{x}_L$.

Period t production possibility set, S^t, for $t = 0$ and 1 can be constructed by the period t input requirement function. It is a feasible set of inputs and outputs attainable from such inputs, defined as follows:

(2) $S^t \equiv \{(\boldsymbol{y}, \boldsymbol{x}_K, \boldsymbol{x}_L) : F^t(\boldsymbol{y}, \boldsymbol{x}_{K,-1}, \boldsymbol{x}_L) \geq x_{K,1}\}$.

We assume that S^t is a closed and convex set that exhibits a free disposal property. *Period t short-run production possibility set*, $S^t(\boldsymbol{x}_K^t)$, for $t = 0$ and 1 is a part of the period t production possibility set that is conditional on the vector of capital services $\boldsymbol{x}_K^t$. It consists of a set of $(\boldsymbol{y}, \boldsymbol{x}_L)$ such that $\boldsymbol{y}$ can be produced by using $\boldsymbol{x}_L$, holding constant the period t technology and capital services $\boldsymbol{x}_K^t$ as follows:

(3) $S^t(\boldsymbol{x}_K^t) \equiv \{(\boldsymbol{y}, \boldsymbol{x}_L) : F^t(\boldsymbol{y}, \boldsymbol{x}_{K,-1}^t, \boldsymbol{x}_L) \geq \boldsymbol{x}_{K,1}^t\}$.

The growth of labour productivity also can be formulated in terms of the short-run production possibility set. The expansion of the short-run production possibility set $S^0(\boldsymbol{x}_K^0) \subset S^1(\boldsymbol{x}_K^1)$ between periods 0 and 1 is equivalent to the improvement in labour productivity.[7][8] Hence, comparing the short-run production frontiers, we can recognize the extent to which labour productivity grows. Given $\boldsymbol{x}_K^*$, the short-run production frontier for the set $S^t(\boldsymbol{x}_K^*)$ is characterized by the input requirement function, $F^t(\mathbf{y}, \boldsymbol{x}_{K,-1}^*, \boldsymbol{x}_L) = x_{K,1}^*$.

3.1 Distance Function Approach

Caves, Christensen and Diewert (1982) introduced a theoretical index for TFP growth, using the output distance function.[9] Following them, we propose the theoretical LP growth index using the output distance function. The *period t output distance function* for $t = 0$ and 1 is defined as follows:

(4) $D^t(\boldsymbol{y}, \boldsymbol{x}_K, \boldsymbol{x}_L) \equiv \min_\delta \left\{\delta : F^t\left(\dfrac{\boldsymbol{y}}{\delta}, \boldsymbol{x}_{K,-1}, \boldsymbol{x}_L\right) \leq x_{K,1}\right\}$.

[7] Fare, Grosskopf, Norris and Zhang (1994) also suggests that the technical progress in the sense of TFP growth can be described by using the production possibility set such that $S^0 \subset S^1$.

[8] The problem associated with the use of average labour productivity in Figure 2 is that average labour productivity declines even though $S^0(\boldsymbol{x}_K^0) \subset S^1(\boldsymbol{x}_K^1)$.

[9] In this paper, we sometimes call it the distance function for simplicity.

It gives the minimum amount by which an output vector $\boldsymbol{y}$ can be deflated and still remain on the production frontier, with given input vectors $\boldsymbol{x}_K$ and $\boldsymbol{x}_L$. Thus, $D^t(\boldsymbol{y}, \boldsymbol{x}_K, \boldsymbol{x}_L)$ is considered to represent the distance between a production plan $(\boldsymbol{y}, \boldsymbol{x}_K, \boldsymbol{x}_L)$ and the period t production frontier in the direction of outputs $\boldsymbol{y}$. The short-run production frontier consists of $(\boldsymbol{y}, \boldsymbol{x}_L)$ such as $\boldsymbol{x}_L$ can produce $\boldsymbol{y}$, holding fixed the current technology and capital services. Hence, we can consider that $D^t(\boldsymbol{y}, \boldsymbol{x}_K^t, \boldsymbol{x}_L)$ represents the distance between a production plan $(\boldsymbol{y}, \boldsymbol{x}_L)$ and the period t short-run production frontier, in the direction of outputs $\boldsymbol{y}$. Comparing the distances between a production plan and the short-run production frontiers, we can measure the extent to that of short-run production frontier shifts. We define the LP growth index between periods 0 and 1 by the ratio between the distances from a production plan to the period 0 and 1 short-run production frontiers. We define a family of the LP growth index as follows:[10]

$$(5)\ LPG(\boldsymbol{x}_K^0, \boldsymbol{x}_K^1, \boldsymbol{y}, \boldsymbol{x}_L) \equiv \frac{D^0(\boldsymbol{y}, \boldsymbol{x}_K^0, \boldsymbol{x}_L)}{D^1(\boldsymbol{y}, \boldsymbol{x}_K^1, \boldsymbol{x}_L)}.$$

The distance function in the LP growth index is conditional on the reference production plan: vectors of outputs and labour inputs, $\boldsymbol{y}$ and $\boldsymbol{x}_L$. Thus, each choice of reference vectors $\boldsymbol{y}$ and $\boldsymbol{x}_L$ might generate a different measure of the shift in technology going from period 0 to period 1. We choose special reference vectors of outputs and capital services to specify for the labour productivity growth index defined by (5): a *Laspeyres type measure*, LPG_L that chooses the period 0 reference vectors of outputs and capital services $\boldsymbol{y}^0$ and $\boldsymbol{x}_L^0$ and a *Paasche type measure*, LPG_P that chooses the period 1 reference vectors of outputs and capital services $\boldsymbol{y}^1$ and $\boldsymbol{x}_L^1$.

$$(6)\ LPG_L \equiv LPG(\boldsymbol{x}_K^0, \boldsymbol{x}_K^1, \boldsymbol{y}^0, \boldsymbol{x}_L^0) = \frac{D^0(\boldsymbol{y}^0, \boldsymbol{x}_K^0, \boldsymbol{x}_L^0)}{D^1(\boldsymbol{y}^0, \boldsymbol{x}_K^1, \boldsymbol{x}_L^0)};$$

$$(7)\ LPG_P \equiv LPG(\boldsymbol{x}_K^0, \boldsymbol{x}_K^1, \boldsymbol{y}^1, \boldsymbol{x}_L^1) = \frac{D^0(\boldsymbol{y}^1, \boldsymbol{x}_K^0, \boldsymbol{x}_L^1)}{D^1(\boldsymbol{y}^1, \boldsymbol{x}_K^1, \boldsymbol{x}_L^1)}.$$

Since both measures of labour productivity growth are equally plausible, we treat the two measures symmetrically. We define the *Malmquist labour productivity growth index* as the geometric mean of the two indexes
(6) and (7);[11]

$$(8)\ LPG_M \equiv \sqrt{LPG_L \cdot LPG_P}.$$

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[10] Strictly speaking, $D^t(\boldsymbol{y}, \boldsymbol{x}_K^t, \boldsymbol{x}_L)$ is considered as the reciprocal of the distance between a production plan $(\boldsymbol{y}, \boldsymbol{x}_L)$ and the period t short-run production frontier. Thus, we need to compare $1/D^t(\boldsymbol{y}, \boldsymbol{x}_K^t, \boldsymbol{x}_L)$ between periods $t = 0$ and 1 so as to capture the extent to that the short-run production frontier shifts. Thus, the shift between periods $t = 0$ and 1 is defined as $D^0(\boldsymbol{y}, \boldsymbol{x}_K^0, \boldsymbol{x}_L)/D^1(\boldsymbol{y}, \boldsymbol{x}_K^1, \boldsymbol{x}_L)$ rather than $D^1(\boldsymbol{y}, \boldsymbol{x}_K^1, \boldsymbol{x}_L)/D^0(\boldsymbol{y}, \boldsymbol{x}_K^0, \boldsymbol{x}_L)$.

[11] Since the firm's profit maximization is assumed, it is possible to adopt a different formulation for the Malmquist LP growth index: $\mathrm{LPG_M}^1 = (D^0(\boldsymbol{y}^1, \boldsymbol{x}_K^0, \boldsymbol{x}_L^1)/D^0(\boldsymbol{y}^0, \boldsymbol{x}_K^0, \boldsymbol{x}_L^0))^{1/2}(D^1(\boldsymbol{y}^1, \boldsymbol{x}_K^1, \boldsymbol{x}_L^1)/ D^1(\boldsymbol{y}^0, \boldsymbol{x}_K^1, \boldsymbol{x}_L^0))^{1/2}$. This formulation is closer to the original Malmquist productivity (TFP growth) index.

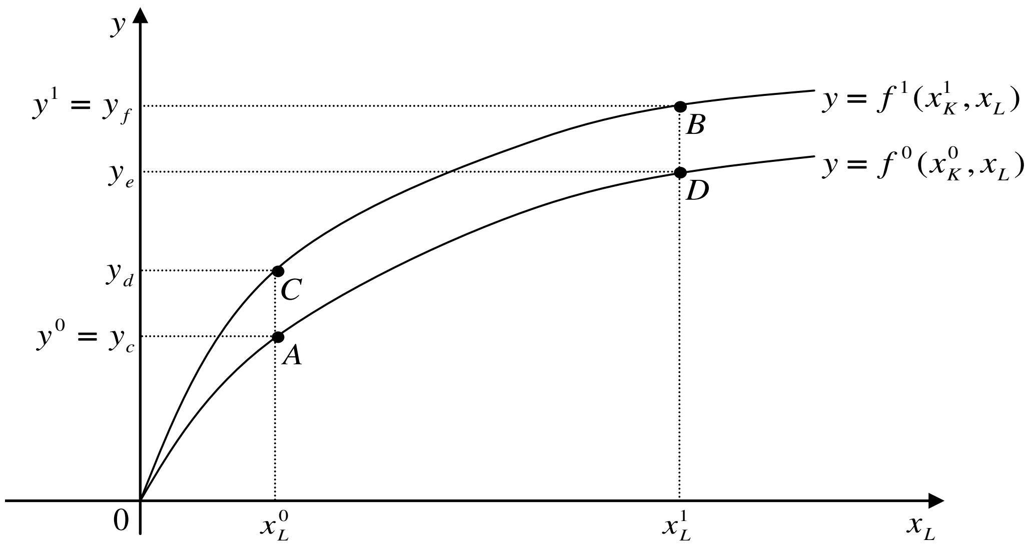

Figure 3: The Malmquist and Diewert-Morrison LP growth indexes

In the case of one output and two inputs, it is easy to give a graphical interpretation of the Malmquist LP growth index. It coincides with the following formula, as shown in Figure 3.

