Bugs in the proofs of revelation principle

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Abstract

In the field of mechanism design, the revelation principle has been known for decades. Myerson, Mas-Colell, Whinston and Green gave formal proofs of the revelation principle respectively. However, in this paper, I argue that there are bugs hidden in their proofs.

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Key words: Revelation principle; Mechanism design; Implementation theory.

The revelation principle is well-known in the economics literature. See Page 884, Line 24 [1]: “The implication of the revelation principle is ... to identify the set of implementable social choice functions, we need only identify those that are truthfully implementable.” But, in this paper I will argue that there are bugs in the proofs given by Mas-Colell, Whinston and Green [1] and Myerson [2] respectively. Coincidently, the bugs are relevant to the same word “imply”. Related definitions and proofs are given in Appendices, which are cited from Section 8.E, 23.B and 23.D [1] and Ref. [2]. Two remarks are added in Appendix 1 and 3 respectively.

1 The bug in the proof by Mas-Colell, Whinston and Green

Here, the notation is referred to Ref. [1]. See the proof of Proposition 23.D.1: “... Condition (23.D.2) implies that for all i and all θ_i ∈ Θ_i,....”. To derive formula (23.D.3), the term “\hat{s}_i” (\forall \hat{s}_i \in S_i, i = 1, \cdots, J) in formula (23.D.2) is replaced

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by “$s^*_i(\hat{\theta}_i)$” ($\forall \hat{\theta}_i \in \Theta_i, i = 1, \cdots, I$). Since formula (23.D.2) holds for all $\hat{s}_i \in S_i$, it looks reasonable to do so at first sight.

However, as formula (23.D.2) specifies, the expectation is taken over realizations of the other players’ random types conditional on player $i$’s realized type $\theta_i$ (also see Proposition 8.E.1). Note that the input of the function $s^*_i(\cdot)$ should be a realized type of player $i$ (see Remark 1), but none of $\hat{\theta}_i$ ($\forall \hat{\theta}_i \in \Theta_i, \hat{\theta}_i \neq \theta_i$) can be such realized type since agent $i$’s type has been realized as $\theta_i$. Therefore, in formula (23.D.3), the term “$s^*_i(\hat{\theta}_i)$” ($\forall \hat{\theta}_i \in \Theta_i, \hat{\theta}_i = \theta_i$) is actually illegal. Put differently, formula (23.D.3) is illegal. That is the bug.

2 The bug in the proof by Myerson

Here, the notation is referred to Ref. [2]. See the proof of Theorem 2: “... Furthermore, the equilibrium inequalities (14) for $\pi$ imply the incentive compatible inequalities (6) for $\pi'$. ...” Let us consider the right part of the incentive compatible inequalities (6) for $\pi'$. For all $i$, $a_i \in A_i$, $b_i \in A_i$,

$$Z_i(\pi', b_i | a_i) = \sum_{\alpha \in A_1 \times \cdots \times A_n} \sum_{c \in C} P_i(\alpha | a_i) \pi'(c | \alpha_{-i}, b_i) U_i(c, \alpha)$$

$$= \sum_{\alpha \in A_1 \times \cdots \times A_n} \sum_{s_1 \in S_1 \times \cdots \times S_n} \sum_{c \in C} P_i(\alpha | a_i) \cdot \pi(c | s) \cdot \left[ \prod_{j=1, j \neq i}^n \sigma_j(s_j | \alpha_j) \times \sigma_i(s_i | b_i) \right] \cdot U_i(c, \alpha)$$

As specified in the left term “$Z_i(\pi', b_i | a_i)$”, agent $i$’s type is realized as $a_i$. Therefore, according to Remark 2, the term “$\sigma_i(s_i | b_i)$” (for all $b_i \in A_i, b_i \neq a_i$) is actually illegal. Put differently, the incentive compatible inequalities (6) for $\pi'$ is illegal. That is the bug.

Appendix 1: Definitions and proof in Section 8.E [1]

According to page 255 [1], formally, in a Bayesian game, each player $i$ has a payoff function $u_i(s_i, s_{-i}, \theta_i)$, where $\theta_i \in \Theta_i$ is a random variable chosen by nature that is observed only by player $i$. The joint probability distribution of the $\theta_i$’s is given by $F(\theta_1, \cdots, \theta_I)$, which is assumed to be common knowledge among the players. Letting $\Theta = \Theta_1 \times \cdots \times \Theta_I$, a Bayesian game is summarized by $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$.

