A nonparametric hypothesis test via the Bootstrap resampling

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Abstract

This paper adapts an already existing nonparametric hypothesis test to the bootstrap framework. The test utilizes the nonparametric kernel regression method to estimate a measure of distance between the models stated under the null hypothesis. The bootstraped version of the test allows to approximate errors involved in the asymptotic hypothesis test. The paper also develops a Mathematica Code for the test algorithm.

JEL: C12, C14, C15

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1. Introduction[1]

Recently, there has been wide interest in testing the significance of a subset of explanatory variables through a nonparametric regression technique. Although this technique generates estimates robust to misspecification, their precision varies inversely with the number of explanatory variables (see Härdle, 1991; Cao-Abad, 1991; Yatchew, 1998; among others), and hence parsimony is important when the nonparametric regression is applied. Researchers have been aware of this pitfall for some time now, only recently, however, have they turned to developing hypothesis testing procedures to identify those variables significant.

The most recent studies in the literature adopt the nonparametric kernel regression technique as the main element of the hypothesis testing procedure. For instance, Delgado and Manteiga (1999) and Fan and Li (1996) propose tests based on the conditional expectation function given only those variables which are significant under the null hypothesis. The latter opts for the asymptotic approach, while the former develops a test adopting the bootstrap framework because of an analytically intractable distribution for the test statistic constructed. Exploiting a traditional approach, Lavergne and Vuong (1996) base their test on the empirical mean-squared error, which is very often adopted in parametric hypothesis tests, from the kernel regression, and suggest a consistent test for discriminating between two sets of regressors. Remaining in spirit of Lavergne and Vuong's approach, Hall and Hart (1990) construct a test for differences between means in nonparametric regressions and adapt it to the bootstrap resampling scheme. More consistent tests are introduced by Lewbel (1995) who tests Slutsky symmetry using U.K. survey data and by Gozalo (1993) who constructs a theoretical test for omitted variables. Last but not least, Yatchew (1992) proposes a test based on comparison of unrestricted and restricted sums of squares, using residuals from the nonparametric regression model. One can easily extend the list at will.

Often adopted in the literature on nonparametric hypothesis tests has been the approach that approximates the finite-sample null distribution ($f.s.n.d.$) of a test statistic by its asymptotic distribution. This is unfortunate since such approximations are usually subject to serious errors when the empirical distribution of the observed sample significantly departs from the true unknown distribution (Singh, 1981; Hall and Horowitz, 1996). Efron (1979) offers the bootstrap re-

[1]The author would like to thank Michiel Keyzer not only for his commensts and suggestions on the earlier version of the paper but also for his enthusiastic ideas that pulled my attention towards this exponentially growing area of econometrics.

sampling scheme as a way to avoid the approximation errors and to identify some unknown characteristics of the test statistic associated with the observed sample.[2] The bootstrap assumes that the unknown relationship between the population and the actual sample is preserved in the relationship between the actual sample and the bootstraped samples. Accordingly, the $f.s.n.d.$ is approximated through the bootstraped distribution based on resamples from the actual sample. To date, the bootstrap technique has found many useful applications especially in situations where variance or confidence limits of test statistics cannot at all or can only with undue effort be calculated by analytical means or where test statistics depend on unknown characteristics of the underlying distribution of variables of interest or where there is a need to estimate the distribution of test statistics in high dimensional linear models or to measure the goodness of fit of a regression model (Delgado and Manteiga, 1999; Stute, Manteiga, and Quindimil, 1998; Mammen, 1993; among others).

The present study aims at adapting Fan and Li's (1996) nonparametric asymptotic hypothesis test procedure to the bootstrap framework as this framework, as shown by Hall and Horowitz (1996), promises smaller approximation errors than those associated with the asymptotic analysis. Such adaptation is relevant because, as proven by Huskova and Janssen (1993), the bootstrap is consistent for degenerate $U-statistics$ which are the building blocks of the nonparametric kernel regression we apply. The current study contributes the literature through the adaptation of an asymptotic test to the bootstrap framework.

The rest of this study is organized as follows. In the following section we discuss the main approaches adopted in hypothesis testing and outline the main advantages of the bootstrap over the asymptotic approximation. In Section 3, we describe and modify Fan and Li's test. Section 4 explains how to calculate the bootstrap test statistic and how to make a decision by using it. Finally, Section 5 concludes the paper.

[2] Here is the simplest example to illustrate why and how the bootstrap is applied. Suppose that a real-valued parameter, η, for example, the unknown population mean of n $i.i.d.$ random variables, can be written as a functional of some common cumulative distribution function F; that is, $\eta = \eta(F)$. The objective is to obtain information from the actual sample $\{X_i : i = 1, ..., n\}$ on η. Put differently, a relationship is sought between η and the sample, and the bootstrap method approximates this relation by utilizing the relationship between $\hat{\eta}$ (i.e., the sample mean) and a bootstrap sample $\{X_i^* : i = 1, ..., n\}$. But, as is clearly seen, this approximation is possible only through the approximation of F, and hence the bootstrap is applied to approximate F.

2. Motivation for the bootstrap

How robust is a decision if it is based only on one sample of data? This has been the most prominent research question in statistical theory and still occupies the first seat in the theory. To date, decision rules have been constructed as follows. First of all, the sample at hand is reduced to a single observation, which is in statistical theory called test statistic; next, the asymptotic distribution of that statistic is obtained as the number of observations goes to infinity; and finally, an arbitrary confidence level, conventionally set at the 5 percent, is used to make a decision as to whether the statement under the null hypothesis is valid. What is unfortunate in this context is the fact that that single test statistic contains no information in the continuous sample space, giving rise to different approaches to the development of a robust decision rule.

