Linear efficient and symmetric values for TU-games: sharing the joint gain of cooperation

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Abstract

Two well-known single valued solutions for TU-games are the *Shapley value* and *Solidarity value,* which verify three properties: Linearity, Symmetry and Efficiency, and the null player axiom. On the other hand, the interpretation of the two values is usually related on the marginal contribution of a player that joins a coalition.

The paper generalizes the approach. First, the marginal contribution concept is extended to any valued solution that satisfies the three properties. Second, the null player axiom is also generalized and it is shown that any single valued solution satisfying the three properties is uniquely characterized by a null player axiom. In particular, the paper provides new interpretations, in the sense of marginal contribution, for other well-known single values such as Egalitarian value and Consensus value and also offers the opportunity for recasting in extensive form some well-established results.

Keywords: TU-games, single valued solution, Shapley value, marginal contribution, null player axiom.

JEL code: C71, D46, D70

1. Introduction

In this paper, we study, in an extensive form, the problem of sharing the joint gains of cooperation and we characterize in a constructive interpretational approach, the whole set of values for TU-games that verify the three properties: linearity, efficiency and symmetry.

We focus on the interpretation based on the marginal contribution concept such as found in Shapley value and Solidarity value.

The Shapley value takes all coalitions into account and computes for each player the expected marginal contribution. In this way, the Shapley value attributes to the entrant player the whole marginal contribution due to its entrance. In fact, this could be debatable since the marginal contribution is jointly created by the entire coalition but not by the entrant player solely. According to this reasoning, it may be exaggerated in some cases, to give a later entrant the whole marginal value.

The Solidarity value, on the other hand, considers the same scheme of marginal contribution and also computes for each player, the expected marginal contribution. But the difference here is that, the marginal contribution due to the entrance of a player is equally shared by all members of the coalition. Of course there is also critique of this approach. The natural question that arises at this case is: do all members of the coalition merit the same values as the later entrance?

These questions open up the problem of sharing the marginal contribution among the later entrant and the incumbents (original members of the coalition).

The paper considers the same line as the Shapley value and the Solidarity value to solve the problem. The average marginal contribution scheme is maintained, but we propose to share the marginal contribution to the entrant and the incumbents according to the cardinality of the coalition. When a player joins a coalition, he or she takes a part (fraction or multiple) of the marginal contribution and the remainder is equally distributed among the original members of the coalition. The formulation of the part taken by the entrant could change from a coalition to another but are always the same for two coalitions with equal cardinality. Furthermore, we accept the part of the entrant as being a negative or positive multiple of the marginal contribution. If the part of the entrant is greater than the entire marginal contribution, the incumbents will lose in the cooperation with the entrant and they will gain in the opposite case.

Of course, the procedure basically generalizes the Shapley value and the Solidarity value. Also, we obtain new interpretations for others well established solution concepts such as the so called Consensus value and Egalitarian value. According to our approach, the Consensus value considers that, the part taken by the entrant is $\frac{n}{2}$ times its marginal contribution in case the coalition cardinality is 2, and equally shares the marginal contribution with the incumbents in other cases. While in the Egalitarian value, the entrant player takes nothing and the whole marginal contribution is equally share between the incumbents.

The paper shows that our procedure coincides with the whole set of the value for TU-games, that are linear, efficient and symmetric. In other words, we demonstrate that every value satisfying the three properties can be seen as a procedure to split the marginal contributions between the entrant and the incumbents as we have described above; and conversely. The null player axiom is also generalized and we show that any single valued solution satisfying the three properties is associated to a null player model.

The question of finding a characteristic expression of value that are linear, efficient and symmetric has been recently solve by L. Hermandez, R. Juarez, and Sanchez (2008) and C. Chameni and N. Andjiga (2008). The authors gave interpretations for their formulas but they did not relate it to the problem of sharing the joint gains of cooperation.

The paper is organized following three sections in addition to the present one. Section 2 contains some preliminaries and introduces the new entity which reflects the total gain due to the cooperation of a player in a coalition. In section 3, we state the main results of the paper. We show how replacing the marginal contribution of the Shapley value by the total gain due to the coalitional membership of a player yields a characterization of the whole class of values that satisfy linearity, symmetry and efficiency. The general null player axiom is introduced and the discrimination via the axiom is demonstrated. Section 4 concludes the paper.