A pure strategy for player $i$ in a Bayesian game is a function $s_i(\theta_i)$, or decision rule, that gives the player’s strategy choice for each realization of his type
\( \theta_i \). Player \( i \)'s pure strategy set \( \mathcal{S}_i \) is therefore the set of all such functions. Player \( i \)'s expected payoff given a profile of pure strategies for the \( I \) players \( (s_1(\cdot), \ldots, s_I(\cdot)) \) is then given by:

\[
\tilde{u}_i(s_1(\cdot), \ldots, s_I(\cdot)) = E_\theta[u_i(s_1(\theta_1), \ldots, s_I(\theta_I), \theta_i)], \tag{8.1E.1}
\]

**Remark 1:** Following page 148 [3], the timing of a static Bayesian game is as follows:

Step 1: Nature chooses a type vector \( \theta = (\bar{\theta}_1, \ldots, \bar{\theta}_I) \), where \( \bar{\theta}_i \) is the realized type of agent \( i \);
Step 2: Nature reveals \( \bar{\theta}_i \) to player \( i \) but not to any other player;
Step 3: The players simultaneously output \( (s_1(\bar{\theta}_1), \ldots, s_I(\bar{\theta}_I)) \);
Step 4: Each player \( i \) receives the payoff \( u_i(s_1(\bar{\theta}_1), \ldots, s_I(\bar{\theta}_1), \bar{\theta}_i) \).

For each player \( i = 1, \ldots, I \), consider his strategy function \( s_i(\cdot) \), then:
1) \( s_i(\cdot) \) is chosen (or controlled) by player \( i \), and is his private information;
2) In a static Bayesian game, player \( i \)'s type can be realized as any element of \( \Theta_i \). The realized type of player \( i \) is his private information;
3) The input of \( s_i(\cdot) \) must be a realized type \( \bar{\theta}_i \) in \( \Theta_i \), and the output of \( s_i(\cdot) \) is \( s_i(\bar{\theta}_i) \) which is observable to the outside agent (either principal or mediator).
4) Suppose player \( i \)'s type has been realized as \( \bar{\theta}_i \) in Step 1, then in Step 3, it is illegal to let player \( i \) output \( s_i(\theta_i) \) for any \( \theta_i \in \Theta_i \), \( \theta_i \neq \bar{\theta}_i \).

**Definition 8.1E.1:** A (pure strategy) Bayesian Nash equilibrium for the Bayesian game \( [I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)] \) is a profile of decision rules \( (s_1(\cdot), \ldots, s_I(\cdot)) \) that constitutes a Nash equilibrium of game \( \Gamma_N = [I, \{\mathcal{S}_i\}, \{\bar{u}_i(\cdot)\}] \). That is, for every \( i = 1, \ldots, I \),

\[
\tilde{u}_i(s_i(\cdot), s_{-i}(\cdot)) \geq \tilde{u}_i(s_i'(\cdot), s_{-i}(\cdot))
\]

for all \( s_i'(\cdot) \in \mathcal{S}_i \), where \( \tilde{u}_i(s_i(\cdot), s_{-i}(\cdot)) \) is defined as in Eq(8.1E.1).

A very useful point to note is that in a (pure strategy) Bayesian Nash equilibrium each player must be playing a best response to the conditional distribution of his opponents’ strategies for each type that he might end up having.

**Proposition 8.1E.1:** A profile of decision rules \( (s_1(\cdot), \ldots, s_I(\cdot)) \) is a Bayesian Nash equilibrium in Bayesian game \( [I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)] \) if and only if, for all \( i \) and all \( \bar{\theta}_i \in \Theta_i \) occurring with positive probability,

\[
E_{\theta_i}[u_i(s_i(\bar{\theta}_i), s_{-i}(\theta_{-i}), \bar{\theta}_i)|\tilde{\theta}_i)] \geq E_{\theta_i}[u_i(s_i'(\cdot), s_{-i}(\theta_{-i}), \bar{\theta}_i)|\tilde{\theta}_i)], \tag{8.1E.2}
\]

for all \( s_i' \in S_i \), where the expectation is taken over realizations of the other
players’ random variables conditional on player $i$’s realization of his signal $\tilde{\theta}_i$.