(9) $LPG_M = \sqrt{\left(\frac{y_d}{y_c}\right)\left(\frac{y_f}{y_e}\right)}$.

Given a quantity of labour input, the ratio of the output attainable from such a labour input at period 1 to the output attainable at period 0 represents the extent to which the short-run production function expands. LPG_M is interpreted as the geometric mean of the ratios conditional on the period 0 labour input and the period 1 labour input.

The Malmquist LP growth index is a theoretical index in the sense that it is defined as the ratio of the distance functions. At this point, it is not clear how we will obtain empirical estimates for the theoretical labour productivity growth indexes defined by (8). One obvious way is econometric approach. In this approach, we assumes a functional form for the distance function $D^t(\boldsymbol{y}, \boldsymbol{x}_K, \boldsymbol{x}_L)$, collect data on prices and quantities of outputs and inputs for a number of years, add error terms and use econometric techniques to estimate the unknown parameters in the assumed functional form. However, econometric techniques are generally not completely straightforward. Different econometricians will make different stochastic specifications and will choose different functional forms.[12] Moreover, as the number of outputs and inputs grows, it will be impossible to estimate a flexible functional form. Thus in the following section, we will suggest methods for estimating LP growth index (8) that are based on exact index number techniques.

Caves, Christensen and Diewert (1982) have shown that the first-order derivatives of the distance function D^t with respect to quantities at the period t actual production plan are computable from observable prices and quantities of inputs and outputs.

[12] "The estimation of GDP functions such as (19) can be controversial, however, since it raises issues such as estimation technique and stochastic specification. ... We therefore prefer to opt for a more straightforward index number approach." Kohli (2004).

They used these relationships to show that the Malmquist productivity index coincides with the Törnqvist productivity index, which is a formula of prices and quantities.[13] We use the same relationships to show that the Malmquist LP growth index coincides with a formula of observable prices and quantities. Although all these relationships have been already derived by Caves, Christensen and Diewert (1982), we outline how to compute the first-order derivatives of the distance functions below for completeness of discussion. The implicit function theorem is applied to the input requirement function $F^t(\boldsymbol{y}/\delta, \boldsymbol{x}_{K,-1}, \boldsymbol{x}_L) = x_{K,1}$ to solve for $\delta = D^t(\boldsymbol{y}, \boldsymbol{x}_K, \boldsymbol{x}_L)$ around $(\boldsymbol{y}^t, \boldsymbol{x}_K^t, \boldsymbol{x}_L^t)$.[14] In this case, $D^t(\boldsymbol{y}, \boldsymbol{x}_K, \boldsymbol{x}_L)$ is differentiable around the point $(\boldsymbol{y}^t, \boldsymbol{x}_K^t, \boldsymbol{x}_L^t)$. Its derivatives are represented by the derivatives of $F^t(\boldsymbol{y}, \boldsymbol{x}_K, \boldsymbol{x}_L)$. We have the following equations for periods $t = 0$ and 1:

$$(10)\ \nabla_y D^t(\boldsymbol{y}^t, \boldsymbol{x}_K^t, \boldsymbol{x}_L^t) = \frac{1}{\boldsymbol{y}^t \cdot \nabla_y F^t(\boldsymbol{y}^t, \boldsymbol{x}_K^t, \boldsymbol{x}_L^t)} \nabla_y F^t(\boldsymbol{y}^t, \boldsymbol{x}_K^t, \boldsymbol{x}_L^t)\,;$$

$$(11)\ \nabla_{x_K} D^t(\boldsymbol{y}^t, \boldsymbol{x}_K^t, \boldsymbol{x}_L^t) = \frac{1}{\boldsymbol{y}^t \cdot \nabla_y F^t(\boldsymbol{y}^t, \boldsymbol{x}_{K,-1}^t, \boldsymbol{x}_L^t)} \begin{bmatrix} -1 \\ \nabla_{x_{K,-1}} F^t(\boldsymbol{y}^t, \boldsymbol{x}_{K,-1}^t, \boldsymbol{x}_L^t \end{bmatrix};$$

$$(12)\ \nabla_{x_L} D^t(\boldsymbol{y}^t, \boldsymbol{x}_K^t, \boldsymbol{x}_L^t) = \frac{1}{\boldsymbol{y}^t \cdot \nabla_y F^t(\boldsymbol{y}^t, \boldsymbol{x}_K^t, \boldsymbol{x}_L^t)} \nabla_{x_L} F^t(\boldsymbol{y}^t, \boldsymbol{x}_K^t, \boldsymbol{x}_L^t)\,.$$

We assume that $(\boldsymbol{y}^t, \boldsymbol{x}_K,^t \boldsymbol{x}_L^t) >> 0_{N+P+Q}$ is a solution to the following period t profit maximization problem for $t = 0$ and 1:

$$(13)\ \max\{\, \boldsymbol{p}^t \cdot \boldsymbol{y} - r_1^t F^t(\boldsymbol{y}, \boldsymbol{x}_{K,-1}, \boldsymbol{x}_L) - \boldsymbol{r}_{-1}^t \cdot \boldsymbol{x}_{K,-1} - \boldsymbol{w}^t \cdot \boldsymbol{x}_L \}\,.$$

The period t profit maximization problem yields the following first order conditions for $t = 0$ and 1:

$$(14)\ \boldsymbol{p}^t = r_1^t \nabla_y F^t(\boldsymbol{y}^t, \boldsymbol{x}_{K,-1}^t, \boldsymbol{x}_L^t)\,;$$

$$(15)\ \boldsymbol{r}_{-1}{}^t = -r_1^t \nabla_{x_{K,-1}} F^s(\boldsymbol{y}^t, \boldsymbol{x}_{K,-1}^t, \boldsymbol{x}_L^t)\,;$$

$$(16)\ \boldsymbol{w}^t = -r_1^t \nabla_{x_L} F^s(\boldsymbol{y}^t, \boldsymbol{x}_{K,-1}^t, \boldsymbol{x}_L^t)\,.$$

By substituting (14), (15) and (16) into (10), (11) and (12), we obtain the following equations for t = 0 and 1:

$$(17)\ \nabla_y D^t(\boldsymbol{y}^t, \boldsymbol{x}_L^t, \boldsymbol{x}_L^t) = \boldsymbol{p}^t / \boldsymbol{p}^t \cdot \boldsymbol{y}^t\,;$$

$$(18)\ \nabla_{x_K} D^t(\boldsymbol{y}^t, \boldsymbol{x}_K^t, \boldsymbol{x}_L^t) = [r_1^t /(\boldsymbol{p}^t \cdot \boldsymbol{y}^t)] \begin{bmatrix} -1 \\ \nabla_{x_{K,-1}} F^t(\boldsymbol{y}^t, \boldsymbol{x}_{K,-1}^t, \boldsymbol{x}_L^t) \end{bmatrix} = -\boldsymbol{r}^t / \boldsymbol{p}^t \cdot \boldsymbol{y}^t\,;$$

$$(19)\ \nabla_{x_L} D^t(\boldsymbol{y}^t, \boldsymbol{x}_L^t, \boldsymbol{x}_L^t) = \boldsymbol{w}^t / \boldsymbol{p}^t \cdot \boldsymbol{y}\,.$$

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[13] The Malmquist productivity index is TFP growth index.

[14] We assume the following three conditions are satisfied for $t = 0$ and 1: 1) F^t is differentiable at the point $(\boldsymbol{y}^t, \boldsymbol{x}_K^t, \boldsymbol{x}_L^t)$,: 2) $\boldsymbol{y}^t >> 0_M$ and 3) $\boldsymbol{y}^t \cdot \nabla_y F^t(\boldsymbol{y}^t, \boldsymbol{x}_K^t, \boldsymbol{x}_L^t) > 0$.

The above equations allow us to compute the derivatives of the distance function, without knowing the distance function itself. They will be useful to implement the theoretical LP growth index. However, one disadvantage is that the derivatives of the period t distance function need to be evaluated at the period t actual production plan in equations (17), (18) and (19) for t = 0 and 1. LPG_L and LPG_P are represented by the distance function with the hypothetical production plan such that ($\boldsymbol{y}^1, \boldsymbol{x}_K{}^0, \boldsymbol{x}_L{}^1$) and ($\boldsymbol{y}^0$, $\boldsymbol{x}_K{}^1$, $\boldsymbol{x}_L{}^0$). Hence, the above equations cannot be directly applied to calculate the theoretical productivity index. In addition to the firm's profit maximization, we further assume a following translog functional form for the period t distance function.

$$
\begin{aligned}
\ln h^t(\boldsymbol{y}, \boldsymbol{x}_K, \boldsymbol{x}_L) \equiv\ & a_0{}^t + \sum_{m=1}^{M} a_m{}^t \ln y_m + (1/2)\sum_{i=1}^{M}\sum_{j=1}^{M} a_{i,j} \ln y_i \ln y_j \\
& + \sum_{p=1}^{P} b_p{}^t \ln x_{K,p} + (1/2)\sum_{i=1}^{P}\sum_{j=1}^{P} b_{i,j} \ln x_{K,i} \ln x_{K,j} \\
& + \sum_{q=1}^{Q} c_q{}^t \ln x_{L,q} + (1/2)\sum_{i=1}^{Q}\sum_{j=1}^{Q} c_{i,j} \ln x_{L,i} \ln x_{L,j} \qquad (20) \\
& + \sum_{m=1}^{M}\sum_{p=1}^{P} d_{m,p} \ln y_m \ln x_{K,p} + \sum_{m=1}^{M}\sum_{q=1}^{Q} e_{m,q} \ln y_m \ln x_{L,q} \\
& + \sum_{p=1}^{P}\sum_{q=1}^{Q} f_{p,q} \ln x_{K,p} \ln x_{L,q}
\end{aligned}
$$