The classical and Bayesian approaches are often adopted in the literature as a way to bridge the gap between the actual observation and a decison rule. The fundamental difference between these approaches lies in the way the parameters of the model of interest are treated. The classical approach treats them as unknown constants to be estimated, and the OLS method provides the best linear unbiased estimators. These estimators are then evaluated for qualities, such as unbiasedness and consistency, by repeated sampling from the population assumed to be available. On the contrary, the Bayesian approach treats them as random variables about which the analyst has or can obtain information before observing the actual sample. This information, called prior information, is characterized by a prior probability distribution. The task then becomes to incorporate this information into the analysis, but unfortunately its update might vary across individuals.

Research on establishing "good" decison rules has not yet been conclusive because in practice the population is very often unavailable and because prior information makes inferences highly subjective. A proper interpretation of a single test statistic requires knowledge of its $f.s.n.d.$, which is available only in very specific and less realistic circumstances, and it seems that it would be wrong if such knowledge is derived from approximations through its asymptotic distribution. There are few good reasons not to rely on the asymptotic approximations. First, asymptotic theory pertains to the hypothetical situation of infinitely many observations, while, in fact, there are only few observations. Second, asymptotic distributions are independent of any feedback mechanism, whereas the $f.s.n.d.$ of a test statistic is in general affected by such mechanisms. Third, various types of misspecifications, such as wrong distributional assumptions and dynamic misspecification, may have important effects on the accuracy of asymptotic distributions.[3] All in all, in his paper, Efron shows that the bootstrap might offer some insights when situations of the above kinds arise. In this paper, we give it a try to see whether the bootstrap really generates better results relative to the asymptotic results.

3. A nonparametric hypothesis test procedure

A statistical test is a decision problem involving unknown parameters that must lie in a certain parameter space. However, this parameter space can be divided into two disjoint subsets, and one must figure out, perhaps using a random sample of data, the subset that is more likely to contain the unknown parameters. Following Rabinson (1989), we develop a hypothesis test that involves the seven main steps. In the first step we specify a data-generating process to characterize the data at hand. A model is constructed in the second step. The hypotheses of interest are formulated in the third step: a null hypothesis is maintained until evidence to the contrary is shown, and an alternative hypothesis is adopted if the null is rejected. In the fourth step we establish asymptotic distributions of distance measures implied by the two hypotheses. A test statistic is defined in the fifth step - a single condensed value that has a known distribution under the null and has some other distribution under the alternative hypothesis. The test is carried out using this single statistic rather than by considering the multidimensional sample space. In the sixth step we define a critical region associated with those values of the test statistic for which the null will be rejected. Finally, we establish a decision rule. In the subsequent paragraphs these steps are explored.

Step 1. The data-generating process (DGP)

Assumption 1. Let $\{(Y, X)\}$ be an independent and identically distributed *(i.i.d.)* random sample (r.s.) of n observations drawn from $(1+k)-$ dimensional distribution with density $f(.,.)$, where Y is a scalar and $X \equiv (X_1, ..., X_k)$.[4]

[3] For a more detailed discussion, the reader is referred to Delgado and Manteiga (1999), Stute, Manteiga, and Quindimil (1998), Giersbergen (1998), Hall and Horowitz (1996), Phillips and Park (1988).

[4] Formally, a random variable, X_j, $j = 1, 2, ...k$, is defined as a function of events denoted by ϖ; that is, $x_j = X_j(\varpi)$, where x_j is a realization of X_j when the event ϖ occurs, and likewise, (y, x) is a particular realization of (Y, X), where $x \equiv (x_1, ..., x_k)$.

Assumption 1 makes explicit the way the data should be generated. Independence ensures that the product of marginal distributions, $f_Y(y)$, $f_{X_1}(x_1), ..., f_{X_k}(x)$, is equal to the joint distribution,

$$f(y, x_1, ..., x_k) \overset{ind.}{=} f_Y(y) f_{X_1}(x_1)...f_{X_k}(x_k) \text{ for each } (y, x) \in \Re^{k+1},$$

while identical distribution ensures that the product of all of the marginal distributions with the same functional form is equal to the joint distribution,

$$f(y, x_1, ..., x_k) \overset{i.i.d.}{=} f(y) f(x_1)...f(x_k) \text{ for each } (y, x) \in \Re^{k+1},$$

where $x \equiv (x_1, ..., x_k) \in \Re^k$. The independence imposed is crucial: if the r.v.'s are normally and identically distributed only they are not necessarily stationary because it is possible to construct different joint distributions that all have normal marginal distributions. By changing the joint distributions, we could violate the stationarity condition while preserving marginal normality. Thus, stationarity strengthens the assumption of identical distribution, since it applies to joint and not to simply marginal distributions.[5] On the other hand, stationarity is weaker than the *i.i.d.* assumption, since *i.i.d.* sequences are stationary, but stationary sequences do not have to be independent.

Step 2. The model

Consider the nonparametric regression model,

$$y_i = r(x_i) + \epsilon_i, \tag{3.1}$$

where $x_i \equiv (x_{i1}, ..., x_{ik}) \in \Re^k$ is a vector of k variables, ϵ_i the disturbance term assumed to satisfy $E(\epsilon_i|X_i) = 0$ almost surely $(a.s.)$ (or with probability 1). Let $r : \Re^k \to \Re$ be a real valued Borel measurable true but unknown regression function. The goal is to estimate, $r(x_i)$, without making explicit assumptions about its functional form. Assumption 1 further implies that Y satisfies $E|Y| < \infty$ and $E(\bar{Y}) = \mu_Y$, and that X satisfies $E|X| < \infty$ and $E(\bar{X}_j) = \mu_j$ for all j.[6] These conditions ensure the existence of the conditional expectation of y_i given $X_i = x_i$; that is, $E(y_i|X_i = x_i) = r(x_i)$ for all x_i.