**Proof:** For necessity, note that if Eq(8.E.2) did not hold for some player $i$ for some $\tilde{\theta}_i \in \Theta_i$ that occurs with positive probability, then player $i$ could do better by changing his strategy choice in the event he gets realization $\tilde{\theta}_i$, contradicting $(s_1(\cdot), \ldots, s_I(\cdot))$ being a Bayesian Nash equilibrium. In the other direction, if condition Eq(8.E.2) holds for all $\tilde{\theta}_i \in \Theta_i$ occurring with positive probability, then player $i$ cannot improve on the payoff he receives by playing strategy $s_i(\cdot)$. $\square$

**Appendix 2: Definitions and proof in Section 23.B and 23.D [1]**

(P858) Consider a setting with $I$ agents, indexed by $i = 1, \cdots, I$. These agents make a collective choice from some set $X$ of possible alternatives. Prior to the choice, each agent $i$ privately observes his type $\theta_i$ that determines his preferences. The set of possible types for agent $i$ is denoted as $\Theta_i$. The vector of agents’ types $\theta = (\theta_1, \ldots, \theta_I)$ is drawn from set $\Theta = \Theta_1 \times \cdots \times \Theta_I$ according to probability density $\phi(\cdot)$. Each agent $i$’s Bernoulli utility function when he is of type $\theta_i$ is $u_i(x, \theta_i)$.

**Definition 23.B.1:** A social choice function is a function $f : \Theta_1 \times \cdots \times \Theta_I \rightarrow X$ that, for each possible profile of the agents’ types $(\theta_1, \ldots, \theta_I)$, assigns a collective choice $f(\theta_1, \ldots, \theta_I) \in X$.

**Definition 23.B.3:** A mechanism $\Gamma = (S_1, \cdots, S_I, g(\cdot))$ is a collection of $I$ strategy sets $S_1, \cdots, S_I$ and an outcome function $g : S_1 \times \cdots \times S_I \rightarrow X$.

**Definition 23.B.4:** The mechanism $\Gamma = (S_1, \cdots, S_I, g(\cdot))$ implements social choice function $f(\cdot)$ if there is an equilibrium strategy profile $(s_1^*(\cdot), \cdots, s_I^*(\cdot))$ of the game induced by $\Gamma$ such that $g(s_1^*(\theta_1), \cdots, s_I^*(\theta_I)) = f(\theta_1, \cdots, \theta_I)$ for all $(\theta_1, \cdots, \theta_I) \in \Theta_1, \cdots, \Theta_I$.

**Definition 23.B.5:** A direct revelation mechanism is a mechanism in which $S_i = \Theta_i$ for all $i$ and $g(\theta) = f(\theta)$ for all $\theta \in \Theta_1 \times \cdots \times \Theta_I$.

**Definition 23.B.6:** The social choice function $f(\cdot)$ is truthfully implementable (or incentive compatible) if the direct revelation mechanism $\Gamma = (S_1, \cdots, S_I, f(\cdot))$ has an equilibrium $(s_1^*(\cdot), \cdots, s_I^*(\cdot))$ in which $s_i^*(\theta_i) = \theta_i$ for all $\theta_i \in \Theta_i$ and all $i = 1, \cdots, I$; that is, if truth telling by each agent $i$ constitutes an equilibrium of $\Gamma = (S_1, \cdots, S_I, f(\cdot))$.

**Definition 23.D.1:** The strategy profile $s^*(\cdot) = (s_1^*(\cdot), \cdots, s_I^*(\cdot))$ is a Bayesian Nash equilibrium of mechanism $\Gamma = (S_1, \cdots, S_I, g(\cdot))$ if, for all $i$ and all
\[ \theta_i \in \Theta_i, \]

\[ E_{\theta_i}[u_i(g(s^*_i(\theta_i), s^*_{-i}(\theta_{-i})), \theta_i)|\theta_i] \geq E_{\hat{\theta}_i}[u_i(g(\hat{s}_i, s^*_{-i}(\theta_{-i})), \theta_i)|\theta_i] \]

for all \( \hat{s}_i \in S_i \).

