Perfect numbers - a lower bound for an odd perfect number

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PERFECT NUMBERS-A LOWER BOUND FOR AN ODD PERFECT NUMBER

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Abstract: In this work we construct a lower bound for an odd perfect number in terms of the number of its distinct prime factors. We further generalize the formula for any natural number for which the number of its distinct prime factors is known.

Keywords: perfect numbers, odd perfect numbers, lower bound.

1. INTRODUCTION: The quest for odd perfect numbers has started with Euler in the 18th century. He put some arithmetical restrictions on the form of odd perfect numbers. If n is an odd perfect number with $\omega(n) = k$ distinct prime factors than $n = p^\alpha Q^2$ where $p \equiv \alpha \equiv 1\ (mod\ 4)$ and $Q \in \mathbb{N}$. Later on many other restrictions have been made on the properties of o.p.n (odd perfect number). In 1896, Stuyvaert showed that an o.p.n must be the sum of two squares, a result which follows directly from the $4k + 1$ Fermat's Theorem. It has been proved by Gradshtein that an o.p.n must have at least 6 distinct prime factors. In 1980 Hagis showed that an o.p.n must have 8 distinct prime factors and in this case it must also be divisible by 15. Neilsen (2006), improving the bound of Hagis 1980, showed that if an odd perfect number is not divisible by 3, it must have at least 12 distinct prime factors. Nielsen (2006) also showed that a general odd perfect number, if it exists, must have at least 9 distinct prime factors. It has been checked through algorithms run in computers that there is no o.p.n up to 10^{300}. This makes their existence appear unlikely. In 1977, Pomerance gave an explicit upper bound in terms k (the number of distinct prime factors). Heath-Brown later improved the bound to $n < 4^{4^k}$. Later on Nielsen improved the upper bound to $n < 2^{4^k}$. Furthermore, Pomerance has given a heuristic idea on the non-existence of such numbers. However no proof or disproof is known to present day. In this work we aim at finding a lower bound for an o.p.n in terms of the number of its distinct prime factors.

2. PRELIMINARY RESULTS:

Definition 2.1: The sum of positive divisors of a natural number n is an arithmetical function $\sigma(n) = \sum_{d/n} d$, where d runs from 1 to n.

Definition 2.2: Let n be a positive integer. If $\sigma(n) = 2n$ then n is called perfect number. If $\sigma(n) < 2n$ then n is called deficient and if $\sigma(n) > 2n$ then n is called abundant number.

Definition 2.3: Euler totient function $\varphi(n)$ is an arithmetical function which counts the number of positive integers smaller and coprime to n.

Proposition 2.1: For any natural number n, $\sigma(n)$ and $\varphi(n)$ satisfy the following inequality,

$$1 > \frac{\sigma(n) \cdot \varphi(n)}{n^2} > \frac{\pi^2}{6}$$

Corollary 2.1: If n is a perfect number and

$n = \prod_{p/n} p^\alpha$, then

$1 > 2 \cdot \prod_{p/n}\left(1-\frac{1}{p}\right) = \prod_{p/n}\left(1-\frac{1}{p^{\alpha+1}}\right) > \frac{\pi^2}{6}$ and in general, for any positive integer n, such than $\sigma(n) = \nu n$, then,

$$1 > \nu \cdot \prod_{p/n}\left(1-\frac{1}{p}\right) = \prod_{p/n}\left(1-\frac{1}{p^{\alpha+1}}\right) > \frac{\pi^2}{6}$$

3. CONSTRUCTING A LOWER BOUND:

Lemma 3.1: Let $f(x) = \ln(1+x)$ and $g(x) = x$ be defined on the interval $(0, a]$ then,

$$\mathrm{f(x)} \geq \frac{\ln(1+a)}{a} \cdot g(x)$$

Proposition 3.1: If n is an o.p.n and $n = \prod_{p/n} p^{\alpha}$, then,

$$\sum_{p^{\alpha}/n} \frac{1}{p^{\alpha}} \leq \frac{1}{p_{min}^{\alpha^*}} \cdot \frac{\ln(2)}{\ln(1+\frac{1}{p_{min}^{\alpha^*}})}$$

, where $\alpha^* = \alpha_{min}$ and $p_{min} = \min\{p: p/n\}$.