2. Preliminaries

An n-person game in characteristic function form or a transferable utility game (TU game) is a pair (N, v) where $N = \{1,2,3, \dots, n\}$is the set of n players and v is a mapping $v: 2^n \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$.

For any subset (coalition) of N, $v(S)$ is the worth of coalition S when all the players in S collaborate. The number of elements of S is denoted $|S|$.

Γ denoted the linear space of n-person transferable utility game v on N:

$$\Gamma = \{v/v: 2^n \longrightarrow \mathbb{R}, v(\emptyset) = 0\}$$

It is well known that Γ is a $2^n - 1$ dimension linear space.

A value $\varphi(v) = (\varphi_1(v), \varphi_2(v), \dots, \varphi_n(v))$ on Γ, is a vector-valued mapping $\varphi: \Gamma \longrightarrow \mathbb{R}^n$ which uniquely determines, for each $v \in \Gamma$, a distribution of wealth or payoff available to the players $1,2,3, \dots, n$ through their participation in the game (N, v).

We are ready to introduce basic axioms widely used in the literature. *Efficiency* requires that the payoffs assigned to players sum to all benefits generated by forming the grand coalition. *Symmetry* requires that the only information of players in determining their payoffs is summarized in their participation through the game v so that if two players make the same contribution in v, then they should receive the same amount. Finally, *Linearity* requires that the payoffs assigned to players in a linear combination of two games are the corresponding linear combination of the payoffs assigned to them in each of the two games separately.

- Linearity

  φ is linear if for any games $v, w \in \Gamma$and for any $\lambda, \mu \in \mathbb{R}$:

  $\varphi(\lambda v + \mu w) = \lambda\varphi(v) + \mu\varphi(w)$

- Symmetry

  φ is symmetric if for all $v \in \Gamma$, for any automorphism π of v, $\varphi_i(v) = \varphi_{\pi(i)}(v)$.

  We recall that π is an automorphism of the game $v \in \Gamma$,if $v(\pi(S)) = v(S)$ for each coalition $S \subseteq N$.

- Efficiency

  φ is efficient if for all $v \in \Gamma$, $\sum_{i \in N} \varphi_i(v) = v(N)$.

Next, we introduce a new entity which reflects the total gain due to the cooperation of a player in a coalition. To do this, we define for any non empty coalition $S \subseteq N$ with $|S| = k$, for any $i \in S$ and for any fixed $v \in \Gamma$, the quantity

$$A_i^{\alpha(k)}(S) = \begin{cases} \alpha(k)[v(S) - v(S-i)] + \frac{1-\alpha(k)}{k-1} \sum_{j \in S-i} [v(S) - v(S-j)] & if\ k > 1 \\ \alpha(k)v(i) & if\ k = 1 \end{cases} \quad (1)$$

Where $\alpha(k)_{k=1,2,\dots,n}$ is any real sequence satisfying $\alpha(1) = 1$.

The interpretation of $A_i^{\alpha(k)}(S)$ can be obtained as followed:

When the player i becomes a member of coalition S, he/she shares the marginal contribution $v(S) - v(S - i)$, due to its entrance, with the original members of S. He/She takes a part (fraction or multiple) of this quantity, that is $\alpha(k)[v(S) - v(S - i)]$ and the rest is equally distributed among the original members $S - i$, each of them takes $\frac{1-\alpha(k)}{k-1}[v(S) - v(S - i)]$.

$\alpha(1) = 1$ means that, the player always takes the whole marginal contribution when he/she is alone in the coalition.

Thus, $A_i^{\alpha(k)}(S)$ is the total gain of the player i due to its S-membership when the successive entrances of all players j in $S - j$ are considered.

It is worth noting that $A_i^{\alpha(k)}(S)$ has the same expression for coalitions with the same size. On the other hand, the coefficient $\alpha(k)$, for $k > 1$, could be taken as any real number, negative or positive, greater than 1 or not. Thus, $A_i^{\alpha(k)}(S)$ could be greater or less than $v(S) - v(S - i)$.