[5] A sequence is stationary if the joint distribution of the variables in the sequence is identical, regardless of the date of the first observation.

[6] See Theorem 3.1 in White (1984, p.30).

Step 3. The null and alternative hypotheses

We consider a model with $k(= q+p)$ independent variables and aim at testing the significance of a total of p variables. The null and alternative hypotheses are expressed as a moment restriction,[7]

$$\begin{aligned} H_0 &: r(x_i) = m(x_i^q) \quad a.s. \\ H_1 &: r(x_i) \neq m(x_i^q), \end{aligned} \tag{3.2}$$

where $x_i^q \equiv (x_{i1}, ..., x_{iq}) \in \Re^q$. The null hypothesis states that given x_i, the contribution of p variables to the explanation of the variation in y_i is insignificant; that is, $E(y_i|x_i) = E(y_i|x_i^q)$. Defining $\nu_i \equiv [y_i - m(x_i^q)]$, we have the following restricted model under H_0,

$$y_i = m(x_i^q) + \nu_i, \tag{3.3}$$

where $E[\nu_i|X_i] = E[(y_i - m(x_i^q))|X_i] = E[y_i|X_i] - E[m(x_i^q)|X_i] = r(x_i) - m(x_i^q)$ $= 0$. Since $E[\nu_i|X_i] = 0$ under H_0, we have

$$T \equiv E[\nu_i E(\nu_i|X_i)] = E\{[E(\nu_i|X_i)]^2\} = 0. \tag{3.4}$$

On the contrary, since $E[\nu_i|X_i] = r(x_i) - m(x_i^q) \neq 0$ under H_1, we have $T > 0$. Using the sample analogue of T, which is some measure of distance between the two nonparametric regression models, $r(x_i)$ and $m(x_i^q)$, we form a consistent test. This measure has a non-degenerate $U-$ distribution under H_0, while having a degenerate $U-$ distribution under $E[\nu_i|X_i] = 0$ for all X_i.[8]

An estimator of T. The idea is to estimate T and test its significance. Rejection of $E[\nu_i|X_i] = 0$ would imply rejection of H_0. Obviously, if ν_i and $E[\nu_i|X_i]$ were available, we could estimate (3.4) by $\frac{1}{n}\Sigma_{i=1}^n \nu_i E[\nu_i|X_i]$. Unfortunately, they are not available, and therefore to obtain a feasible test statistic, we estimate it by

$$T_n \equiv \frac{1}{n}\sum_{i=1}^{n}[\nu_i f(x_i^q)] E[\nu_i f(x_i^q)|X_i] f(x_i), \tag{3.5}$$

[7] See Gozalo (1993), Fan and Li (1996), and Delgado and Manteiga (1999) for a similar formulation of the hypotheses.

[8] It should be noted that H_0, a conditional first-moment restriction, is translated into a conditional second-moment restriction, because this allows for the exploitation of $U-$*structures*.

where $f(x_i^q)$ and $f(x_i)$ stand for the joint probability density functions $(p.d.f.)$ of x_i^q and x_i, respectively.[9] The kernel regression method is applied and (3.5) estimated by its sample analogue.[10] The term ν_i is estimated by $\tilde{\nu}_i \equiv (y_i - \hat{y}_i)$ and a kernel estimator of $m(x_i^q)$, denoted by $\hat{y}_i$, by

$$\hat{y}_i = \frac{[(n-1)\eta^q]^{-1} \sum_{j=1\&j\neq i}^{n} y_j K_{ij}^q}{\hat{f}(x_i^q)}, \tag{3.6}$$

where $\hat{f}(x_i^q) = [(n-1)\eta^q]^{-1} \sum_{j=1\&j\neq i}^{n} K_{ij}^q$, $K_{ij}^q \equiv \Pi_{d=1}^{q} k(\frac{x_{id}-x_{jd}}{\eta})$, and $k(.)$ a univariate kernel with band width $\eta \equiv \eta_n$. Next, we calculate $E[\tilde{\nu}_i \hat{f}(x_i^q)|X_i]\hat{f}(x_i)$ as

$$[(n-1)\theta^k]^{-1} \sum_{j=1\&j\neq i}^{n} [\tilde{\nu}_j \hat{f}(x_i^q)] K_{ij} \tag{3.7}$$

where $\hat{f}(x_i) = [(n-1)\theta^k]^{-1} \sum_{j=1\&j\neq i}^{n} K_{ij}$, $K_{ij} \equiv \Pi_{d=1}^{k} k(\frac{x_{id}-x_{jd}}{\theta})$, and $\theta \equiv \theta_n$ band width corresponding to the unrestricted regression model (3.1). Lastly, substitution of $\tilde{\nu}_i$, $\hat{f}(x_i^q)$, and $\hat{f}(x_i)$ into (3.5) yields the sample analogue of T_n :

$$\hat{T}_n = [n(n-1)\theta^k]^{-1} \sum_{i=1}^{n} \sum_{j=1\&j\neq i}^{n} \left[\tilde{\nu}_i \hat{f}(x_i^q)\right] \left[\tilde{\nu}_j \hat{f}(x_j^q)\right] K_{ij}, \tag{3.8}$$

where $\tilde{\nu}_i \equiv (y_i - \hat{y}_i) = [m(x_i^q) + \nu_i] - [\hat{m}(x_i^q) + \hat{\nu}_i]$, and $\hat{m}(x_i^q)$ and $\hat{\nu}_i$ are defined in the same way as $\hat{y}_i$ in which y_j is replaced by $m(x_j^q)$ and ν_j, respectively.