**Definition 23.D.2:** The mechanism \( \Gamma = (S_1, \cdots , S_I, g(\cdot)) \) implements the social choice function \( f(\cdot) \) in Bayesian Nash equilibrium if there is a Bayesian Nash equilibrium of \( \Gamma \), \( s^*(\cdot) = (s^*_1(\cdot), \cdots , s^*_I(\cdot)) \), such that \( g(s^*(\theta)) = f(\theta) \) for all \( \theta \in \Theta \).

**Definition 23.D.3:** The social choice function \( f(\cdot) \) is truthfully implementable in Bayesian Nash equilibrium if \( s^*_i(\theta_i) = \theta_i \) (for all \( \theta_i \in \Theta_i \) and \( i = 1, \cdots , I \)) is a Bayesian Nash equilibrium of the direct revelation mechanism \( \Gamma = (\Theta_1, \cdots , \Theta_I, f(\cdot)) \). That is, if for all \( i = 1, \cdots , I \) and all \( \theta_i \in \Theta_i \),

\[ E_{\theta_i}[u_i(f(\theta_i, \theta_{-i}), \theta_i)|\theta_i] \geq E_{\hat{\theta}_i}[u_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i)|\theta_i], \quad (23.D.1) \]

for all \( \hat{\theta}_i \in \Theta_i \).

**Proposition 23.D.1** (The Revelation Principle for Bayesian Nash Equilibrium) Suppose that there exists a mechanism \( \Gamma = (S_1, \cdots , S_I, g(\cdot)) \) that implements the social choice function \( f(\cdot) \) in Bayesian Nash equilibrium. Then \( f(\cdot) \) is truthfully implementable in Bayesian Nash equilibrium.

**Proof:** Since \( \Gamma = (S_1, \cdots , S_I, g(\cdot)) \) implements \( f(\cdot) \) in Bayesian Nash equilibrium, then there exists a profile of strategies \( s^*(\cdot) = (s^*_1(\cdot), \cdots , s^*_I(\cdot)) \) such that \( g(s^*(\theta)) = f(\theta) \) for all \( \theta \), and for all \( i \) and all \( \theta_i \in \Theta_i \),

\[ E_{\theta_i}[u_i(g(s^*_i(\theta_i), s^*_{-i}(\theta_{-i})), \theta_i)|\theta_i] \geq E_{\hat{\theta}_i}[u_i(g(\hat{s}_i, s^*_{-i}(\theta_{-i})), \theta_i)|\theta_i], \quad (23.D.2) \]

for all \( \hat{s}_i \in S_i \). Condition (23.D.2) implies that for all \( i \) and all \( \theta_i \in \Theta_i \),

\[ E_{\theta_i}[u_i(g(s^*_i(\theta_i), s^*_{-i}(\theta_{-i})), \theta_i)|\theta_i] \geq E_{\hat{\theta}_i}[u_i(g(s^*_i(\hat{\theta}_i), s^*_{-i}(\theta_{-i})), \theta_i)|\theta_i], \quad (23.D.3) \]

for all \( \hat{\theta}_i \in \Theta_i \). Since \( g(s^*(\theta)) = f(\theta) \) for all \( \theta \), (23.D.3) means that, for all \( i \) and all \( \theta_i \in \Theta_i \),

\[ E_{\theta_i}[u_i(f(\theta_i, \theta_{-i}), \theta_i)|\theta_i] \geq E_{\hat{\theta}_i}[u_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i)|\theta_i], \quad (23.D.4) \]

for all \( \hat{\theta}_i \in \Theta_i \). But, this is precisely condition (23.D.1), the condition for \( f(\cdot) \) to be truthfully implementable in Bayesian Nash equilibrium. Q.E.D.
Appendix 3: Definitions and proof in Ref. [2]

The arbitrator’s problem is described by a Bayesian collective choice problem, an object of the form:

\[(C, A_1, A_2, \ldots, A_n, U_1, U_2, \ldots, U_n, P), \tag{1}\]

The individual members of the group, or players, are numbered 1, 2, \ldots, n. C is the set of choices available to the group. For each player i, A_i is the set of possible types for player i. Each \(U_i : C \times A_1 \times \cdots \times A_n \rightarrow \mathbb{R}\) is a utility function such that each \(U_i(c, a_1, \cdots, a_n)\) is the payoff which player i would get if \(c \in C\) were chosen and if \((a_1, \cdots, a_n)\) were the true vector of player types. P is a probability distribution on \(A_1 \times \cdots \times A_n\) such that \(P(a_1, \cdots, a_n)\) is the probability, as judged by the arbitrator, that \((a_1, \cdots, a_n)\) is the true vector of types for the n players.