If $A_i^{\alpha(k)}(S) < v(S) - v(S - i)$ player i offers some part of its marginal contribution (it is possible that i lost more than the whole marginal contribution) to others members of the coalition. If in contrary, $A_i^{\alpha(k)}(S) > v(S) - v(S - i)$, player i benefits from its

S-membership by obtaining more than its marginal contribution.

Let us now focus on some algebraic properties of the entity $A_i^{\alpha(k)}(S)$. We considerer the particular cases where $(k) = a, k = 2, \dots, n$; $\alpha(1) = 1$ (a is any real number) and denote the corresponding $A_i^{\alpha(k)}(S)$ by $A_i^a(S)$.

Proposition 1: *For any game (N, v), for any coalition $S \ni i$ with $|S| = k$, if a is any real number, $\alpha(k)_{k=1,2,\dots,n}$, and $\beta(k)_{k=1,2,\dots,n}$ real sequences such that $\alpha(1) = 1$ and $\beta(1) = 1$.*

1) $A_i^{a\alpha(k)+(1-a)\beta(k)}(S) = aA_i^{\alpha(k)}(S) + (1-a)A_i^{\beta(k)}(S)$
2) *Consider the product* $a.\alpha(k) = \begin{cases} a\alpha(k) & if\ k > 1 \\ 1 & if\ k = 1 \end{cases}$ *then*

$$A_i^{a.\alpha(k)}(S) = aA_i^{\alpha(k)}(S) + (1-a)A_i^0(S)$$

3) $A_i^a(S) = aA_i^1(S) + (1-a)A_i^0(S)$

Proof: Since 2) and 3) can directly be obtained from 1), we only give the proof of 1).

- For $k > 1$

$A_i^{a\alpha(k)+(1-a)\beta(k)}(S) =$

$a\alpha(k) + (1-a)\beta(k)[v(S) - v(S-i)] + \frac{1-[a\alpha(k)+(1-a)\beta(k)]}{k-1}\sum_{j\in S-i}[v(S) - v(S-j)]$

$a\left[\alpha(k)[v(S) - v(S-i)] + \frac{1-\alpha(k)}{k-1}\sum_{j\in S-i}[v(S) - v(S-j)]\right] + \frac{1-a}{k-1}\sum_{j\in S-i}[v(S) - v(S-j)] + (1-a)\left[\beta(k)[v(S) - v(S-i)] + \frac{1-\beta(k)}{k-1}\sum_{j\in S-i}[v(S) - v(S-j)]\right] - \frac{1-a}{k-1}\sum_{j\in S-i}[v(S) - v(S-j)]$

$= a\left[\alpha(k)[v(S) - v(S-i)] + \frac{1-\alpha(k)}{k-1}\sum_{j\in S-i}[v(S) - v(S-j)]\right] +$

$(1-a)\left[\beta(k)[v(S) - v(S-i)] + \frac{1-\beta(k)}{k-1}\sum_{j\in S-i}[v(S) - v(S-j)]\right]$

$= aA_i^{\alpha(k)}(S) + (1-a)A_i^{\beta(k)}(S)$

- For $k = 1$,

$A_i^{a\alpha(k)+(1-a)\beta(k)}(S) = v(i) = av(i) + (1-a)v(i) = aA_i^{\alpha(k)}(S) + (1-a)A_i^{\beta(k)}(S)$. ■

3. Definitions and main results

We define the value of the TU-game (N, v) by replacing in the Shapley value expression, the marginal contribution when a player joins a coalition S by the total gain defined in (1):

$$\varphi_i^{\alpha}(N, v) = \sum_{k=1}^{n}\sum_{\substack{S\ni i\\ |S|=k}} \frac{(n-k)!\,(k-1)!}{n!} A_i^{\alpha(k)}(S) \qquad (2)$$

Indeed, φ^{α} defines a family of values based on the parameter α where $\alpha(k)_{k=1,2,\dots,n}$ is any real sequence satisfying $(1) = 1$. One of the main results of this paper is to show that, the family coincides with the set of linear, symmetric and efficient values. But, before tackle the issue, let us focus on algebraic properties of φ^{α} and some computational classical examples.