Assumption 2. The kernel function $K(X)$ is a Borel measurable real-valued bounded function on a Euclidean space such that (a) $\int K(X)dX = 1$, (b)

[9] A density-weighted version of $\frac{1}{n}\Sigma_{i=1}^{n} \nu_i E[\nu_i|X_i]$, which was first introduced by Powell, Stock, and Stoker (1989), is commonly used in the literature for its two useful consequences. First, its multiplication by $f(x_i^q)$ avoids trimming the small values of the density function; and second, this multiplication yields a degenerate $U - structure$, whose asymptotic properties have been well-established.

[10] The nonparametric kernel regression has several advantages. First, rather than imposing a particular class of functional and distributional forms to the data which may or may not be correctly specified, it allows the data to reveal the data-generating process. Second, it can be designed to keep bias small enough not to compromise the asymptotic validity of test statistics. Third, in the presence of serial dependence, it is easier to handle mathematically than some others via estimating the density function by the the drop-one method. Dropping-one observation at a time yields a density estimate for x_i, $\hat{f}(x_i)$, which is independent of x_i.

$\int |K(X)|dX < \infty$ (i.e., boundedness), (c) $\sup_X |K(X)| < \infty$ (i.e., K vanishes outside $\mathbf{X}$), and (d) $K(X) = K(-X)$ and $\lim_{\|X\|\to\infty} \| X \| K(X) = 0$, where $\| X \|$ is the Eucledian norm of X in $\Re^k$.

Assumption 3. (a) $r(X)$ and $f(X)$ are Lipschitz continuous in their respective arguments and (b) $\sup_{X\in\mathbf{X}} |r(X)| < \infty$ and $\sup_{X\in\mathbf{X}} f(X) < \infty$ (i.e., $r(.)$ and $f(.)$ vanish outside the compact support $\mathbf{X}$).

Assumption 4. Let $\{\theta_n, \eta_n\}$ be an a priori chosen sequence of positive numbers satisfying $\lim_{n\to\infty} \theta_n = 0$, $\lim_{n\to\infty} \eta_n = 0$, $\lim_{n\to\infty} n\theta_n^k = \infty$, $\lim_{n\to\infty} n\eta_n^q = \infty$, $\lim_{n\to\infty} n\eta_n^{2\delta}\theta_n^{k/2} = 0$, $\lim_{n\to\infty} \left(\frac{\theta_n^k}{\eta_n^{2q}}\right) = 0$, where $\delta = \min(\lambda+1, \mu)$, $\lambda > 0$, and $\mu \geq 0$ (Fan and Li, 1996).

Assumption 2 characterizes the kernel K which vanishes outside the Euclidean space $\mathbf{X} \subseteq \Re^k$. The test proposed is still valid if $\mathbf{X}$ is a finite convex subset of $\Re^k$ and $f(X)$ vanishes on the boundary of $\mathbf{X}$. However, if $\mathbf{X}$ is a compact subset of $\Re^k$ and $f(X)$ is bounded away from zero on $\mathbf{X}$, then the proposed test needs some modification. Some trimming method is needed to overcome the boundary effect. One way to accomplish this is to use a fixed weight function such that the support of the weight is a proper subset of X. For consistency of such tests, the weight function is required to be a function of n such a way that its support approaches $\mathbf{X}$ as $n \to +\infty$ (Fan and Li, 1996). Assumption 3 guarantees that there exists two unique continuous functions, $r(X)$ and $f(X)$, defined for all values of X such that their derivatives exist and reduces to $r(X_0)$ and $f(X_0)$ at $X = X_0$. The first four conditions in Assumption 4 simply state that (i) band widths should be small if n is large, (ii) the kernel estimators involved are consistent, and (iii) the limiting distribution of $n\theta_n^{k/2}\hat{T}_n$ under H_0 ia centered correctly at zero. Also implied by Assumption 4, as suggested by Rabinson (1988), Fan and Li (1996), and Delgado and Manteiga (1999), are the necessary conditions, $\mu = q/2$ and $\lambda = (q/2) - 1$, required for bias reduction using higher order kernels.

Step 4. The asymptotic distribution of $\hat{T}_n$

The asymptotic distribution of $\hat{T}_n$ needs to be determined to tell how far $\hat{T}_n$ must be from zero to reject H_0, and a value of $\hat{T}_n$ far from zero should be regarded as evidence against H_0.[11] The key to establishing $\sqrt{n}-$ consistency

[11] A lengthy proof for the existence of the asymptotic distribution of $\hat{T}_n$ is given in Fan and Li (1996). The proof heavily exploits the key features of $U-statistics$ which are commonly

and asymptotic normality of $\hat{T}_n$ is to note that Equ. (3.8) can be written as a $U - statistic$, whose structure permits proper accounting of the "overlaps" in the density estimators. These overlaps result from the fact that each data point is used in the estimation of several density estimates.[12]

The finite sample distribution, defined as $D_n(x) = P[\sqrt{n\theta_n^{k/2}}(\hat{T}_n - T)/\sqrt{2}\hat{\sigma}_{\hat{T}_n} < x]$, has no closed-form expression because it depends on certain features of the distribution of (Y, X). Luckily, by the central limit theorem the studentized root, $[\sqrt{n\theta_n^{k/2}}(\hat{T}_n - T)/\sqrt{2}\hat{\sigma}_{\hat{T}_n}]$, is asymptotically standard normally distributed under weak regularity conditions. Define $\hat{Z}_n \equiv n\theta_n^{k/2}\hat{T}_n \xrightarrow{d} N(0, 2\sigma_T^2)$ since $T = 0$ under H_0. Next, a law of large numbers is invoked to show consistency in probability of $\hat{Z}_n$ for Z (i.e., $E(\hat{Z}_n) \xrightarrow{p} Z$) and unbiasedness in probability of Z_n to center $\hat{Z}_n$ correctly at zero (i.e., $P[\{[E(\hat{Z}_n) - Z] < \varepsilon\}] = 1$ where $\varepsilon > 0$).[13]