The translog functional form is a flexible functional form so that it can approximate an arbitrary twice continuously differentiable function to the second order at an arbitrary point. Note that the coefficients for the quadratic terms are assumed to be constant over time. There are enough parameters so that we can choose them in order for h^t to satisfy the linear homogeneity properties with respect to output quantity vector $\boldsymbol{y}$:[15]

Proposition 1:
Assume that the distance functions D^0 and D^1 have the translog functional form defined by (20) and there is competitive profit maximizing behaviour in each period. Then, the *Malmquist labour productivity growth index*, LPG_M, can be computed from observable prices and quantities as follows:

$$(21)\ \ln LPG_M = \sum_{m=1}^{M} s_m \ln\left(\frac{y_m^1}{y_m^0}\right) - \sum_{q=1}^{Q} s_{L,q} \ln\left(\frac{x_{L,q}^1}{x_{L,q}^0}\right)$$

where s_m and $s_{L,q}$ are the average value-added shares of output m and labour input q between periods 0 and 1 such that;

$$s_m = \frac{1}{2}\left(\frac{p_m^0 y_m^0}{\boldsymbol{p}^0 \cdot \boldsymbol{y}^0} + \frac{p_m^1 y_m^1}{\boldsymbol{p}^1 \cdot \boldsymbol{y}^1}\right) \text{ and } s_{L,q} = \frac{1}{2}\left(\frac{w_q^0 x_{L,q}^0}{\boldsymbol{p}^0 \cdot \boldsymbol{y}^0} + \frac{w_q^1 x_{L,q}^1}{\boldsymbol{p}^1 \cdot \boldsymbol{y}^1}\right).$$

The index number formula in (21) can be interpreted as the ratio of a volume measure of outputs to a volume measure of labour input. Note that no data of price and quantity of capital inputs appear in this formula. It is found that the shift in the short-run production frontier can be calculated, independent of the information of capital services.

[15] We can choose coefficients satisfying the following restrictions; $a_{i,j} = a_{j,i}$ for all i and j ; $b_{i,j} = b_{j,i}$ for all i and j; $c_{i,j} = c_{j,i}$ for i and j; $\sum_{n=1}{}^{N} a_n{}^t = 1$ for t = 0,1,2, ...; $\sum_{i=1}{}^{M} a_{i,m} = 0$ for $m = 1,\ldots,M$; $\sum_{m=1}{}^{M} d_{m,p} = 0$ for $p = 1,\ldots,P$; $\sum_{m=1}{}^{M} e_{m,q} = 0$ for $q = 1,\ldots,Q$.

3.2 Profit Function Approach

The research on productivity measurement based on the restricted profit function goes back to Diewert and Morrison (1986).[16] Given prices of outputs and quantities of primary inputs, the change in the profit can be attributed to productivity changes. Diewert and Morrison (1986) define the theoretical TFP growth index as a ratio of the profit function between two periods, given output prices and primary input quantities. In the end, it is shown that their theoretical TFP growth index coincides with the implicit Törnqvist productivity index.[17]

Given an output price vector $\boldsymbol{p}$ and input quantity vectors $\boldsymbol{x}_K$ and $\boldsymbol{x}_L$, we define the *period t restricted profit function*, $g^t(\boldsymbol{p}, \mathbf{x}_{\mathrm{K}}, \boldsymbol{x}_L)$ for $t = 0$ and 1, as follows:

$$(22)\ g^t(\boldsymbol{p},\boldsymbol{x}_K,\boldsymbol{x}_L) \equiv \max_{y,x_L}\{\boldsymbol{p}\cdot\boldsymbol{y} : F^t(\boldsymbol{y},\boldsymbol{x}_{K,-1},\boldsymbol{x}_L) \le x_{K,1}\}.$$

Thus, the profit of the firm depends on the period t technology and the output price vector $\boldsymbol{p}$ and input quantity vectors $\boldsymbol{x}_K$ and $\boldsymbol{x}_L$.

If $\boldsymbol{p}^t$ is the period t output price vector and $\boldsymbol{x}_K{}^t$ and $\boldsymbol{x}_L{}^t$ are the vectors of factor inputs used during period t, and if the profit function is differentiable with respect to the components of $\boldsymbol{p}$ and $\boldsymbol{w}$ at the point $(\boldsymbol{p}^t, \boldsymbol{w}_L{}^t, \boldsymbol{x}_K{}^t)$, then the period t vector of the firm's net outputs $\boldsymbol{y}^t$ and capital inputs $\boldsymbol{x}_L{}^t$ will be equal to the vector of first order partial derivatives of $g^t(\boldsymbol{p}^t, \boldsymbol{w}_L{}^t, \boldsymbol{x}_K{}^t)$ with respect to the components of $\boldsymbol{p}$ and $\boldsymbol{w}$. We will have the following equation for periods $t = 0$ and 1:[18]

$$(23)\ \boldsymbol{y}^t = \nabla_{\boldsymbol{P}} g^t(\boldsymbol{p}^t,\boldsymbol{x}_K^t,\boldsymbol{x}_L^t).$$

If the restricted profit function is differentiable with respect to the quantities of capital inputs $\boldsymbol{x}_K$ at the point $(\boldsymbol{p}^t, \boldsymbol{w}_L{}^t, \boldsymbol{x}_K{}^t)$, then the period t vector of input prices $\boldsymbol{r}^t$ will be equal to the vector of first order partial derivatives of $g^t(\boldsymbol{p}^t, \boldsymbol{w}_L{}^t, \boldsymbol{x}_K{}^t)$ with respect to the components of the quantities of capital services $\boldsymbol{x}_K$. We will have the following equations for periods $t = 0$ and 1:[19]

$$(24)\ \boldsymbol{r}^t = \nabla_{x_K} g^t(\boldsymbol{p}^t,\boldsymbol{x}_K^t,\boldsymbol{x}_L^t);$$
$$(25)\ \boldsymbol{w}^t = \nabla_{x_L} g^t(\boldsymbol{p}^t,\boldsymbol{x}_K^t,\boldsymbol{x}_L^t).$$

The above equations allow us to compute the derivatives of the restricted profit function without knowing the profit function itself. They will be useful to implement the theoretical LP growth index.

We maintain the idea that the LP growth index should reflect the shift in the short-run production frontier. In the dual representation, the shift in the restricted profit function reflects the shift in the short-run production frontier. Thus, we define the LP

[16] In this paper, we sometimes call it the profit function for simplicity.
[17] It equals the implicit Törnqvist output quantity divided by the Törnqvist input quantity index.
[18] These relationships are due to Hotelling (1932).
[19] These relationships are due to Samuelson (1953) and Diewert (1974).

growth index by the growth rate of the restricted profit function caused by technological progress and the increase in capital services. We define a family of the labour productivity growth index as follows:

$$(26)\ LPG(\boldsymbol{x}_K^1, \boldsymbol{x}_K^0, \boldsymbol{p}, \boldsymbol{x}_L) \equiv \frac{g^1(\boldsymbol{p}, \boldsymbol{x}_K^1, \boldsymbol{x}_L)}{g^0(\boldsymbol{p}, \boldsymbol{x}_K^0, \boldsymbol{x}_L)}.$$

The restricted profit function in the productivity index is conditional on the output price vector $\boldsymbol{p}$ and the vector of labour input $\boldsymbol{x}_L$. Thus, each choice of reference output price vector $\boldsymbol{p}$ and reference vector of labour input $\boldsymbol{x}_L$ will generate a possibly different measure of the shift in technology from period 0 to period 1. We choose special reference output price vector $\boldsymbol{p}$ and special reference vector of labour input $\boldsymbol{x}_L$ for the labour productivity growth index defined by (26): a *Laspeyres type measure*, LPG_L that chooses the period 0 reference output price vector $\boldsymbol{p}^0$ and the period 0 reference vector of labour input $\boldsymbol{x}_L{}^0$ and a *Paasche type measure*, LPG_P that chooses the period 1 reference output price vector $\boldsymbol{p}^1$ and the period 1 reference vector of labour input $\boldsymbol{x}_L{}^1$:

$$(27)\ LPG_L \equiv LPG(\boldsymbol{x}_K^1, \boldsymbol{x}_K^0, \boldsymbol{p}^0, \boldsymbol{x}_L^0) = \frac{g^1(\boldsymbol{p}^0, \boldsymbol{x}_K^1, \boldsymbol{x}_L^0)}{g^0(\boldsymbol{p}^0, \boldsymbol{x}_K^0, \boldsymbol{x}_L^0)};$$

$$(28)\ LPG_P \equiv LPG(\boldsymbol{x}_K^1, \boldsymbol{x}_K^0, \boldsymbol{p}^1, \boldsymbol{x}_L^1) = \frac{g^1(\boldsymbol{p}^1, \boldsymbol{x}_K^1, \boldsymbol{x}_L^1)}{g^0(\boldsymbol{p}^1, \boldsymbol{x}_K^0, \boldsymbol{x}_L^1)}.$$

Since both measures of technical progress are equally valid, it is natural to average them to obtain an overall measure of labour productivity growth. If we want to treat the two measures in a symmetric manner and we want the measure to satisfy the time reversal property from the difference approach to index number theory (so that the estimate going backwards is equal to the negative of the estimate going forwards), then the arithmetic mean will be the best simple average to take. Thus, we define the *Diewert-Morrison labour productivity growth index* (hereafter, Diewert-Morrison LP growth index) by the arithmetic mean of (27) and (28) as follows:

$$(29)\ LPG_{DM} \equiv \sqrt{LPG_L \cdot LPG_P}\ .$$

The Diewert-Morrison LP growth index can be also illustrated graphically in Figure 3. It coincides with the Malmquist LP growth index in this simple model of one output and two inputs.

$$(30)\ LPG_{DM} = \sqrt{\left(\frac{p^0 y_d}{p^0 y_c}\right)\left(\frac{p^1 y_f}{p^1 y_e}\right)} = \sqrt{\left(\frac{y_d}{y_c}\right)\left(\frac{y_f}{y_e}\right)}.$$