The following properties are straightforward from proposition 1:

Proposition 2: *For any game* (N, v), *for any player* $i \in N$, *if* a *is any real number,* $\alpha(k)_{k=1,2,\dots,n}$, *and* $\beta(k)_{k=1,2,\dots,n}$ *real sequences such that* $\alpha(1) = 1$ *and* $\beta(1) = 1$.

1) $\varphi_i^{a\alpha+(1-a)\beta}(v) = a\varphi_i^{\alpha}(v) + (1-a)\,\varphi_i^{\beta}(v)$
2) $\varphi_i^{a\alpha}(v) = a\varphi_i^{\alpha}(v) + (1-a)E_i(v)$
3) $\varphi_i^{\alpha}(v) = aSh_i(v) + (1-a)E_i(v)$

Where Sh and E are respectively the Shapley and the Egalitarian value $\left(E_i(v) = \frac{v(N)}{n}\right)$

It is clear that the procedure generalizes the well established values such as Shapley value, Solidarity value and Consensus value; they corresponding $\alpha(k)_{k=1,2,\dots,n}$ are computed in the following table.

Value Name	$\alpha(1)$	$\alpha(2)$	…	$\alpha(k)$	…	…	$\alpha(n)$	**Interpretation**
Shapley	1	1	…	1	…	1	1	The entrant player takes the whole marginal contribution and gives nothing to the incumbents
Equal division or Egalitarian	1	0	…	0	…	0	0	The entrant player takes nothing and the whole marginal contribution is equally share between the incumbents, except the case where he is alone.
Solidarity	1	$\frac{\mathbf{1}}{\mathbf{2}}$	…	$\frac{\mathbf{1}}{\boldsymbol{k}}$	…	…	$\frac{\mathbf{1}}{\boldsymbol{n}}$	The entrant player equally shares the marginal contribution with the incumbents.
Consensus	1	$\frac{\boldsymbol{n}}{\mathbf{2}}$	$\frac{\mathbf{1}}{\mathbf{2}}$	…	$\frac{\mathbf{1}}{\mathbf{2}}$	…	$\frac{\mathbf{1}}{\mathbf{2}}$	The entrant takes $\frac{\boldsymbol{n}}{\mathbf{2}}$ times the marginal contribution in case the coalition cardinality is 2, and takes half the marginal contribution in other cases
C.I.S. or Equal surplus	1	$\boldsymbol{n}-\mathbf{1}$	0	…	0	…	0	The entrant takes $\boldsymbol{n}-\mathbf{1}$ times the marginal contribution in case the coalition cardinality is 2, and takes nothing in other cases.

Note that a similar table is found in C. Chameni and N. Andjiga (2008) but with the difference that the coefficients, though mathematically close, have neither the same substance nor the same interpretation. Here the first column of the table is full of one's while in the referenced tables it is the last column which is ones.

We can now state our first main result:

Theorem1*: Let φ be a value on Γ, φ is linear, symmetric and efficient if and only if there exist a unique sequence $\alpha(k)_{k=1,2,\ldots,n}$ with $\alpha(1) = 1$ such that:*

$$\varphi_i(v) = \varphi_i^{\alpha}(v) = \sum_{\substack{S \ni i \\ |S|=k}} \frac{(n-k)!(k-1)!}{n!} A_i^{\alpha(k)}(S).$$

In others words, φ can be seen as a procedure to distribute the marginal contribution among the entrant and the original members of a coalition.

Proof: For our result, we will use the following lemma recently established in the literature

(L. Hermandez, R. Juarez, and Sanchez (2008)) or (C. Chameni and N. Andjiga (2008)).

Lemma 1: *A value φ on Γ is linear, symmetric and efficient if and only if there exist a unique sequence $\beta(k)_{k=1,2,\ldots,n-1}$ such that:*

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{k=1}^{n-1} \left(\frac{(n-k)!(k-1)!}{n!} \beta(k) \sum_{\substack{S \ni i \\ |S|=k}} v(S) - \frac{(n-k-1)!k!}{n!} \beta(k) \sum_{\substack{S \not\ni i \\ |S|=k}} v(S) \right) \qquad (3)$$

Thus, the proof of the theorem consists in breaking down the expression of $\varphi_i^{\alpha}(N, v)$ in (2) to obtain the expression (3). To do this, we evaluate in (2) the total coefficient of $v(S)$ for each coalition S.