For some collection of response sets \(S_1, \cdots, S_n\), a choice mechanism is defined as a real-valued function \(\pi\) with a domain of the form \(C \times (S_1 \times \cdots \times S_n)\) such that:

\[
\sum_{c' \in C} \pi(c'|s_1, \cdots, s_n) = 1, \quad \text{and} \quad \pi(c|s_1, \cdots, s_n) \geq 0 \quad \text{for all } c, \tag{2}\]

for every \((s_1, \cdots, s_n) \in S_1 \times \cdots \times S_n\).

Given a choice mechanism \(\pi\), for any player i and for any \(a_i \in A_i\) and \(b_i \in A_i\), let:

\[
Z_i(\pi, b_i|a_i) = \sum_{\alpha \in A_1 \times \cdots \times A_n} \sum_{c \in C} P(\alpha|a_i) \pi(c|\alpha_{-i}, b_i) U_i(c, \alpha), \tag{5}\]

where \((\alpha_{-i}, b_i) = (\alpha_1, \cdots, \alpha_{i-1}, b_i, \alpha_{i+1}, \cdots, \alpha_n)\), \(P(\alpha|a_i) = 0\) if \(\alpha_i \neq a_i\). \(Z_i(\pi, b_i|a_i)\) is the conditionally-expected utility payoff for player i, given that his type is \(a_i\), if he says that his type is \(b_i\) when \(\pi\) is the choice mechanism and when all other players are expected to tell the truth.

A choice mechanism \(\pi\) using the standard response sets is said to be Bayesian incentive compatible if

\[
Z_i(\pi, a_i|a_i) \geq Z_i(\pi, b_i|a_i), \quad \text{for all } i, a_i \in A_i, b_i \in A_i, \tag{6}\]

If choice mechanism \(\pi\) is used and if everyone is honest, then player i’s conditionally-expected payoff when he knows \(a_i\) is:

\[
V_i(\pi|a_i) = Z_i(\pi, a_i|a_i), \tag{7}\]

The allocation of conditionally-expected payoffs associated with mechanism \(\pi\)
is the vector:

\[ V(\pi) = \left( (V_i(\pi|a_i))_{a_i \in A_i} \right)_{i=1}^n. \] (8)

This is a vector of \( \sum_{i=1}^n |A_i| \) real numbers, indexed on the disjoint union of the \( A_i \) sets. If the arbitrator could use any choice mechanism and expect honest responses, then we would define the \textit{feasible set} of expected allocation vectors to be:

\[ F = \{ V(\pi) : \pi \text{ is a choice mechanism} \}. \]

The set of \textit{incentive-feasible} expected allocation vectors is defined to be:

\[ F^* = \{ V(\pi) : \pi \text{ is Bayesian incentive compatible} \}. \]

A \textit{response plan} for player \( i \) is a function \( \sigma_i \) mapping each type \( a_i \in A_i \) onto a probability distribution over his response set \( S_i \). That is, if \( \sigma_i \) is player \( i \)'s response plan, then \( \sigma_i(s_i|a_i) \) is the probability that player \( i \) will tell the arbitrator \( s_i \) if his \textit{true} type is \( a_i \).

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Remark 2: Like Remark 1, I list the timing of a static Bayesian game as follows:

Step 1: Nature chooses a type vector \( (\bar{a}_1, \cdots, \bar{a}_n) \), where \( \bar{a}_i \) is the \textit{realized} type of agent \( i \);

Step 2: Nature reveals \( \bar{a}_i \) to player \( i \) but not to any other player;

Step 3: Player \( i \) tells his response \( s_i \) to the arbitrator according to the probability \( \sigma_i(s_i|\bar{a}_i) \). All players tell the arbitrator simultaneously.

Step 4: The arbitrator assigns choice \( c \) to all players according to the probability \( \pi(c|s_1, \cdots, s_n) \).

Step 5: Each player \( i \) receives the payoff \( U_i(c, \bar{a}_1, \cdots, \bar{a}_n) \).