Estimation of the consistent variance of $\hat{T}_n$. Typically, the variance σ_T^2 is unknown. The goal is then to find a consistent estimator $\hat{\sigma}^2_{\hat{T}_n}$ such that $(\hat{\sigma}^2_{\hat{T}_n} \to \sigma_T^2) \xrightarrow{p} 0$ as n goes to ∞. Utilizing the $U - structure$, Fan and Li (1996) derive $\hat{\sigma}^2_{\hat{T}_n}$, a natural estimator of the asymptotic variance of $\hat{T}_n$,

$$\hat{\sigma}^2_{\hat{T}_n} = [n(n-1)\theta^k]^{-1} \sum_{i=1}^{n} \sum_{j=1 \& j \neq i}^{n} \left[\tilde{\nu}_i \hat{f}(x_i^q)\right]^2 \left[\tilde{\nu}_j \hat{f}(x_j^q)\right]^2 K_{ij} \left[\int K^2(u)du\right]. \quad (3.9)$$

Step 5. The test statistic

A touchy point is to calculate an "appropriate" test statistic, τ_n, where $\tau_n = \tau(\hat{Z}_n)$. This functional implies that the distribution of τ_n should agree with that of $\hat{Z}_n$. Hence, we opt for a χ^2 distribution for τ_n since degenerate $U - statistics$

used in the literature when the kernel regression is utilized for hypothesis testing purposes, see, for example, Powell, Stock, and Stoker (1989), Lee (1990, 1992), Horowitz and Hardle (1994), Sherman (1994), Zheng (1996), and Fan and Li (1996). For the arguments in the following paragraphs, the reader is referred to Definitions 1 and 2 and Assumption A, stated by Rabinson (1988) and Fan and Li (1996).

[12] See Appendix for more on the structure of $U - statistics$. and on how one can translate the kernel estimator as a $U - statistic$.

[13] For bias reduction, Robinson (1988), Powell, Stock, and Stoker (1989), and Liu and Singh (1992) suggest the use of a higher order kernel or the generalized jackknife estimator of $\hat{T}_n$, because both the kernel and jackknife estimators maintain maximum rate of convergence in distribution.

built in $\hat{Z}_n$ ordinarily are asymptotically distributed as linear combinations of χ^2 variates (see Horowitz and Hardle (1994)).[14]

Define

$$\tau_n = \left[\frac{\hat{Z}_n}{\sqrt{2}\hat{\sigma}_{\hat{T}_n}} \right]^2 \backsim \chi^2_q \tag{3.10}$$

where $T = 0$ under H_0. But τ_n depends on certain unknown characterisitcs of the distribution of (Y, X), and an asymptotic test cannot be implemented except in exceptional circumstances. That is why we propose a bootstrap test in order to approximate the $f.s.n.d.$ and then estimate the critical values of τ_n.[15]

Step 6. The critical region

A critical region of given size α is defined as $\Pr[\tau^*_{n[\alpha(B+1)]} < \tau^{obs}_n] = \alpha$, where τ^{obs}_n is the test statistic calculated from the observed sample, $\tau^*_{n[\alpha(B+1)]}$ the bootstrap critical value, B the number of the bootstrap samples.

Step 7. The decision rule

For small n, an approximate $\alpha-$level significance test is to reject H_0 if $\tau^{obs}_n > \tau^*_{n[\alpha(B+1)]}$.

4. Hypothesis Test Algorithm

The bootstrap treats the observed data as if they were the population and, by repeatedly sampling the data and computing $\{\tau^*_{nb} : b = 1, ..., B\}$, from the resulting bootstrap samples, develops the empirical distribution of the bootstrap version of τ_n, τ^*_n. The bootstrap estimate of the $\alpha - level$ critical value of τ_n is the $1 - \alpha$ quantile of the empirical distribution of τ^*_n. Three main bootstrap schemes are present to accomplish this: the residual-based bootstrap (RB), the paired-based bootstrap (PB), and the external (or wild) bootstrap (EB).[16] Following Hall and Hart (1990), we opt for the residual-based bootstrap scheme to

[14] Developing an asymptotic test procedure, Fan and Li (1996) define $\tau_n = \left[n\theta_n^{k/2}\hat{T}_n/\sqrt{2}\hat{\sigma}_T \right]$ $\rightarrow N(0,1)$ in distribution under H_0.

[15] See Delgado and Manteiga (1999) for an application of a bootstrap test in a similar context. They employ the Kolmogorov-Smirnov and Cramer-von Mises test statistics.