Consider a coalition S with $|S| = k$

- First, $if\ k = n$, S is the grand coalition and then $i \epsilon S$; therefore, $v(S)$ is present in the two terms of the sum in $A_i^{\alpha(k)}(S)$, so the total coefficient of $v(S)$ in (2) is
$$\frac{(n-n)!(n-1)!}{n!} \left[\alpha(n) + \frac{1-\alpha(n)}{n-1}(n-1) \right] = \frac{1}{n}.$$

- Second, $if\ k = 1$, two alternatives are possible :

- If $i \epsilon S$, $S = \{i\}$ and can be also obtained of the form $S = \{i, j\} - j$ with n-1 possibilities for that, so the total coefficient of $v(S)$ in (2) is :

$$\frac{(n-1)!(1-1)!}{n!}\alpha(1)-\frac{(n-2)!(2-1)!}{n!}\frac{1-\alpha(2)}{2-1}(n-1)=\alpha(2)\frac{(n-1)!(1-1)!}{n!}=\frac{(n-k)!(k-1)!}{n!}\alpha(k+1)$$

- $i \notin S, S=\{i,j\}-i$, the total coefficient of $v(S)$ in (2) is $-\frac{(n-2)!(2-1)!}{n!}\alpha(2)=-\frac{(n-k-1)!k!}{n!}\alpha(k+1)$.

- Third, $1<k<n$, there are also two alternatives:

- If $i \epsilon S$, S can be also obtained of the form $S=S'-j$ where $|S'|=k+1$; there are n-k possibilities for that, so the total coefficient of $v(S)$ in (2) is :

$$\frac{(n-k)!(k-1)!}{n!}[\alpha(k)+(1-\alpha(k))]-\frac{(n-k-1)!k!}{n!}\frac{1-\alpha(k+1)}{k}(n-k)=\frac{(n-k)!(k-1)!}{n!}\alpha(k+1).$$

- If $i \notin S,\ \ S=S'-i$ with $|S'|=k+1$, the total coefficient of $v(S)$ in (2) is : $-\frac{(n-k-1)!k!}{n!}\alpha(k+1)$

Finally, if $i \epsilon S$ with $|S|=k<n$, the total coefficient of $v(S)$ is $\frac{(n-k)!(k-1)!}{n!}\alpha(k+1)$ and if $i \notin S$, the total coefficient is $-\frac{(n-k-1)!k!}{n!}\alpha(k+1)$; for $k=n$, the coefficient of $v(N)=\frac{1}{n}$.

It comes that,

$$\varphi_i^{\alpha}(N,v)=\frac{v(N)}{n}+\sum_{k=1}^{n-1}\left(\frac{(n-k)!(k-1)!}{n!}\alpha(k+1)\sum_{\substack{S\ni i\\|S|=k}}v(S)-\frac{(n-k-1)!k!}{n!}\alpha(k+1)\sum_{\substack{S\not\ni i\\|S|=k}}v(S)\right)$$

Hence, $\varphi_i^{\alpha}(N,v)$ can be put of the form (3) with the relation:

$$\beta(k)=\alpha(k+1), k=1,2,\dots,n-1 \qquad \blacksquare$$

Remark 1: It is worth noting that, the proof of lemma 1 clearly establishes a direct link between the two sequences $\beta(k)_{k=1,2,\dots,n-1}$ and $\alpha(k)_{k=2,\dots,n}$. This in particular offers the possibility for recasting, by using $\alpha(k)_{k=2,\dots,n}$,all the properties (such as monotonicity, covariant property, duality etc.) found in the literature and based on $\beta(k)_{k=1,2,\dots,n-1}$. We will not focus on to the matter as we would like to go straight to our main results.

Having generalized the marginal contribution concept, let us now examine the null player axiom introduced in both the Shapley value and the Solidarity value. Of course we start by a general definition of the null player.