For each player \( i = 1, \cdots, n \), consider his response plan \( \sigma_i(s_i|\cdot) \), then:

1) \( \sigma_i(s_i|\cdot) \) is chosen (or controlled) by player \( i \), and is his private information;

2) In a static Bayesian game, player \( i \)'s type can be realized as any element of \( A_i \). The realized type of player \( i \) is his private information;

3) The input of \( \sigma_i(s_i|\cdot) \) must be a \textit{realized} type \( \bar{a}_i \) in \( A_i \), and the output of \( \sigma_i(s_i|\cdot) \) is the probability that player \( i \) will tell the arbitrator \( s_i \) if his true type is \( \bar{a}_i \);

4) Suppose player \( i \)'s type has been realized as \( \bar{a}_i \) in Step 1, then in Step 3, it is illegal to let player \( i \) act using another response plan \( \sigma_i(s_i|b_i) \) for any \( b_i \in A_i \), \( b_i \neq \bar{a}_i \).

*******************************

If \( (\sigma_1, \cdots, \sigma_n) \) lists the players’ response plans for the choice mechanism \( \pi \), and if player \( i \) knows that \( a_i \) is his true type, then player \( i \)'s expected utility
The payoff is:

\[ W_i(\pi, \sigma_1, \cdots, \sigma_n|a_i) = \sum_{\alpha \in A_1 \times \cdots \times A_n} \sum_{s \in S_1 \times \cdots \times S_n} \sum_{c \in C} P_i(\alpha|a_i) \cdot (\prod_{j=1}^n \sigma_j(s_j|a_j)) \cdot \pi(c|s) \cdot U_i(c, \alpha). \]  

(12)

The vector of conditionally-expected payoffs generated by \((\sigma_1, \cdots, \sigma_n)\) is:

\[ \mathbf{W}(\pi, \sigma_1, \cdots, \sigma_n) = (((W_i(\pi, \sigma_1, \cdots, \sigma_n|a_i))_{a_i \in A_i})^n_{i=1}). \]  

(13)

This is a vector with \(\sum_i^n |A_i|\) components, indexed on the disjoint union of the \(A_i\) sets, like the \(\mathbf{V}(\pi)\). We say that \((\sigma_1, \cdots, \sigma_n)\) is a response-plan equilibrium for the choice mechanism \(\pi\) if, for any player \(i\) and type \(a_i \in A_i\), for every possible alternative response plan \(\sigma'_i\) for player \(i\):

\[ W_i(\pi, \sigma_1, \cdots, \sigma_n|a_i) \geq W_i(\pi, \sigma_1, \cdots, \sigma_{i-1}, \sigma'_i, \sigma_{i+1}, \cdots, \sigma_n|a_i). \]  

(14)

The set of equilibrium-feasible expected allocation vectors is defined to be:

\[ F^{**} = \{ \mathbf{W}(\pi, \sigma_1, \cdots, \sigma_n) : \pi \text{ is a choice mechanism, and} \]

\[ (\sigma_1, \cdots, \sigma_n) \text{ is a response-plan equilibrium for } \pi \}. \]  

(15)

**Theorem 2:** \(F^{**} = F^*\).

**Proof:** If \((\sigma_1, \cdots, \sigma_n)\) is a response-plan equilibrium for a mechanism \(\pi\) on \(S_1, \cdots, S_n\), then we can define an equivalent choice mechanism \(\pi'\) on \(A_1, \cdots, A_n\) by:

\[ \pi'(c|\alpha) = \sum_{s \in S_1 \times \cdots \times S_n} \pi(c|s) \cdot (\prod_{i=1}^n \sigma_i(s_i|a_i)). \]

It is easy to check that \(\mathbf{V}(\pi') = \mathbf{W}(\pi, \sigma_1, \cdots, \sigma_n)\), so that the allocations generated are the same. Furthermore, the equilibrium inequalities (14) for \(\pi\) imply the incentive compatible inequalities (6) for \(\pi'\). Thus \(x = \mathbf{W}(\pi, \sigma_1, \cdots, \sigma_n) \in F^{**}\) implies \(x = \mathbf{V}(\pi') \in F^*\). So \(F^{**} \subseteq F^*\). I omit the rest of proof.

Q.E.D.

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