[16] See Giersbergen (1998) for a comparison of these resampling schemes.

determine critical values for testing because the null postulates $E(\nu_i|X_i) = 0\ \forall_i$. Here is the procedure to apply this scheme. First, the residuals, $\tilde{\nu}_i \equiv (y_i - \hat{y}_i)$, are centered by $\hat{\nu}_i \equiv (\tilde{\nu}_i - \bar{\nu})$, where $\bar{\nu} = n^{-1}\Sigma_{i=1}^{n}\nu_i$. Then, a bootstrap resample, $\{\hat{\nu}_i^* : i = 1, ..., n\}$, is drawn from $\{\hat{\nu}_i : i = 1, ..., n\}$ at random, with replacement. Resampling is done from the centered residuals to ensure $E(\hat{\nu}_i^*|X_i) = 0$ under H_0 and hence $E(\hat{\nu}_i^*|X_i) = E(\tilde{\nu}_i|X_i) = 0\ \forall_i$. The centering is especially important as the alternative hypothesis, $E(\tilde{\nu}_i|X_i) \neq 0\ \forall_i$, holds in the nonparametric regression models.[17]

The residual-based bootstrap test

1. Consider the restricted nonparametric regression model,

$$y_i = m(x_i^q) + \nu_i,$$

   and estimate $m(x_i^q)$ by

$$\hat{y}_i = \frac{[(n-1)\eta^q]^{-1}\sum_{j=1\&j\neq i}^{n} y_j K_{ij}^q}{\hat{f}(x_i^q)}$$

   where $\hat{f}(x_i^q) = [(n-1)\eta^q]^{-1}\sum_{j=1\&j\neq i}^{n} K_{ij}^q$.

2. Define $\tilde{\nu}_i \equiv (y_i - \hat{y}_i)$ and approximate the asymptotic test statistic:

$$\tau_n^{obs} = \left[\frac{\hat{Z}_n}{\sqrt{2}\hat{\sigma}_{\hat{T}_n}}\right]^2$$

   where

$$\hat{Z}_n \equiv n\theta_n^{k/2}\hat{T}_n$$

$$\hat{T}_n = \left[n(n-1)\theta^k\right]^{-1}\sum_{i=1}^{n}\sum_{j=1\&j\neq i}^{n}\left[\tilde{\nu}_i\hat{f}(x_i^q)\right]\left[\tilde{\nu}_j\hat{f}(x_j^q)\right]K_{ij}$$

$$K_{ij} \equiv \Pi_{d=1}^{k} k(\frac{x_{id} - x_{jd}}{\theta})$$

$$\hat{\sigma}_{\hat{T}_n}^2 = \left[n(n-1)\theta^k\right]^{-1}\sum_{i=1}^{n}\sum_{j=1\&j\neq i}^{n}\left[\tilde{\nu}_i\hat{f}(x_i^q)\right]^2\left[\tilde{\nu}_j\hat{f}(x_j^q)\right]^2 K_{ij}\left[\int K^2(u)du\right]$$

[17] The reader is referred to Freedman (1981,1984), Hall (1988), Hall and Hart (1990), and Li and Maddala (1996) for a discussion of choice of appropriate resampling scheme.

$$\int K^2(u)du \approx \left[(n-1)\theta^k\right] \int Var[\hat{f}(x_i)]dx_i$$

$$\hat{f}(x_i) = [(n-1)\theta^k]^{-1} \sum_{j=1\&j\neq i}^{n} K_{ij}.$$

3. Rescale $\tilde{\nu}_i$ as $\hat{\nu}_i \equiv (\tilde{\nu}_i - \bar{\nu})$, where $\bar{\nu} = n^{-1}\Sigma_{i=1}^{n}\tilde{\nu}_i$, and draw the bootstrap sample $\{\hat{\nu}_i^* : i = 1, ..., n\}$ at random, with replacement, from $\{\hat{\nu}_i : i = 1, ..., n\}$.

4. Calculate the bootstrap test statistic:

$$\tau_n^* = \left[\frac{\hat{Z}_n^*}{\sqrt{2}\hat{\sigma}_{\hat{T}_n^*}}\right]^2$$

where

$$\hat{T}_n^* = \left[n(n-1)\theta^k\right]^{-1} \sum_{i=1}^{n} \sum_{j=1\&j\neq i}^{n} \left[\hat{\nu}_i^* \hat{f}(x_i^q)\right] \left[\hat{\nu}_j^* \hat{f}(x_i^q)\right] K_{ij}$$

$$\hat{\sigma}_{T_n^*}^2 = \left[n(n-1)\theta^k\right]^{-1} \sum_{i=1}^{n} \sum_{j=1\&j\neq i}^{n} \left[\hat{\nu}_i^* \hat{f}(x_i^q)\right]^2 \left[\hat{\nu}_j^* \hat{f}(x_i^q)\right]^2 K_{ij} \left[\int K^2(u)du\right].$$

Replicate this calculation for B times to obtain

$$\{\tau_{nb}^* : b = 1, ..., B\}.$$

5. Let $\tau_{n[1]}^* \leq \tau_{n[2]}^* \leq ... \leq \tau_{n[B]}^*$ denote B *ordered* bootstrap realizations. Given α, the bootstrap critical value, $\tau_{n\alpha}^*$, is determined as[18]

$$\tau_{n\alpha}^* = \tau_{n[\alpha(B+1)]}^*,$$

such that $\Pr(\tau_{nb}^* > \tau_{n\alpha}^*) = \alpha$.

6. Reject H_0 if $\tau_n^{obs} > \tau_{n\alpha}^*$.

7. Choose α such that

$$\alpha_n^* = \Pr\left[\tau_n^{obs} \geq \tau_{n\alpha}^*\right]$$

where α_n^* denotes the bootstrap $p-value$ (or the rejection probability) conditional on H_0.

[18] For convenience in applications, B is usually chosen in such a way that $\alpha(B+1)$ is an integer. For example, if $B = 99$ and $\alpha = 0.05$, then $\tau_{n\alpha}^* = \tau_{n[5]}^*$ which is the $5th$ lowest value in the ordered $\tau_{n[M]}^*$.

5. Concluding Remark

In this paper we adapted an asymptotic, nonparametric hypothesis test to the bootstrap framework, applying the kernel regression method for the estimation of a measure of distance between the models under the null hypothesis. Furthermore, with an algorithm, the proposed bootstrap test was operationalized, allowing us to compare the asymptotic with the bootstrap approximations of the test statistic. The paper also developed a Mathematica Code for the test algorithm.