Definition 1:

For any sequence $\alpha(k)_{k=1,2,\ldots,n}$ satisfying $\alpha(1) = 1$, for any game $(N,v) \in \Gamma$. A player $i \in N$ is considered as $\alpha - A$ null player in the game (N,v) if for any coalition $S \ni i$ $with$ $|S| = k$, $A_i^{\alpha(k)}(S) = 0$.

Remark 2: The $\alpha - A$ null player defined here joints together most of the null player models found in the literature. If $\alpha(k) = 1, k = 1,2,\ldots,n$, we obtain the null player of Shapley(1953) value; if $\alpha(k) = \frac{1}{k}, k = 1,2,\ldots,n$, it is the null player of Solidarity value (A.S.Nowack and Tadeusz Radzik ;1994) that is obtained and if $\alpha(k) = 0, k = 2,\ldots,n$; $\alpha(1) = 1$, it is easy to check that we fall on the *zero player* that is the null player of the Egalitarian value (René Van Den Brink, 2007).

Next, we have the general null player axiom :

A value φ on Γ satisfies the $\alpha - A$ null player axiom, if $i \in N$ is a $\alpha - A$ null player in the game (N,v) then $\varphi_i(v) = 0$.

Our purpose now is to demonstrate the discriminatory property of the $\alpha - A$ null player axiom. In other words, we show that φ^α is the unique value that satisfies linearity, efficiency, symmetric and the $\alpha - A$ null player axiom.

To reach our goal, we need to state two lemmas as follows. The first lemma gives, for any sequence $\alpha(k)_{k=1,2,\ldots,n}$ satisfying $\alpha(1) = 1$, a basis $\{v_T^\alpha : T \subseteq N, T \neq \emptyset\}$ for the linear space Γ that can be considered as a canonical basis associated to the value φ^α.

Lemma 2: *For any sequence* $\alpha(k)_{k=1,2,\ldots,n}$ *satisfying* $\alpha(1) = 1$, *for any non empty coalition* T, *we consider the game*

$$v_T^\alpha(S) = \begin{cases} \prod_{p=t+1}^{k}\left(1 - \frac{1-\alpha(p)}{p-1}t\right) & if\ T \subset S\ ,\ |S| = k > |T| = t \\ 1 & if\ \ T = S \\ 0 & otherwise \end{cases} \qquad (4)$$

Then the family $B = \{v_T^\alpha : T \subseteq N, T \neq \emptyset\}$ *of games is a basis for the linear space* Γ.

Proof:

The proof is classical and will be led as made by A.S.Nowack and Tadeusz Radzik (1994) in the case of Solidarity value.

Let $T_1, T_2, \dots T_K$,with $K = 2^n - 1$, be a sequence containing all the non empty subsets of N such that 1= $|T_1| \leq |T_2| \leq, \dots \leq |T_K| = n$.

Consider the KxK matrix $M^\alpha = [m_{lq}^\alpha]$ defined by $m_{lq}^\alpha = v_{T_l}^\alpha(T_q)$ $l, q = 1,2, \dots, K$.

It is clear that M^α is a triangle matrix with all diagonal entries equal to 1. Thus Det$M^\alpha \neq 0$ and it is immediate that the family $B = \{v_T^\alpha : T \subseteq N, T \neq \emptyset\}$ constitutes a set of K linear independent vectors of Γ, thus a basis on Γ. ■

Remark 3 : It might be interesting to explain the source of the basis $B = \{v_T^\alpha : T \subseteq N, T \neq \emptyset\}$. In fact, B basically generalizes some well-known and classical basis found in the literature. For the sequence $(k) = 1, k = 1,2, \dots, n$, it is easy to see that, v_T^α is the unanimity game associated to the coalition T ; thus B is the basis introduced by L.S.Shapley in the case of Shapley value. If $(k) = \frac{1}{k}, k = 1,2, \dots, n$, $v_T^\alpha(S) = \begin{cases} \binom{|S|}{|T|}^{-1} & if\ T \subseteq S \\ 0 & otherwise \end{cases}$ and then B is the basis introduced by A.S.Nowack and Tadeusz Radzik in the case of Solidarity value.

The second lemma establishes particular properties for the basis defined in (4).

