More Properties about Odd Perfect Numbers

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Abstract: As shown by Euler an odd perfect number n must be of the form $n = p^{\alpha}m^2$ where $p \equiv \alpha \equiv 1\ (mod\ 4)$ and p is called the special prime. In this work we show that $p \geq 13$ and if $q \in \{3,5\}$ and $q|n$ then either $\gcd\left(q, \sigma(m^2)\right) = 1$ or $\gcd\left(q, \sigma(p^{\alpha})\right) = 1$.

Keywords: perfect numbers, odd perfect numbers, special prime, greatest common divisor.

1. **INTRODUCTION:** In 1975 G. G. Dandapat, J. L. Hunsucker and Carl Pomerance showed that if n is a is a multiply perfect number $(\sigma(n) = tn\ for\ some\ integer\ t)$, then they asked if there is a prime p with $n = p^{\alpha}k$, $\gcd(p^{\alpha}, k) = 1, \sigma(p^{\alpha}) = tk\ and\ \sigma(k) = p^{\alpha}$. They proved that the only multiply perfect numbers with this property are the even perfect numbers and 672. Hence they settled a problem raised by Suryanarayana who asked if odd perfect numbers necessarily had such a prime factor. This result yields immediately that if $n = p^{\alpha}m^2$ is an o.p.n then $\sigma(p^{\alpha}) \neq 2m^2 and\ \sigma(m^2) \neq p^{\alpha}$. Using $Little\ Fermat\ Theorem$ we obtain the results outlined in the abstract above.

2. **PRELIMINARY RESULTS:**

Theorem (Little Fermat Theorem): If p is a prime number and a is a natural number then,

$$a^p \equiv a\ (mod\ p)$$

Furthermore, if $p \nmid a$, then there exists some smallest exponent d such that

$$a^d - 1 \equiv 0\ (mod\ p)$$

and $d|(p-1)$ therefore,

$$a^{p-1} - 1 \equiv 0\ (mod\ p)$$

Proof: We use induction on a for a proof. Assume that $a^p \equiv a\ (mod\ p)$. Consider the expression $(a+1)^p$, from the binomial theorem one gets,

$$(a+1)^p = a^p + \binom{p}{1}a^{p-1} + \binom{p}{2}a^{p-2} + .. + \binom{p}{p-1}a + 1$$

Rewritten in another form,

$$(a+1)^p - a^p - 1 = \binom{p}{1}a^{p-1} + \binom{p}{2}a^{p-2} + .. + \binom{p}{p-1}a \equiv 0\ (mod\ p)$$

Thus,

$$(a+1)^p - a^p - 1 \equiv a^p - a \equiv 0\ (mod\ p) => (a+1)^p - (a+1) \equiv 0\ (mod\ p)$$

And the theorem is proved.

3. MORE POPERTIES ABOUT O.P.N:

Proposition 3.1: Let $n = p^\alpha m^2 = p^\alpha \prod_{i=1}^{k} q_i^{2\beta_i}$ be an o.p.n. If for some $i \in [1, k], q_i | m^2$, and $q_i | \sigma(p^\alpha)$ then $q_i | \sigma\left(q_j^{2\beta_j}\right)$ for $i \neq j$. Furthermore all such q_i are greater or equal to 7 or 13 for all $q_i \equiv 3 \ (mod\ 4)$ and $q_i \equiv 1 \ (mod\ 4)$ respectively.

Proof: Since $\sigma(p^\alpha) \neq 2m^2 and\ \sigma(m^2) \neq p^\alpha$ then it follows that there exists some $i \in [1, k]$such that $q_i | m^2, q_i | \sigma(p^\alpha)$ and $q_i | \sigma\left(q_j^{2\beta_j}\right)$. Therefore,

$$\left(1 + q_j + .. + q_j^{2\beta_j}\right) \equiv 0 \ (mod\ q_i) \ \Leftrightarrow \ q_j^{2\beta_j + 1} \equiv 0 \ (mod\ q_i)$$

By Little Fermat Theorem $q_j^{q_i - 1} \equiv 1 \ (mod\ q_i)$ and since $q_i \nmid q_j$ then either $(q_i - 1) | (2\beta_j + 1)$ or $\left(2\beta_j + 1\right) | (q_i - 1)$. However one observes that $(q_i - 1) \equiv 0 \ (mod\ 2)$ thus $(q_i - 1) \nmid (2\beta_j + 1)$. We are left with the only case

$$\left(2\beta_j + 1\right) | (q_i - 1) \Leftrightarrow k\left(2\beta_j + 1\right) = (q_i - 1) \ for\ some\ integer\ k$$

At this point, if $q_i \equiv 3 \ (mod\ 4) \Rightarrow k \equiv 2 \ (mod\ 4) \Rightarrow k \geq 2 \Rightarrow (q_i - 1) \ \geq 2\left(2\beta_j + 1\right) \geq 6 \Rightarrow q_i \geq 7$.
And if $q_i \equiv 1 \ (mod\ 4) \Rightarrow k \equiv 0 \ (mod\ 4) \Rightarrow k \geq 4 \Rightarrow (q_i - 1) \ \geq 4\left(2\beta_j + 1\right) \geq 12 \Rightarrow q_i \geq 13$.

Proposition 3.2: Let $n = p^\alpha m^2$ with $p \equiv \alpha \equiv 1 \ (mod\ 4)$ be an o.p.n then $p \geq 13$.

Proof: $\sigma(p^\alpha m^2) = 2p^\alpha m^2 \ \Rightarrow \ \sigma(p^\alpha)\sigma(m^2) = 2p^\alpha m^2 \Rightarrow p^\alpha || \sigma(m^2)$ therefore there exists some prime $q | m^2$ such that $\sigma\left(q^{2\beta}\right) \equiv 0 \ (mod\ p) \Rightarrow q^{2\beta + 1} \equiv 1 \ (mod\ p)$. Using again Little Fermat Theorem yields that

$$k(2\beta + 1) = p - 1$$

Since $p \equiv 1 \ (mod\ 4) \Rightarrow k \equiv 0 \ (mod\ 4) \Rightarrow k \geq 4 \Rightarrow p \geq 13$.

Proposition 3.3: Let n is an o.p.n and if $q \in \{3,5\}$ and $q | n$ then either $\gcd\left(q, \sigma(m^2)\right) = 1$ or $\gcd\left(q, \sigma(p^\alpha)\right) = 1$.

Proof: It is almost immediate from proposition 3.1 that all prime that divide both $\sigma(p^\alpha)$ and $\sigma(m^2)$ must be greater or equal to 7 or 13. Therefore if this prime is 3 or 5 then it must divide only $\sigma(p^\alpha)$ or $\sigma(m^2)$.

4. REFERENCES:

1.G.H.Hardy, E.M. Wright, An Introduction to the Theory of Numbers, Sixth Edition, Oxford University Press, 2008 (revised by D.R. Heath-Brown and J.H. Silverman).

2. G. G. Dandapat, J. L. Hunsucker, Carl Pomerance, Some New Results on Odd Perfect Numbers, Pacific Journal of Mathematics, Vol.57, No.2, 1975.

3. Wolfram MathWorld, Perfect Numbers, http://mathworld.wolfram.com/PerfectNumber.html, 2011.