The Diewert-Morrison LP growth index is a theoretical index in the sense that it is defined by the restricted profit function. LPG_L and LPG_P are represented by the restricted profit function with the hypothetical production plan such that ($\boldsymbol{p}^1, \boldsymbol{x}_K{}^0, \boldsymbol{x}_L{}^1$) and ($\boldsymbol{p}^0, \boldsymbol{x}_K{}^1, \boldsymbol{x}_L{}^0$). Hence, the equations (23)(24)(25) cannot be directly applied to calculate the theoretical productivity index. In addition to the firm's profit maximization, we further assume a following translog functional form for the period t restricted profit function.

$$
\begin{aligned}
\text{(31)} \quad \ln H^t(\boldsymbol{p},\boldsymbol{x}_K,\boldsymbol{x}_L) \equiv\ & a_0{}^t + \sum_{m=1}^{M} a_m{}^t \ln p_m + (1/2)\sum_{i=1}^{M}\sum_{j=1}^{M} a_{i,j}\ln p_i \ln p_j \\
& + \sum_{p=1}^{P} b_p{}^t \ln x_{K,p} + (1/2)\sum_{i=1}^{P}\sum_{j=1}^{P} b_{i,j}\ln x_{K,i}\ln x_{K,j} \\
& + \sum_{q=1}^{Q} c_q{}^t \ln x_{L,q} + (1/2)\sum_{i=1}^{Q}\sum_{j=1}^{Q} c_{i,j}\ln x_{L,i}\ln x_{L,j} \\
& + \sum_{m=1}^{M}\sum_{p=1}^{P} d_{m,p}\ln p_m \ln x_{K,p} + \sum_{m=1}^{M}\sum_{q=1}^{Q} e_{m,q}\ln p_m \ln x_{L,q} \\
& + \sum_{p=1}^{P}\sum_{q=1}^{Q} f_{p,q}\ln x_{K,p}\ln x_{L,q}
\end{aligned}
$$

The translog functional form is a flexible functional form so that it can approximate an arbitrary twice continuously differentiable function to the second order at an arbitrary point. Note that the coefficients for the quadratic terms are assumed to be constant over time. There are enough parameters so that we can choose them in order for H^t to satisfy the linear homogeneity properties with respect to output price vector $\boldsymbol{p}$:[20]

Proposition 2:
Assume that the profit functions g^0 and g^1 have the translog functional form defined by (31).[21] Then, the *Diewert-Morrison labour productivity growth index, LPG_{DM}*, can be computed from observable prices and quantities as follows:

$$\text{(32)} \quad \ln LPG_{DM} = \ln\left(\frac{\boldsymbol{p}^1\cdot\boldsymbol{y}^1}{\boldsymbol{p}^0\cdot\boldsymbol{y}^0}\right) - \sum_{m=1}^{M} s_m \ln\left(\frac{p_m^1}{p_m^0}\right) - \sum_{q=1}^{Q} s_{L,q}\ln\left(\frac{x_{L,q}^1}{x_{L,q}^0}\right)$$

where s_m and $s_{L,q}$ are the average value-added shares of output m and labour input q between periods 0 and 1 such that;

$$s_m = \frac{1}{2}\left(\frac{p_m^0 y_m^0}{\boldsymbol{p}^0\cdot\boldsymbol{y}^0} + \frac{p_m^1 y_m^1}{\boldsymbol{p}^1\cdot\boldsymbol{y}^1}\right) \text{ and } s_{L,q} = \frac{1}{2}\left(\frac{w_q^0 x_{L,q}^0}{\boldsymbol{p}^0\cdot\boldsymbol{y}^0} + \frac{w_q^1 x_{L,q}^1}{\boldsymbol{p}^1\cdot\boldsymbol{y}^1}\right).$$

It turns out that both labour productivity growth indexes based on the distance function and the profit function coincide with the almost identical index number formula. Both are interpreted as the ratio of a quantity index of output to a quantity index of labour input. For labour inputs, both use the same quantity index for labour inputs. On the other hand, there is a difference in the output quantity index. While the Malmquist labour productivity index uses the Törnqvist quantity index, the Diewert-Morrison labour productivity index uses the implicit Törnqvist quantity index. However, the Törnqvist and the implicit Törnqvist quantity indexes are superlative indexes, which are immune from the substitution bias associated with the Laspeyres and Paasche indexes. Since it is known that the difference between superlative

[20] We can choose coefficients satisfying the following restrictions; $a_{i,j} = a_{j,i}$ for all i and j ; $b_{i,j} = b_{j,i}$ for all i and j; $c_{i,j} = c_{j,i}$ for i and j; $\sum_{n=1}^{N} a_n{}^t = 1$ for t = 0,1,2, ...; $\sum_{i=1}^{M} a_{i,m} = 0$ for $m = 1,\ldots,M$; $\sum_{m=1}^{M} d_{m,p} = 0$ for $p = 1,\ldots,P$; $\sum_{m=1}^{M} e_{m,q} = 0$ for $q = 1,\ldots,Q$.

[21] For the case of the restricted profit function, the linear homogeneity with respect to output price vector $\boldsymbol{p}$ is satisfied.

indexes is minor (Diewert, 1978), the difference between the Malmquist and Diewert-Morrison LP growth indexes is negligible.[22]

3.3 Comparison with Average Labour Productivity Growth

We compare two new indexes of labour productivity growth, LPG_M and LPG_{DM}, with the standard productivity measure of average labour productivity. We treat the growth in average labour productivity as an index of labour productivity growth and call it the average labour productivity (LP) growth index, which is denoted by *ALPG*. By definition, the average LP growth index equals the ratio of the growth rate of output quantity to the growth rate of labour input quantity. Given multiple outputs and labour inputs, it is necessary to use the quantity index to aggregate the growth of multiple outputs and labour inputs. We consider two types of the average LP growth indexes, denoted by $ALPG_T$ and $ALPG_{ImT}$. Both apply the Törnqvist quantity index to aggregating the growth rates of labour inputs. However, for aggregating changes of outputs, one index, $ALPG_T$, uses the Törnqvist quantity index, and the other index, $ALPG_{ImT}$, uses the implicit Törnqvist quantity index.

$$(33)\ \ln ALPG_T = \sum_{m=1}^{M} s_m \ln\left(\frac{y_m^1}{y_m^0}\right) - \sum_{q=1}^{Q} \bar{s}_{L,q} \ln\left(\frac{x_{L,q}^1}{x_{L,q}^0}\right);$$

$$(34)\ \ln ALPG_{ImT} = \ln\left(\frac{\boldsymbol{p}^1\cdot\boldsymbol{y}^1}{\boldsymbol{p}^0\cdot\boldsymbol{y}^0}\right) - \sum_{m=1}^{M} s_m \ln\left(\frac{p_m^1}{p_m^0}\right) - \sum_{q=1}^{Q} \bar{s}_{L,q} \ln\left(\frac{x_{L,q}^1}{x_{L,q}^0}\right);$$

where s_m is the average value-added shares of output m and $\bar{s}_{L,q}$ is the average labour-compensation share of labour input q between periods 0 and 1 such that;

$$s_m = \frac{1}{2}\left(\frac{p_m^0 y_m^0}{\boldsymbol{p}^0\cdot\boldsymbol{y}^0} + \frac{p_m^1 y_m^1}{\boldsymbol{p}^1\cdot\boldsymbol{y}^1}\right) \text{ and } \bar{s}_{L,q} = \frac{1}{2}\left(\frac{w_q^0 x_{L,q}^0}{\boldsymbol{w}^0\cdot\boldsymbol{x}_L^0} + \frac{w_q^1 x_{L,q}^1}{\boldsymbol{w}^1\cdot\boldsymbol{x}_L^1}\right).$$

While $ALPG_T$ corresponds to the Malmquist LP growth index, LPG_M, using the Törnqvist quantity index for aggregating the quantity growth of outputs, the $ALPG_{ImT}$ corresponds to the Diewert-Morrison LP growth index, LPG_{DM}, using the implicit Törnqvist quantity index for aggregating the quantity growth of outputs. As we discussed for the difference between Malmquist and Diewert-Morrison LP growth indexes, the difference between two average LP growth indexes, $ALPG_T$ and $ALPG_{ImT}$ is negligible.

$$(35)\quad \begin{aligned} LPG_M - ALPG_T &= LPG_{DM} - ALPG_{\mathrm{Im}T} \\ &= \sum_{q=1}^{Q}\left(\left(\frac{\boldsymbol{r}^0\cdot\boldsymbol{x}_K^0}{\boldsymbol{p}^0\cdot\boldsymbol{y}^0}\right)\frac{w_q^0 x_{L,q}^0}{\boldsymbol{w}^0\cdot\boldsymbol{x}_L^0} + \left(\frac{\boldsymbol{r}^1\cdot\boldsymbol{x}_K^1}{\boldsymbol{p}^1\cdot\boldsymbol{y}^1}\right)\frac{w_q^1 x_{L,q}^1}{\boldsymbol{w}^1\cdot\boldsymbol{x}_L^1}\right)\ln\left(\frac{x_{L,q}^1}{x_{L,q}^0}\right) \end{aligned}$$

The difference between the average LP growth indexes ($ALPG_T$ and $ALPG_{ImT}$) and the new LP growth indexes (Malmquist and Diewert-Morrison LP growth indexes, LPG_M and LPG_{DM}) comes from the weight attached to the growth of labour inputs. While $ALPG_T$ and $ALPG_{ImT}$ weight different types of labour inputs with their shares in total labour compensation, LPG_M and LPG_{DM} weight different types of labour inputs with

[22] The Fisher quantity index is another superlative index. See Diewert (1976).

their shares in value-added. This is reflected in the differences between LPG_M and $ALPG_T$ or between LPG_M and $ALPG_T$, as shown in equation (35). Since $ALPG_T$ and $ALPG_{ImT}$ gives more weight to each type of labour input than LPG_M and LPG_{DM}, $ALPG_T$ and $ALPG_{ImT}$ are larger than LPG_M and LPG_{DM}, as long as the quantities of labour inputs grow. The larger capital share, the larger the difference between them becomes.