Lemma 3: *For any sequence* $\alpha(k)_{k=1,2,\dots,n}$ *satisfying* $\alpha(1) = 1$, *for any non empty coalition* T, v_T^α *has the following properties:*

1) *For any player* $i \in N - T$, *for any coalition* $S \ni i$, *with* $|S| = k$,
$$v_T^\alpha(S) = \left(1 - \frac{1-\alpha(k)}{k-1} t\right) v_T^\alpha(S - i)$$
2) *Every player* $i \in N - T$ *is* $\alpha - A$ *null in the game* v_T^α.

Proof:

1) Consider a player $i \in N - T$; for any coalition $S \ni i$, with $|S| = k$,

- If $T \nsubseteq S$, then $v_T^\alpha(S) = v_T^\alpha(S - i) = 0$ and the assertion is true.
- If $T \subseteq S$ then $T \subseteq S - i$ and $k > t$. Two alternatives are possible

   a) $k > t + 1$

$$v_T^\alpha(S) = \prod_{p=t+1}^{k} \left(1 - \frac{1-\alpha(p)}{p-1} t\right) = \left(1 - \frac{1-\alpha(k)}{k-1} t\right) \prod_{p=t+1}^{k-1} \left(1 - \frac{1-\alpha(p)}{p-1} t\right)$$
$$= \left(1 - \frac{1-\alpha(k)}{k-1} t\right) v_T^\alpha(S - i).$$

b) $k = t + 1$

Then $T = S - i$ and $v_T^\alpha(S - i) = 1$

$v_T^\alpha(S) = \left(1 - \frac{1-\alpha(t+1)}{t+1-1} t\right) = \left(1 - \frac{1-\alpha(k)}{k-1} t\right) = \left(1 - \frac{1-\alpha(k)}{k-1} t\right) v_T^\alpha(S - i).$

2) For any player $i \in \mathrm{N} - \mathrm{T}$, for any coalition $\mathrm{S} \ni \mathrm{i}$, with $|\mathrm{S}| = \mathrm{k}$, we have:

- If $k = 1$,
    - $T \nsubseteq S \Longrightarrow v_T^\alpha(S) = 0$, thus $A_i^{\alpha(k)}(S) = v_T^\alpha(S) = 0$.
    - $T \subseteq S$ is impossible since $i \notin T$.

- If $k > 1$,
    - $T \nsubseteq S \Longrightarrow T \nsubseteq S - j$ for all $j \in$ N, thus $v_T^\alpha(S) = v_T^\alpha(S - j) = 0$

$$A_i^{\alpha(k)}(S) = \alpha(k)[v_T^\alpha(S) - v_T^\alpha(S - i)] + \frac{1 - \alpha(k)}{k - 1} \sum_{j \in S - i} [v_T^\alpha(S) - v_T^\alpha(S - j)] = 0$$

- $T \subseteq S$

$$A_i^{\alpha(k)}(S) = \alpha(k)[v_T^\alpha(S) - v_T^\alpha(S - i)] + \tfrac{1-\alpha(k)}{k-1} \textstyle\sum_{j \in S-i} [v_T^\alpha(S) - v_T^\alpha(S - j)]$$

$$= v_T^\alpha(S) - \alpha(k) v_T^\alpha(S - i) - \tfrac{1-\alpha(k)}{k-1} \textstyle\sum_{j \in S-i} v_T^\alpha(S - j)$$

Since $v_T^\alpha(S - j) = v_T^\alpha(S - i)$ for all $j \in S - T$, and $v_T^\alpha(S - j) = 0$ for all $j \in T$,

$$A_i^{\alpha(k)}(S) = v_T^\alpha(S) - \alpha(k) v_T^\alpha(S - i) - \tfrac{1-\alpha(k)}{k-1} (k - 1 - t) v_T^\alpha(S - i)$$

$$= v_T^\alpha(S) - v_T^\alpha(S - i) + t\,\tfrac{1-\alpha(k)}{k-1} v_T^\alpha(S - i)$$

$$= v_T^\alpha(S) - \left(1 - \tfrac{1-\alpha(k)}{k-1} t\right) v_T^\alpha(S - i)$$

$= 0$ according to the property *1)* of the lemma. ■

We are now ready to state the second main result of the paper:

Theorem 2*: Let φ be a value on Γ, φ is linear, symmetric , efficient and* satisfies the $\alpha - A$ null player axiom *if and only if* $\varphi = \varphi^\alpha$.