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```
(****************************************************)
(****************************************************)
(*  Bootstrap Hypothesis Test Algorithm              *)
(*  Developed by Tugrul Temel,  June 28, 2011        *)
(*  Development Research Institute, Tilburg University *)
(*  Tilburg, The Netherlands                         *)
(****************************************************)
(****************************************************)
```

The Residual-Based Bootstrap Test

```
<< Statistics`DescriptiveStatistics`
<< Statistics`NormalDistribution`

Clear[X, Y, n, k, p, q, η, θ, σ, kern, prob, Ŷ, f̂, ṽ, kernel,
      probability, F̂, VarK, T̂, Var T̂, VarT, τ, v̂, v̂*, f̂*, X*,
      k*, pr*, T̂*, F̂*, VarK*, τ*, OrderedTS, α, M, τ*_nα];

SetDirectory["u:\Andre\Bootstrap"];
Data = Import["Georgia.dat", "Table"];
Y    = Data[[1]];
X[1] = Data[[2]];
X[2] = Data[[3]];
X[3] = Data[[4]];

(***************************************************)
(*        Variables, Parameters, and Indices        *)
(***************************************************)
(* X[d]   d-th independent variable, d=1,...,k      *)
(* Y      dependent variable                        *)
(* n      # of observations, i,j=1,...,n            *)
(* p      # of variables tested for significance    *)
(* q      # of variables in the restricted model    *)
(* α      significance level                        *)
(* η      band width for the restricted model       *)
(* θ      band width for the unrestricted model     *)
(* μ,λ,γ  parameters for biased reduction           *)
(* σ[d,θ] standard deviation * θ                    *)
(* σ[d,η] standard deviation * η                    *)
(* θ=N[(4/(n (k+2)))^(1/(4+k))] optimal band width  *)
(* necessary conditions for biased reduction:       *)
(*          μ=N[q/2+1]; λ=N[q/2]; γ=N[Min[μ,λ+1]]   *)
(***************************************************)

n = Length[Y];
```

```
k = 3;
p = 1;
q = k - p;
η = N[(4/(n (q + 2)))^(1/(4+q))];
θ = N[1.3 η];
Do[
    σ[d, θ] = N[√Variance[X[d]] θ];
    σ[d, η] = N[√Variance[X[d]] η], {d, 1, k}
   ];

(************************************************)
(*  Step 1: Estimate Y by the N-W kernel, Ŷ    *)
(************************************************)

Do[
    Do[
      Do[
          kern[d, i] = N[NormalDistribution[X[d][[i]], σ[d, η]]];
          prob[d, i, j] = N[PDF[kern[d, i], X[d][[j]]]], {j, 1, n}
         ], {i, 1, n}
      ], {d, 1, q}
   ];

Do[
    Ŷ[i] = N[(Sum[Y[[j]] Product[prob[d, i, j], {d, 1, q}], {j, 1, n}] - Y[[i]] Product[prob[d, i, i], {d, 1, q}]) /
      (Sum[Product[prob[d, i, j], {d, 1, q}], {j, 1, n}] - Product[prob[d, i, i], {d, 1, q}])], {i, 1, n}
   ];

Do[
    f̂[i] = N[1/((n - 1) η^q) (Sum[Product[prob[d, i, j], {d, 1, q}], {j, 1, n}] - Product[prob[d, i, i], {d, 1, q}])];
    ṽ[i] = N[Y[[i]] - Ŷ[i]], {i, 1, n}
    ];

Do[
    Do[
```

```
        Do[
            kernel[d, i] = N[NormalDistribution[X[d][[i]], σ[d, θ]]];
            probability[d, i, j] = N[PDF[kernel[d, i], X[d][[j]]]], {j, 1, n}
          ], {i, 1, n}
       ], {d, 1, k}
   ];
```

Do[

$$\hat{F}[i] = N\left[\frac{1}{(n-1)\,\theta^{k}}\left(\sum_{j=1}^{n}\prod_{d=1}^{k}\text{probability}[d, i, j] - \prod_{d=1}^{k}\text{probability}[d, i, i]\right)\right], \{i, 1, n\}$$

];

$$\text{VarK} = (n-1)\,\theta^{k}\,\text{Variance}\left[\text{Table}\left[\hat{F}[i], \{i, 1, n\}\right]\right];$$

$$\hat{T}[n] = N\left[\frac{1}{n\,(n-1)\,\theta^{k}}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\tilde{v}[i]\,\hat{f}[i]\right)\left(\tilde{v}[j]\,\hat{f}[j]\right)\prod_{d=1}^{k}\text{probability}[d, i, j] - \sum_{i=1}^{n}\left(\tilde{v}[i]\,\hat{f}[i]\right)\left(\tilde{v}[i]\,\hat{f}[i]\right)\prod_{d=1}^{k}\text{probability}[d, i, i]\right)\right];$$

$$\text{VarT} = N\left[\frac{1}{n\,(n-1)\,\theta^{k}}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\tilde{v}[i]\,\hat{f}[i]\right)^{2}\left(\tilde{v}[j]\,\hat{f}[j]\right)^{2}\prod_{d=1}^{k}\text{probability}[d, i, j] - \sum_{i=1}^{n}\left(\tilde{v}[i]\,\hat{f}[i]\right)^{2}\left(\tilde{v}[i]\,\hat{f}[i]\right)^{2}\prod_{d=1}^{k}\text{probability}[d, i, i]\right)\right]\text{VarK};$$

$$\tau[n] = N\left[\frac{n\,\theta^{k/2}\,\hat{T}[n]}{\sqrt{2\,\text{VarT}}}\right];$$