Proof: Set $p_{min} = \min\{p: p/n\}$ and $\alpha = \alpha_{min}$, then

$$\frac{1}{p^{\alpha}} \leq \frac{1}{p_{min}^{\alpha^*}}$$

Using Lemma 3.1 on the interval $(0.\frac{1}{p_{min}^{\alpha^*}})$ one obtains

$$\ln(1+\frac{1}{p^{\alpha}}) \geq p_{min}^{\alpha^*} \ln(1+\frac{1}{p_{min}^{\alpha^*}}) \cdot \frac{1}{p^{\alpha}}$$

Summing over all prime numbers that divide n yields

$$\sum_{p^{\alpha}/n} \ln(1+\frac{1}{p^{\alpha}}) \geq p_{min}^{\alpha^*} \ln(1+\frac{1}{p_{min}^{\alpha^*}}) \cdot \sum_{p^{\alpha}/n} \frac{1}{p^{\alpha}}$$

By corollary 2.1 if n is a perfect number then

$$1 > 2 \cdot \prod_{p/n}\left(1-\frac{1}{p}\right) = \prod_{p/n}\left(1-\frac{1}{p^{\alpha+1}}\right) \geq \prod_{p/n}\left(1-\frac{1}{p^2}\right)$$

Therefore

$$2 \geq \prod_{p/n}\left(1+\frac{1}{p}\right) \geq \prod_{p/n}\left(1+\frac{1}{p^{\alpha}}\right)$$

Taking the natural logarithm both sides gives

$$\ln(2) \geq \sum_{p/n} \ln\left(1+\frac{1}{p^{\alpha}}\right)$$

$$\leftrightarrow$$

$$\ln(2) \geq p_{min}^{\alpha^*} \ln(1+\frac{1}{p_{min}^{\alpha^*}}) \cdot \sum_{p^{\alpha}/n} \frac{1}{p^{\alpha}}$$

And the result follows.

Theorem 3.1: If n is an o.p.n and $n = \prod_{p/n} p^{\alpha}$ then,

$$n \geq \left\lceil \left(p_{min}^{\alpha^*} k \cdot log_2\left(1+\frac{1}{p_{min}^{\alpha^*}}\right)\right)^k \right\rceil$$

, where $\alpha^* = \alpha_{min}$ and $p_{min} = \min\{p: p/n\}$ and $\lceil \ \ \rceil$ stands for the greatest integer function.

Proof: In the derivation of this formula we use an elementary inequality, the AM-GM (arithmetic mean-geometric mean) inequality, which states that the arithmetic mean of a set of positive integers is greater or equal to their geometric mean. Therefore,

$$\sum_{p^{\alpha}/n} \frac{1}{p^{\alpha}} \geq k \cdot (\prod_{p^{\alpha}/n} \frac{1}{p^{\alpha}})^{\frac{1}{k}} = k \cdot n^{\frac{1}{k}}$$

Using proposition 3.1 we obtain

$$\frac{1}{p_{min}^{\alpha^*}} \cdot \frac{\ln(2)}{\ln(1+\frac{1}{p_{min}^{\alpha^*}})} \geq k \cdot n^{\frac{1}{k}}$$

$$\leftrightarrow$$

$$n \geq \left(p_{min}^{\alpha^*} k \cdot log_2 (1 + \frac{1}{p_{min}^{\alpha^*}}) \right)^k$$

Since n is an integer then the above statement is equivalent to,

$$n \geq \left\lceil \left(p_{min}^{\alpha^*} k \cdot log_2 \left(1 + \frac{1}{p_{min}^{\alpha^*}} \right) \right)^k \right\rceil$$

4. CONCLUSIONS: This lower bound is applicable not only to o.p.n, but actually to all perfect numbers. This formula gives a close approximation to the smallest even perfect number. Since the form of all even perfect numbers is $n = 2^{m-1}(2^m - 1)$ where m and $(2^m - 1)$ are primes, then by setting $\alpha^* = 1\ \ p_{min} = 2$ and $k = 2$, yields

$$n \geq \lceil 5{,}47489804 \rceil = 6$$

Indeed the first even perfect number is 6. It was shown by Nielsen (2006) that a general odd perfect number, if it exists, it must have at least 9 distinct prime factors. Therefore if one plugs into the formula $\alpha^* = 1, p_{min} = 3$ and $k = 9$, then

$$n \geq \lceil 2786505575.61076 \rceil = 2786505576$$

However this lower bound is too much lower than the actual lower bound 10^{300}, found by running algorithms in computers. Nevertheless one may be able to improve this bound by taking advantage of the form an odd perfect number must have, as showed by Euler. Also we can take advantage of the corollary 2.1 and state in more generally that for any natural number $n > 1$ with k distinct prime factors, and $\alpha = \alpha_{min}$ and $p_{min} = \min\{p : p/n\}$, such that $\sigma(n) = \nu n$ the following boundary holds,

$$n \geq \left\lceil \left(p_{min}^{\alpha^*} k \cdot log_\nu \left(1 + \frac{1}{p_{min}^{\alpha^*}} \right) \right)^k \right\rceil$$

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