As we pointed out in the previous section, the difference between the average LP growth index and the Malmquist LP growth index is attributed to the effect of scale economies. Equation (35) shows when its effect is enhanced. Under the concave production frontier, the more labour input increases, the more average labour productivity declines. It is possible to interpret that the larger share of capital inputs indicates the flatter slope of the short-run production frontier.[23] Along the short-run production frontier that has a flatter slope, the impact of the change in labour input will be strengthened.

4. Aggregation over Industries

We discuss a good aggregation property which the Malmquist and Diewert-Morrison LP growth indexes satisfy. Aggregation of the LP growth indexes is necessary for many cases. The aggregation property of the LP growth indexes, which we will discuss below, holds for any type of aggregation problem. However, we restrict our discussion to the aggregation over industries in particular for simplicity.

We have followed discrete time approach to the productivity measurement up to the previous section. In this approach, the price and quantity data are defined only for integer values of t, which denote discrete unit time periods. There is another approach called the Divisia approach. In this approach, the price and quantity data are defined as functions of continuous time.[24] Thus, the logarithm of the ratio of some variable between period 0 and 1 is replaced by the time derivative of that variable. The average share of revenue from each output in total value-added $(1/2)(p_m^0 y_m^0/\boldsymbol{p}^0\cdot\boldsymbol{y}^0 + p_m^1 y_m^1/\boldsymbol{p}^1\cdot\boldsymbol{y}^1)$ is now identically specified by $p_m y_m/\boldsymbol{p}\cdot\boldsymbol{y}$. We apply the Divisia approach to the two LP growth indexes, LPG_M and LPG_{DM}, and discuss their aggregation properties.[25]

There are J types of industries. For each industry, $\boldsymbol{y}^j$ is output vector for an industry j, $\boldsymbol{x}_K^j$ and $\boldsymbol{x}_L^j$ are input quantity vectors of capital services and labour inputs for an industry j such as $\boldsymbol{y}^j \equiv [y_1^j,\ldots, y_M^j]^T$, $\boldsymbol{x}_K^j \equiv [x_{K,1}^j,\ldots, x_{K,P}^j]^T$ and $\boldsymbol{x}_L^j \equiv [x_{L,1}^j,\ldots, x_{L,Q}^j]^T$ where y_m^j is quantity of output m produced by an industry j, $x_{K,p}^j$ and $x_{L,q}^j$ are quantities of capital service p and labour input q utilized by an industry j. The quantities of output m produced by each industry sum up to the aggregate quantity of output m, y_m and similarly, the quantities of capital service p used by each industry sum up to the aggregate quantity of capital service p and the quantity of labour input q

[23] If we assume a Cobb-Douglas production function, the slope of the short-run production function is a function of the share of capital income to total value added. The large share of capital income makes the slope flatter, holding fixed capital services and labour inputs.
[24] The Divisia approach is coined by Diewert and Nakamura (2007). See Hulten (1973) and Balk (2000) for detailed in Divisia approach.
[25] In the Divisia approach, since there is no difference between the Törnqvist quantity index and the implicit Törnqvist quantity index, LPG_M and LPG_{DM} are the same. Thus, although we only discuss LPG_M, the aggregation property we discuss here is also shared by LPG_{DM}.

used by each industry sum up to the aggregate quantity of labour input q such that $y_m = \sum_{j=1}^{J} y_m^j$, $x_{K,p} = \sum_{j=1}^{J} x_{K,p}^j$ and $x_{L,q}^j = \sum_{j=1}^{J} x_{L,q}^j$. We assume that the prices of the same output and the same input are constant across industries. Applying equation (34), we can calculate the economy-wide Malmquist LP growth index $LPG_{M,T}$ as well as the industry j Malmquist LP growth index $LPG_{M,j}$ as follows:

$$LPG_{M,T} = \sum_{j=1}^{J} \sum_{m=1}^{M} \left(\frac{p_m^j y_m^j}{\sum_{j=1}^{J} \boldsymbol{p}^j \cdot \boldsymbol{y}^j} \right) \left(1 + \frac{\dot{y}_m^j}{y_m^j} \right) \tag{36}$$
$$- \sum_{j=1}^{J} \sum_{q=1}^{Q} \left(\frac{w_q^j x_{L,q}^j}{\sum_{j=1}^{J} \boldsymbol{p}^j \cdot \boldsymbol{y}^j} \right) \left(1 + \frac{\dot{x}_{K,q}^j}{x_{K,q}^j} \right)$$

$$LPG_{M,j} = \sum_{m=1}^{M} \left(\frac{p_m^j y_m^j}{\boldsymbol{p} \cdot \boldsymbol{y}} \right) \left(1 + \frac{\dot{y}_m^j}{y_m^j} \right) - \sum_{q=1}^{Q} \left(\frac{w_q^j x_{L,q}^j}{\boldsymbol{p} \cdot \boldsymbol{y}} \right) \left(1 + \frac{\dot{x}_{K,q}^j}{x_{K,q}^j} \right). \tag{37}$$

From equations (36) and (37), we can derive the following relationship between the economy-wide LP growth and the industry LP growth indexes:

$$LPG_{M,T} = \sum_{j=1}^{J} \left(\frac{\boldsymbol{p}^j \cdot \boldsymbol{y}^j}{\sum_{j=1}^{J} \boldsymbol{p}^j \cdot \boldsymbol{y}^j} \right) LPG_{M,j}\,. \tag{38}$$

Equation shows that the economy-wide LP growth index is the average of the industry LP growth index, weighted by the industry's value-added share. Thus, the use of these two LP growth indexes, LPG_M or LPG_{DM}, enables us to precisely identify the contribution of each industry to the economy-wide LP growth. It enables us to investigate the industry origins of the economy-wide LP growth.

5. An Application to the U.S. Industry Data

We apply the labour productivity growth indexes to investigate the industry productivity performance of the U.S. for the period 1970-2005. The U.S. industry data is taken from the comprehensive industry dataset called the EU KLEMS Growth and Productivity Accounts.[26] Industry accounts that we used consist of gross outputs and intermediate inputs at current and constant prices, and hours worked of employment by 30 industries.[27] These industry data are organized according to the System of Industry Classification (SIC) adopted by the U.S. official statistics.

[26] Data are downloaded from the EU KLEMS website (http://www.euklems.net/). The detailed explanation about this comprehensive international database is found in O'Mahony and Timmer (2009). The U.S. industry data of EU KLEMS is constructed by Dale Jorgenson and his research group. See Jorgenson, Ho and Stiroh (2008).

[27] For each industry, there exist one type of gross output and one type of intermediate input. Their deflator varies across industries. Labour input is hours worked by total employment. Total employment in each industry includes employees and the self-employed engaged in the production of the industry.

Figure 4 compares the economy-wide average and Malmquist LP growth indexes for the entire sample period 1970-2005. Since the Diewert-Morrison LP growth index is almost identical to the Malmquist LP growth index, we exclude the former index from the figure.[28] Equation (35) shows that the difference between the average LP growth index and the Malmquist LP growth index, which can be attributed to the effect of scale economies, depends on the growth rate of hours worked and the nominal share of capital input in total value added. Total hours worked for the U.S. was stagnated in some years but it was on the overall upward trend.[29] The widening gap between the average LP growth index and the Malmquist LP growth index reflects this increasing trend of hours worked. The economy-wide capital share also increased over years from 33.0 per cent in 1970 to 39.7 per cent in 2005.

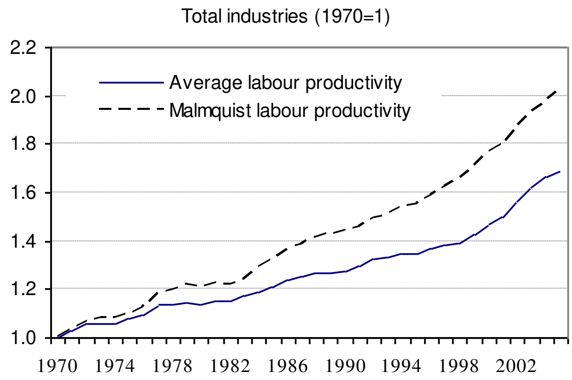

Figure 4: Economy-wide Average and Malmquist LP growth indexes, 1970-2005

We investigate the industry origins of the economy-wide labour productivity growth. The entire sample period 1970-2005 can be usefully divided into three periods: 1970-1995, 1995-2000 and 2000-2005.[30] Industry productivity performance has been stagnated since the early 1970s. 1995 is the watershed year when the productivity of the U.S. industries revived again. Productivity growth even accelerated in 2000s.

Table 1: Annual Average: Average and Malmquist LP Growth Indexes

	Average	Malmquist	Average	Malmquist	Average	Malmquist	Average	Malmquist
	1970-2005		1970-1995		1995-2000		2000-2005	
Total industries	1.5%	2.0%	1.2%	1.8%	1.8%	2.7%	2.8%	2.7%
Electrical and optical equipment	11.0%	10.9%	9.2%	9.3%	20.0%	20.3%	11.4%	9.8%
Other manufacturing	2.4%	2.2%	1.8%	1.8%	3.2%	3.2%	4.7%	3.2%
Other production	-0.5%	0.1%	-0.5%	0.0%	-0.2%	1.2%	-1.1%	-0.6%
Post and communication	5.1%	5.4%	4.8%	5.1%	-0.3%	2.5%	12.1%	9.5%
Other market services	1.7%	2.3%	1.4%	2.0%	1.7%	2.6%	3.1%	3.1%
Distribution	2.7%	3.1%	2.5%	2.8%	2.5%	3.1%	4.4%	4.2%
Finance and business, except real estate	0.6%	1.7%	0.2%	1.3%	0.8%	2.4%	2.8%	2.7%
Personal services	0.1%	0.5%	-0.3%	0.2%	-0.1%	0.4%	1.8%	2.0%
Non-market services	0.8%	1.6%	0.7%	1.6%	0.7%	1.3%	1.5%	2.1%

[28] From the same reason, we did not report the Diewert-Morrison LP growth index in any figures and tables, hereafter.

[29] Hours worked decreased in 1971, 1975, 1980, 1982, 1991 and 2000-03.

[30] It is known that the stagnation of the U.S. economy started since 1973. Since the dataset is available since 1970 and the period 1970-1973 is small enough to know the overall trend, we deal with the period 1970-1995.