```
(*****************************************************)
(* Step 3: Define standardized residual, v̂_i        *)
(*****************************************************)

Do[
   v̂[i] = ṽ[i] - Mean[Table[ṽ[i], {i, 1, n}]], {i, 1, n}
  ];

(**********************************************************)
(* Step 4: Bootstrap samples {v̂*_i} from v̂_i             *)
```

```
(***********************************************************)
(* asterix *         superscript for bootstrap variables     *)
(* B                 # of bootstrap samples, b=1,...,B       *)
(* ṽ[i]=ṽ_i          estimated ν_i from the restricted model  *)
(* v̂[i]=(ṽ_i - v̄)=v̂_i  standardized ṽ_i, where v̄=Mean[ṽ_i]   *)
(* v̂*[b]={v̂*_i}      b-th bootstrap sample from {v̂_i}        *)
(* Ŷ*[b]=Y*_i=(Ŷ_i+v̂*_i)  b-th bootstrap sample from {Ŷ_i}    *)
(* ŶY[b]=Ŷ_i         rearranged Ŷ_i according to b-th sample  *)
(* X*[j,b]=X*_j      rearranged X_j according to b-th sample  *)
(************************************************************)

B = 39;
Do[
   rndSmpl = Table[Random[Integer, {1, n}], {n}];
   Bsmpl[b] = rndSmpl, {b, 1, B}
  ];

Do[
   v̂*[b] = Table[ v̂[Bsmpl[b][[i]]], {i, 1, n}];
   f̂*[b] = Table[f̂[Bsmpl[b][[i]]], {i, 1, n}], {b, 1, B}
  ];

Do[
   Do[
      X*[d, b] = Table[X[d][[Bsmpl[b][[i]]]], {i, 1, n}], {b, 1, B}
     ], {d, 1, k}
  ];

(****************************************************)
(* Step 5: Calculate the bootstrap test stat., τ*_n *)
(****************************************************)

Do[
    Do[
       σ*[d, b, θ] = N[Sqrt[Variance[X*[d, b]]] θ];
       σ*[d, b, η] = N[Sqrt[Variance[X*[d, b]]] η], {d, 1, k}
    ], {b, 1, B}
 ];

Do[
  Do[
    Do[
```

```
          Do[
             k*[b, d, i] = N[NormalDistribution[X*[d, b][[i]], σ*[d, b, θ]]];
             pr*[b, d, i, j] = N[PDF[k*[b, d, i], X*[d, b][[j]]]], {j, 1, n}
            ], {i, 1, n}
         ], {d, 1, k}
      ], {b, 1, B}
   ];
```

$$\text{Do}\Big[\hat{T}^*[b] = N\Big[\frac{1}{n\ (n-1)\ \theta^k}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\hat{v}^*[b][[i]]\ \hat{f}^*[b][[i]]\right)\left(\hat{v}^*[b][[j]]\ \hat{f}^*[b][[j]]\right)\prod_{d=1}^{k}\text{pr}^*[b, d, i, j] - \sum_{i=1}^{n}\left(\hat{v}^*[b][[i]]\ \hat{f}^*[b][[i]]\right)^2\prod_{d=1}^{k}\text{pr}^*[b, d, i, i]\right)\Big], \{b, 1, B\}\Big];$$

$$\text{Do}\Big[\text{Do}\Big[\hat{F}^*[b, i] = N\Big[\frac{1}{(n-1)\ \theta^k}\left(\sum_{j=1}^{n}\prod_{d=1}^{k}\text{pr}^*[b, d, i, j] - \prod_{d=1}^{k}\text{pr}^*[b, d, i, i]\right)\Big], \{i, 1, n\}\Big], \{b, 1, B\}\Big];$$

$$\text{Do}\Big[\text{VarK}^*[b] = (n-1)\ \theta^k\ \text{Variance}\Big[\text{Table}\Big[\hat{F}^*[b, i], \{i, 1, n\}\Big]\Big], \{b, 1, B\}\Big];$$

$$\text{Do}\Big[\text{VarT}^*[b] = N\Big[\frac{1}{n\ (n-1)\ \theta^k}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\hat{v}^*[b][[i]]\ \hat{f}^*[b][[i]]\right)^2\left(\hat{v}^*[b][[j]]\ \hat{f}^*[b][[j]]\right)^2\prod_{d=1}^{k}\text{pr}^*[b, d, i, j] - \sum_{i=1}^{n}\left(\hat{v}^*[b][[i]]\ \hat{f}^*[b][[i]]\right)^2\prod_{d=1}^{k}\text{pr}^*[b, d, i, i]\right)\Big]\ \text{VarK}^*[b], \{b, 1, B\}$$

```
];

Do[
    τ*[b] = N[ n θ^(k/2) T̂*[b] / Sqrt[2 VarT*[b]] ], {b, 1, B}
  ];

(*************************************************)
(* Step 6: Order the bootstrap realizations,     *)
(*         τ*n[1] ≤ τ*n[2] ≤ ... ≤ τ*n[B], and calculate  *)
(*         the bootstrap critical value, τ*nα     *)
(*         α = significance level                *)
(*************************************************)

OrderedTS = Table[τ*[b], {b, 1, B}];
α = 0.05;
M = α (B + 1);
τ*nα = Sort[OrderedTS][[M]];

(*************************************************)
(* Step 7 : Establish a decision rule            *)
(*************************************************)

If[τ[n] > τ*nα, Print["Reject H0 because τ[n]>τ*nα"],
  Print["Accept H0 because τ[n]<τ*nα"]];
```