Proof:

Let φ be a value on Γ, if $\varphi = \varphi^{\alpha}$, it is clear that φ is linear, symmetric , efficient and satisfies the $\alpha - A$ null player axiom.

Conversely. If φ is linear, symmetric, efficient and satisfies the $\alpha - A$ null player axiom, then for any non empty coalition T, with $|T| = t$; by efficiency ,symmetry and *2)* of lemma 3, we have:

$$\varphi_i^{\alpha}(v_T^{\alpha}) = \varphi_i(v_T^{\alpha}) = \begin{cases} \dfrac{v_T^{\alpha}(N)}{t} & if\ i \in T \\ 0 & if\ \ i \notin T \end{cases}$$

Thus, φ and φ^{α} coincide on the basis B=$\{v_T^{\alpha}: T \subseteq N, T \neq \emptyset\}$; therefore $\varphi = \varphi^{\alpha}$. ■

Theorem 2 clearly generalizes the case of Shapley value and Solidarity value. On the other hand, the proof of the theorem suggests an extension of its assertion and we actually deal with the issue. Let us denote $A_0^{\alpha}(v) = \{i \in N | i\ is\ \alpha - A\ null\ in\ the\ game\ v\}$, then we have the following result.

Theorem 3: *Let φ and ψ be any two values on Γ which are linear, symmetric, and efficient.*

Then, $\varphi = \psi$ if and only if φ and ψ coincide on $A_0^{\alpha}(v)$ for any $v \in \Gamma$.

Proof:

Following the proof of theorem 2, we lead to φ *and* ψ coincide on the basis $B = \{v_T^{\alpha}: T \subseteq N, T \neq \emptyset\}$; thus $\varphi = \psi$ ■

Corollary: *Let φ and ψ be any two values on Γ which are linear, symmetric and efficient.*

Then, for any $v \in \Gamma$, for any reel number a,

$$(\ \psi_i(v) = a\varphi_i(v)\ \ for\ any\ \alpha - A\ null\ null\ player\ i) \Leftrightarrow \psi = a\varphi + (1-a)\varphi^{\alpha}$$

Proof:

If $i \in A_0^{\alpha}(v)$, $\psi_i(v) = a\varphi_i(v) = a\varphi_i(v) + (1-a)\varphi_i^{\alpha}(v)$ *since* $\varphi_i^{\alpha}(v) = 0$. *Hence* ψ *and* $a\varphi + (1-a)\varphi^{\alpha}$ *coincide on* $A_0^{\alpha}(v)$*; thus* $\psi = a\varphi + (1-a)\varphi^{\alpha}$. ■

Note that the corollary may be useful for characterizing the class of solutions values that are obtained as convex combinations of others values (see Brink, R. Van Den and Yukihiko Funaki, 2008).

4. Conclusion:

The class of values for transferable utility games defined on (N, v) that are simultaneously linear, symmetric and efficient is wide and defines a $|N| - 1$ dimension affine space. It contains, among others, the well-known values such as Shapley value, Solidarity value, Egalitarian value and Consensus value etc. A better understanding patterns of the class of values is essential to understand the difference (similarity) among them and the difference among the members in the class of values and others values.

In this paper we have outlined the common expression of values which satisfy the three properties. In particular, we have obtained that every value in the class can be put in the random order form, similar to the well-known form of the Shapley value and Solidarity value. In other words, we have demonstrated that every value satisfying the three properties can be seen as a procedure to distribute the marginal contributions between the player who joins a coalition and the initial members of the coalition. The generalization of the null player axiom has been established and we have shown that every value in the class is connected to a null player model. The discrimination by the treatment of null players has been studied; and we have shown that: the answer to the question *What does a value give to null players?* Uniquely characterizes the value. Finally the paper offers the opportunity for recasting in extensive form some well-established results such as those concerning the characterization of the class of solutions values that are obtained as convex combinations of others values.

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