The 30 industries are classified into 6 representative industries: 1) *Electrical and other equipment*, 2) *Other manufacturing*, 3) *Other production*, 4) *Post and communication*, 5) *Other market services*, and 6) *Non-market services*. *Electrical and other equipment* is manufacturing and *post and communication* showed is service sector. However, since they show very different performance within the manufacturing and service sectors, they are isolated from *other manufacturing* and *other market services* and *non-market services*. Table 1 compares the annual average growth rates of labour productivity based on the average and Malmquist LP growth indexes.

There is a hypothesis so called "Baumol's disease" stating that labour productivity growth in the service sector is likely to be stagnated and lower than that of goods producing industries, especially the manufacturing sector. It has been widely advocated by Triplett and Bosworth (2004), (2006) and Bosworth and Triplett (2007). Except for *distribution* in *other market services*, the average LP growth index among the service sectors is lower than *other manufacturing*, as shown in Table 1. On average over the entire sample period 1970-2005, the average growth rates of *other market sectors* and *non-market services* are 1.7 per cent and 0.8 per cent, which are smaller than that of *other manufacturing*, which is 2.4 per cent. However, if we compare these industries by the Malmquist LP growth index, the average growth rates of *other market sectors* and *non-market services* are 2.3 per cent and 1.6 per cent, while that of *other manufacturing* is 2.2 per cent. The difference in the growth of labour productivity between the service sectors and *other manufacturing* based on the average LP index becomes much smaller under the comparison based on the Malmquist LP growth index. This underestimation of the industry labour productivities by the average LP growth index is even more severe during the low productivity growth period 1970-1995. Two indexes are almost the same for manufacturing. However, moving from the average LP growth index to the Malmquist LP growth index, the labour productivity growth in the service sectors becomes much larger. For the period 1970-1995, the average growth rate of the Malmquist LP growth index of *other market services* is 2.0 per cent, even higher than that of *other manufacturing* 1.8 per cent. The average growth rate for *non-market services* is 1.6 per cent, close to that of *other manufacturing*. The productivity resurgence of service sectors since 1995 made Triplett and Bosworth (2007) state that Baumol's disease has been cured. Under the comparison of industry labour productivity based on the Malmquist LP growth index, we conclude that although Baumol's disease exists before 1995, this disease has not been as serious as it appeared.

Table 2: Growth Rate of Hours Worked and Share of Capital Input in Value Added

	Growth of Hours Worked				Share of Capital
	1970-2005	1970-1995	1995-2000	2000-2005	1970-2005
Total industries	1.4%	1.6%	2.2%	-0.3%	36.4%
Electrical and optical equipment	-0.2%	0.6%	0.8%	-5.4%	25.4%
Other manufacturing	-0.7%	-0.2%	-0.1%	-4.2%	31.4%
Other production	1.5%	1.2%	3.5%	1.0%	41.4%
Post and communication	0.5%	0.6%	4.9%	-4.3%	54.4%
Other market services	2.2%	2.5%	3.2%	-0.1%	25.7%
Distribution	1.3%	1.6%	2.0%	-0.7%	25.8%
Finance and business, except real estate	3.6%	4.1%	5.1%	-0.1%	29.2%
Personal services	2.4%	2.6%	2.9%	0.7%	17.1%
Non-market services	1.7%	1.8%	1.3%	1.3%	47.8%

The difference between two LP growth indexes comes from the large share of capital input and the high growth rate of hours worked. Table 2 shows that the difference between two indexes is significant for the sectors whose hours worked steadily increased over the period. The growth rate of hours worked is much higher in the service sectors than in the manufacturing sector. This flow of labour inputs from the manufacturing sectors to the service sectors can explain the pessimistic view on the service sector, expressed by Baumol's disease.

Figure 5 shows the long term trend of the average and Malmquist LP growth indexes for the period 1970-2005. Overall trend are quite similar in two indexes. However, the movements of two indexes are different in shorter period of time by reflecting the drastic changes in hours worked in the service sector.

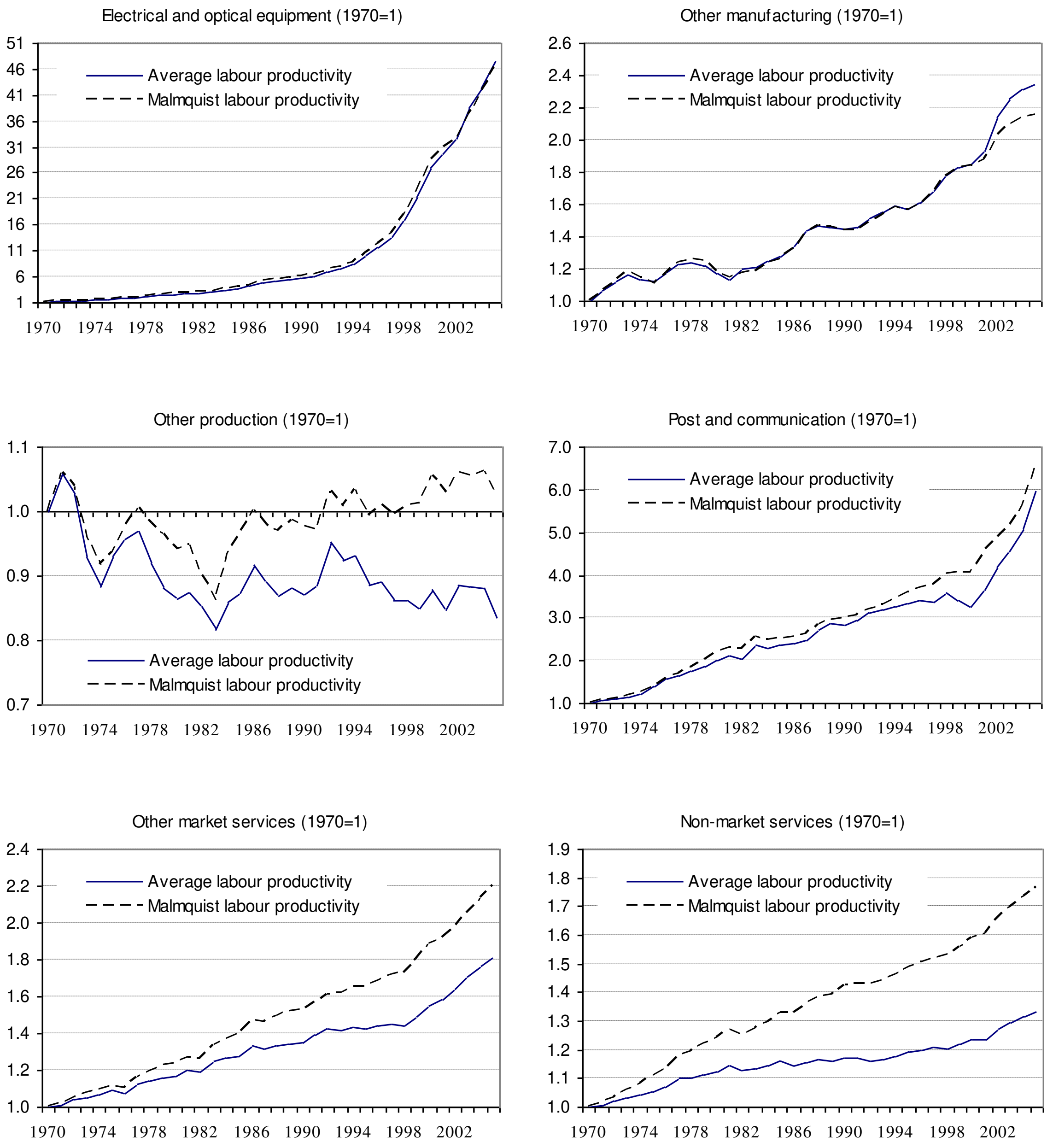

Figure 5: Average and Malmquist LP Growth Indexes by Sector, 1970-2005

We examine the productivity performance of the service sector in more detailed. Table 3 compares labour productivity of 13 sub-industries in the service sector. Table 4 shows the annual average growth rate of hours worked and the share of capital input in nominal industry value added for each sub-industry. The difference between the average LP growth index and the Malmquist LP growth index diverges especially for *renting equipment and other business activities*, and *real estate activities* for the period 1970-2005. However, its reason differs between two sub-industries. Hours worked for *renting equipment and other business services* grew at average annual rate of 4.7 per cent. This is far above the average growth rate of other service industries, widening the gap between two LP growth indexes in *renting equipment and other business activities*. The average growth rate of hours worked for *real estate activities* is 2.6 per cent. There are other sub-industries such as *hotels and restaurants* and *health and social work* whose hours worked grew much faster than *real estate activities*. The reason why the difference between two LP growth indexes significantly widens only for *real estate activities* is the large share of capital input in the value added for this sub-industry. It is almost 90 per cent and is incomparably high within all the industries. The impact of the growth in hours worked of *real estate activities* is strengthened with the larger capital share.

Table 3: Annual Average: Average and Malmquist LP Growth Indexes in Service Sector

	Average	Malmquist	Average	Malmquist	Average	Malmquist	Average	Malmquist
	1970-2005		1970-1995		1995-2000		2000-2005	
Other market services								
Distribution								
Sale, maintenance and repair of motor vehicles and motorcycles	3.7%	4.3%	2.8%	3.4%	5.0%	6.4%	6.4%	6.2%
Wholesale trade and commission trade	3.8%	4.2%	4.3%	4.7%	2.6%	3.0%	3.0%	2.7%
Retail trade	2.0%	2.2%	0.8%	1.1%	3.4%	3.7%	6.3%	6.2%
Transport and storage	1.7%	2.1%	1.6%	2.0%	0.1%	0.9%	3.9%	3.7%
Finance and business services								
Financial intermediation	2.8%	3.6%	2.6%	3.4%	4.5%	5.8%	2.1%	2.2%
Renting of equipment and other business activities	-0.8%	0.2%	-1.6%	-0.4%	-1.0%	0.1%	3.1%	3.1%
Personal services								
Hotels and restaurants	-1.5%	-0.9%	-2.6%	-1.9%	-0.6%	0.0%	3.0%	2.9%
Other community, social and personal services	1.0%	1.4%	1.3%	1.7%	-0.3%	0.3%	0.9%	1.2%
Private households with employed persons	0.6%	0.6%	0.7%	0.7%	0.0%	0.0%	0.2%	0.2%
Non-market services								
Public administration and defence	1.0%	1.1%	1.2%	1.3%	1.0%	0.9%	0.4%	0.5%
Education	0.2%	0.8%	0.3%	0.9%	-0.2%	0.4%	0.5%	0.8%
Health and social work	-0.3%	0.3%	-1.0%	-0.4%	0.8%	1.0%	2.5%	2.8%
Real estate activities	0.9%	3.3%	0.7%	3.3%	-0.1%	2.4%	2.8%	4.0%

Table 4: Growth Rate of Hours Worked and Share of Capital Input in Value Added in Service Sector

	Growth of Hours Worked				Share of Capital
	1970-2005	1970-1995	1995-2000	2000-2005	1970-2005
Other market services					
Distribution					
Sale, maintenance and repair of motor vehicles and motorcycles	1.2%	1.1%	3.1%	-0.4%	43.8%
Wholesale trade and commission trade	1.3%	1.7%	1.7%	-1.2%	26.5%
Retail trade	1.4%	1.6%	1.7%	-0.3%	15.2%
Transport and storage	1.3%	1.5%	2.5%	-1.0%	29.1%
Finance and business services					
Financial intermediation	2.0%	2.3%	2.6%	0.3%	41.8%
Renting of equipment and other business activities	4.7%	5.4%	6.2%	-0.3%	20.2%
Personal services					
Hotels and restaurants	3.0%	3.8%	2.4%	-0.3%	21.2%
Other community, social and personal services	2.5%	2.4%	3.8%	1.5%	15.3%
Private households with employed persons	-0.9%	-1.5%	-1.8%	2.7%	0.0%
Non-market services					
Public administration and defence	0.2%	0.2%	-0.2%	0.4%	33.3%
Education	2.2%	2.2%	2.9%	1.5%	24.0%
Health and social work	3.3%	3.9%	1.4%	2.0%	16.7%
Real estate activities	2.6%	2.8%	2.8%	1.4%	89.8%

6. Conclusion

Total factor productivity growth has been theoretically defined as the shift in the production frontier caused by technological progress. We examine the same reasoning applied to the measurement of labour productivity growth. We start from the viewpoint that the labour productivity growth index should capture the shift in the short-run production frontier caused by technological progress and the change in capital services. We propose the Malmquist and Diewert-Morrison labour productivity growth indexes, which capture the shift, by using the distance function as well as the profit function. Following the index number techniques initiated by Caves, Christensen and Diewert (1982) and Diewert and Morrison (1987), we show that these two indexes equal the index number formulae consisting of observable prices and quantities. These indexes also have a good aggregation property that the standard average labour productivity growth index does not satisfy. In the end, we apply the average and Malmquist labour productivity growth indexes to the industry data of the U.S. for the period 1970-2005. It is well known that the low labour productivity growth of the service sector drags down the growth rate of labour productivity for the entire U.S. economy. However, we found that the difference in the Malmquist labour productivity growth index between the service sector and other sectors is much smaller than the difference in the average labour productivity between them. The underestimation of the labour productivity growth in the service sector by average labour productivity is even more serious in the low productivity era before 1995. The difference between the average and the Malmquist labour productivity growth indexes can be attributed to the effect of scale economies, which grows throughout the change in labour inputs. The flow of employment from the manufacturing sector to the service sector accounts for the underestimation of productivity performance of the service sector during the low productivity era before 1995.

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Appendix
Proof of Proposition 1

$$\ln LPG_M = \left(\frac{1}{2}\right)\ln\left(\frac{D^0(\boldsymbol{y}^0, \boldsymbol{x}_K^0, \boldsymbol{x}_L^0)}{D^1(\boldsymbol{y}^0, \boldsymbol{x}_K^1, \boldsymbol{x}_L^0)}\right) + \left(\frac{1}{2}\right)\ln\left(\frac{D^0(\boldsymbol{y}^1, \boldsymbol{x}_K^0, \boldsymbol{x}_L^1)}{D^1(\boldsymbol{y}^1, \boldsymbol{x}_K^1, \boldsymbol{x}_L^1)}\right)$$

$$= \left(\frac{1}{2}\right)\ln\left(\frac{D^1(\boldsymbol{y}^1, \boldsymbol{x}_K^1, \boldsymbol{x}_L^1)}{D^1(\boldsymbol{y}^0, \boldsymbol{x}_K^1, \boldsymbol{x}_L^0)}\right) + \left(\frac{1}{2}\right)\ln\left(\frac{D^0(\boldsymbol{y}^1, \boldsymbol{x}_K^0, \boldsymbol{x}_L^1)}{D^0(\boldsymbol{y}^0, \boldsymbol{x}_K^0, \boldsymbol{x}_L^0)}\right)$$

Since the firm's profit maximization is assumed, the period t production plan is on the period t production frontier for periods $t = 0$ and 1.

$$= \left(\frac{1}{4}\right)\sum_{m=1}^{M}\left(\frac{\partial \ln D^1(\boldsymbol{y}^1, \boldsymbol{x}_K^1, \boldsymbol{x}_L^1)}{\ln y_m} + \frac{\partial \ln D^1(\boldsymbol{y}^0, \boldsymbol{x}_K^1, \boldsymbol{x}_L^0)}{\ln y_m} + \frac{\partial \ln D^0(\boldsymbol{y}^1, \boldsymbol{x}_K^0, \boldsymbol{x}_L^1)}{\ln y_m} + \frac{\partial \ln D^0(\boldsymbol{y}^0, \boldsymbol{x}_K^0, \boldsymbol{x}_L^0)}{\ln y_m}\right)\ln\left(\frac{y_m^1}{y_m^0}\right)$$

$$\left(\frac{1}{4}\right)\sum_{m=1}^{M}\left(\frac{\partial \ln D^1(\boldsymbol{y}^1, \boldsymbol{x}_K^1, \boldsymbol{x}_L^1)}{\ln x_{L,q}} + \frac{\partial \ln D^1(\boldsymbol{y}^0, \boldsymbol{x}_K^1, \boldsymbol{x}_L^0)}{\ln x_{L,q}} + \frac{\partial \ln D^0(\boldsymbol{y}^1, \boldsymbol{x}_K^0, \boldsymbol{x}_L^1)}{\ln x_{L,q}} + \frac{\partial \ln D^0(\boldsymbol{y}^0, \boldsymbol{x}_K^0, \boldsymbol{x}_L^0)}{\ln x_{L,q}}\right)\ln\left(\frac{x_{L,q}^1}{x_{L,q}^0}\right)$$

using the translog identity in Caves, Christensen and Diewert (1982)

$$= \left(\frac{1}{2}\right)\sum_{m=1}^{M}\left(\frac{\partial \ln D^1(\boldsymbol{y}^1, \boldsymbol{x}_K^1, \boldsymbol{x}_L^1)}{\ln y_m} + \frac{\partial \ln D^0(\boldsymbol{y}^0, \boldsymbol{x}_K^0, \boldsymbol{x}_L^0)}{\ln y_m}\right)\ln\left(\frac{y_m^1}{y_m^0}\right)$$

$$+ \left(\frac{1}{2}\right)\sum_{q=1}^{Q}\left(\frac{\partial \ln D^1(\boldsymbol{y}^1, \boldsymbol{x}_K^1, \boldsymbol{x}_L^1)}{\ln x_{L,q}} + \frac{\partial \ln D^0(\boldsymbol{y}^0, \boldsymbol{x}_K^0, \boldsymbol{x}_L^0)}{\ln x_{L,q}}\right)\ln\left(\frac{x_{L,q}^1}{x_{L,q}^0}\right)$$

$$+ \left(\frac{1}{4}\right)\sum_{m=1}^{M}\left(-\frac{\partial \ln D^1(\boldsymbol{y}^1, \boldsymbol{x}_K^1, \boldsymbol{x}_L^1)}{\ln y_m} + \frac{\partial \ln D^1(\boldsymbol{y}^0, \boldsymbol{x}_K^1, \boldsymbol{x}_L^0)}{\ln y_m} + \frac{\partial \ln D^0(\boldsymbol{y}^1, \boldsymbol{x}_K^0, \boldsymbol{x}_L^1)}{\ln y_m} - \frac{\partial \ln D^0(\boldsymbol{y}^0, \boldsymbol{x}_K^0, \boldsymbol{x}_L^0)}{\ln y_m}\right)\ln\left(\frac{y_m^1}{y_m^0}\right)$$

$$+ \left(\frac{1}{4}\right)\sum_{m=1}^{M}\left(-\frac{\partial \ln D^1(\boldsymbol{y}^1, \boldsymbol{x}_K^1, \boldsymbol{x}_L^1)}{\ln x_{L,q}} + \frac{\partial \ln D^1(\boldsymbol{y}^0, \boldsymbol{x}_K^1, \boldsymbol{x}_L^0)}{\ln x_{L,q}} + \frac{\partial \ln D^0(\boldsymbol{y}^1, \boldsymbol{x}_K^0, \boldsymbol{x}_L^1)}{\ln x_{L,q}} - \frac{\partial \ln D^0(\boldsymbol{y}^0, \boldsymbol{x}_K^0, \boldsymbol{x}_L^0)}{\ln x_{L,q}}\right)\ln\left(\frac{x_{L,q}^1}{x_{L,q}^0}\right)$$

$$= \left(\frac{1}{2}\right)\sum_{m=1}^{M}\left(\frac{\partial \ln D^1(\boldsymbol{y}^1, \boldsymbol{x}_K^1, \boldsymbol{x}_L^1)}{\ln y_m} + \frac{\partial \ln D^0(\boldsymbol{y}^0, \boldsymbol{x}_K^0, \boldsymbol{x}_L^0)}{\ln y_m}\right)\ln\left(\frac{y_m^1}{y_m^0}\right)$$

$$+ \left(\frac{1}{2}\right)\sum_{q=1}^{Q}\left(\frac{\partial \ln D^1(\boldsymbol{y}^1, \boldsymbol{x}_K^1, \boldsymbol{x}_L^1)}{\ln x_{L,q}} + \frac{\partial \ln D^0(\boldsymbol{y}^0, \boldsymbol{x}_K^0, \boldsymbol{x}_L^0)}{\ln x_{L,q}}\right)\ln\left(\frac{x_{L,q}^1}{x_{L,q}^0}\right)$$

from the equation (20).

$$= \left(\frac{1}{2}\right)\sum_{m=1}^{M}\left(\frac{p_m^0 y_m^0}{\boldsymbol{p}^0 \cdot \boldsymbol{y}^0} + \frac{p_m^1 y_m^1}{\boldsymbol{p}^1 \cdot \boldsymbol{y}^1}\right)\ln\left(\frac{y_m^1}{y_m^0}\right) - \left(\frac{1}{2}\right)\sum_{q=1}^{Q}\left(\frac{w^0 x_{L,q}^0}{\boldsymbol{p}^0 \cdot \boldsymbol{y}^0} + \frac{w^1 x_{L,q}^1}{\boldsymbol{p}^1 \cdot \boldsymbol{y}^1}\right)\ln\left(\frac{x_{L,q}^1}{x_{L,q}^0}\right)$$

substituting equations (17), (18) and (19).

Proof of Proposition 2

$$\ln LPG_{DM} = \left(\frac{1}{2}\right)\ln\left(\frac{g^1(\boldsymbol{p}^0,\boldsymbol{x}_K^1,\boldsymbol{x}_L^0)}{g^0(\boldsymbol{p}^0,\boldsymbol{x}_K^0,\boldsymbol{x}_L^0)}\right)+\left(\frac{1}{2}\right)\ln\left(\frac{g^1(\boldsymbol{p}^1,\boldsymbol{x}_K^1,\boldsymbol{x}_L^1)}{g^0(\boldsymbol{p}^1,\boldsymbol{x}_K^0,\boldsymbol{x}_L^1)}\right)$$

$$= \ln\left(\frac{\boldsymbol{p}^1\cdot\boldsymbol{y}^1}{\boldsymbol{p}^0\cdot\boldsymbol{y}^0}\right)-\left(\frac{1}{2}\right)\ln\left(\frac{g^0(\boldsymbol{p}^1,\boldsymbol{x}_K^0,\boldsymbol{x}_L^1)}{g^0(\boldsymbol{p}^0,\boldsymbol{x}_K^0,\boldsymbol{x}_L^0)}\right)-\left(\frac{1}{2}\right)\ln\left(\frac{g^1(\boldsymbol{p}^1,\boldsymbol{x}_K^1,\boldsymbol{x}_L^1)}{g^1(\boldsymbol{p}^0,\boldsymbol{x}_K^1,\boldsymbol{x}_L^0)}\right)$$

$$\begin{aligned}
&= \ln\left(\frac{\boldsymbol{p}^1\cdot\boldsymbol{y}^1}{\boldsymbol{p}^0\cdot\boldsymbol{y}^0}\right)\\
&-\left(\frac{1}{4}\right)\sum_{m=1}^{M}\left(\frac{\partial\ln g^0(\boldsymbol{p}^1,\boldsymbol{x}_K^0,\boldsymbol{x}_L^1)}{\ln p_m}+\frac{\partial\ln g^0(\boldsymbol{p}^0,\boldsymbol{x}_K^0,\boldsymbol{x}_L^0)}{\ln p_m}\right.\\
&\qquad\left.+\frac{\partial\ln g^1(\boldsymbol{p}^1,\boldsymbol{x}_K^1,\boldsymbol{x}_L^1)}{\ln p_m}+\frac{\partial\ln g^1(\boldsymbol{p}^0,\boldsymbol{x}_K^1,\boldsymbol{x}_L^0)}{\ln p_m}\right)\ln\left(\frac{p_m^1}{p_m^0}\right)\\
&-\left(\frac{1}{4}\right)\sum_{q=1}^{Q}\left(\frac{\partial\ln g^0(\boldsymbol{p}^1,\boldsymbol{x}_K^0,\boldsymbol{x}_L^1)}{\ln x_{L,q}}+\frac{\partial\ln g^0(\boldsymbol{p}^0,\boldsymbol{x}_K^0,\boldsymbol{x}_L^0)}{\ln x_{L,q}}\right.\\
&\qquad\left.+\frac{\partial\ln g^1(\boldsymbol{p}^1,\boldsymbol{x}_K^1,\boldsymbol{x}_L^1)}{\ln x_{L,q}}+\frac{\partial\ln g^1(\boldsymbol{p}^0,\boldsymbol{x}_K^1,\boldsymbol{x}_L^0)}{\ln x_{L,q}}\right)\ln\left(\frac{x_{L,q}^1}{x_{L,q}^0}\right)
\end{aligned}$$

using the translog identity in Caves, Christensen and Diewert (1982)

$$\begin{aligned}
&= \ln\left(\frac{\boldsymbol{p}^1\cdot\boldsymbol{y}^1}{\boldsymbol{p}^0\cdot\boldsymbol{y}^0}\right)-\left(\frac{1}{2}\right)\sum_{m=1}^{M}\left(\frac{\partial\ln g^0(\boldsymbol{p}^0,\boldsymbol{x}_K^0,\boldsymbol{x}_L^0)}{\ln p_m}+\frac{\partial\ln g^0(\boldsymbol{p}^1,\boldsymbol{x}_K^1,\boldsymbol{x}_L^1)}{\ln p_m}\right)\ln\left(\frac{p_m^1}{p_m^0}\right)\\
&\qquad-\left(\frac{1}{2}\right)\sum_{q=1}^{Q}\left(\frac{\partial\ln g^0(\boldsymbol{p}^0,\boldsymbol{x}_K^0,\boldsymbol{x}_L^0)}{\ln x_{L,q}}+\frac{\partial\ln g^0(\boldsymbol{p}^1,\boldsymbol{x}_K^1,\boldsymbol{x}_L^1)}{\ln x_{L,q}}\right)\ln\left(\frac{x_{L,q}^1}{x_{L,q}^0}\right)\\
&-\left(\frac{1}{4}\right)\sum_{m=1}^{M}\left(\frac{\partial\ln g^0(\boldsymbol{p}^1,\boldsymbol{x}_K^0,\boldsymbol{x}_L^1)}{\ln p_m}+\frac{\partial\ln g^0(\boldsymbol{p}^0,\boldsymbol{x}_K^0,\boldsymbol{x}_L^0)}{\ln p_m}\right.\\
&\qquad\left.+\frac{\partial\ln g^1(\boldsymbol{p}^1,\boldsymbol{x}_K^1,\boldsymbol{x}_L^1)}{\ln p_m}+\frac{\partial\ln g^1(\boldsymbol{p}^0,\boldsymbol{x}_K^1,\boldsymbol{x}_L^0)}{\ln p_m}\right)\ln\left(\frac{p_m^1}{p_m^0}\right)\\
&-\left(\frac{1}{4}\right)\sum_{q=1}^{Q}\left(\frac{\partial\ln g^0(\boldsymbol{p}^1,\boldsymbol{x}_K^0,\boldsymbol{x}_L^1)}{\ln x_{L,q}}+\frac{\partial\ln g^0(\boldsymbol{p}^0,\boldsymbol{x}_K^0,\boldsymbol{x}_L^0)}{\ln x_{L,q}}\right.\\
&\qquad\left.+\frac{\partial\ln g^1(\boldsymbol{p}^1,\boldsymbol{x}_K^1,\boldsymbol{x}_L^1)}{\ln x_{L,q}}+\frac{\partial\ln g^1(\boldsymbol{p}^0,\boldsymbol{x}_K^1,\boldsymbol{x}_L^0)}{\ln x_{L,q}}\right)\ln\left(\frac{x_{L,q}^1}{x_{L,q}^0}\right)\\
&= \ln\left(\frac{\boldsymbol{p}^1\cdot\boldsymbol{y}^1}{\boldsymbol{p}^0\cdot\boldsymbol{y}^0}\right)-\left(\frac{1}{2}\right)\sum_{m=1}^{M}\left(\frac{\partial\ln g^0(\boldsymbol{p}^0,\boldsymbol{x}_K^0,\boldsymbol{x}_L^0)}{\ln p_m}+\frac{\partial\ln g^0(\boldsymbol{p}^1,\boldsymbol{x}_K^1,\boldsymbol{x}_L^1)}{\ln p_m}\right)\ln\left(\frac{p_m^1}{p_m^0}\right)\\
&\qquad-\left(\frac{1}{2}\right)\sum_{q=1}^{Q}\left(\frac{\partial\ln g^0(\boldsymbol{p}^0,\boldsymbol{x}_K^0,\boldsymbol{x}_L^0)}{\ln x_{L,q}}+\frac{\partial\ln g^0(\boldsymbol{p}^1,\boldsymbol{x}_K^1,\boldsymbol{x}_L^1)}{\ln x_{L,q}}\right)\ln\left(\frac{x_{L,q}^1}{x_{L,q}^0}\right)
\end{aligned}$$

from the equation (31).

$$= \ln\left(\frac{\boldsymbol{p}^1 \cdot \boldsymbol{y}^1}{\boldsymbol{p}^0 \cdot \boldsymbol{y}^0}\right) - \left(\frac{1}{2}\right)\sum_{m=1}^{M}\left(\frac{p_m^0 y_m^0}{\boldsymbol{p}^0 \cdot \boldsymbol{y}^0} + \frac{p_m^1 y_m^1}{\boldsymbol{p}^1 \cdot \boldsymbol{y}^1}\right)\ln\left(\frac{p_m^1}{p_m^0}\right)$$

$$- \left(\frac{1}{2}\right)\sum_{q=1}^{Q}\left(\frac{w_m^0 x_{L,q}^0}{\boldsymbol{p}^0 \cdot \boldsymbol{y}^0} + \frac{w_m^1 x_{L,q}^1}{\boldsymbol{p}^1 \cdot \boldsymbol{y}^1}\right)\ln\left(\frac{x_{L,q}^1}{x_{L,q}^0}\right)$$

substituting equations (23), (24) and (25).