An Algorithm for payoff space in C1 parametric games

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Abstract

We present a novel algorithm to determine the payoff-space of certain normal-form C^1 parametric games, and - more generally - of continuous families of normal-form C^1 games. The algorithm has been implemented by using MATLAB, and it has been applied to several examples. The implementation of the algorithm gives the parametric expressions of the critical zone of any game in the family under consideration both in the bistrategy space and in the payoff space and the graphical representations of the disjoint union (with respect to the parameter set of the parametric game) of the family of all payoff spaces. We have so the parametric representation of the union of all the critical zones. One of the main motivations of our paper is that, in the applications, many normal-form games appear naturally in a parametric fashion; moreover, some efficient models of coopetition are parametric games of the considered type. Specifically, we have realized an algorithm that provides the parametric and graphical representation of the payoff space and of the critical zone of a parametric game in normal-form, supported by a finite family of compact intervals of the real line. Our final goal is to provide a valuable tool to study simply (but completely) normal-form C^1-parametric games in two dimensions.

1 Introduction and motivations

Our study is based on a method able to determine the payoff space of normal-form C^1-games in n dimensions, that is for n-players normal-form games whose payoff functions are defined on compact intervals of the real line and of class at least C^1. In the particular case of two dimensions, the payoff space of normal-form C^1-games is determined. In this paper we delineate the procedure for the representation of the payoff space in normal-form C^1 parametric games.
The complete study of a C^1 parametric games is strongly motivated not only by theoretical and pure mathematical reasons but especially by the applications to Economics, Finance and Social Sciences. Indeed, many real interactions - between competitive or cooperative subjects - are modeled by games with time dependence on strategy sets and payoff functions. Moreover, recently, one operative and particularly efficient model of coopetition has been proposed and

applied by D. Carfì and others. The model is given by a particular kind of parametric game, in which the parameter set is the cooperative strategy set of the game (see [8, 9, 10, 11, 12, 13]). That is the basic object which allows to pass from the standard normal-form definition of game([1, 2, 3, 14]) to their coopetitive extension. Our algorithm permits a general vision of the payoff space of a parametric game and of the Nash paths, knowledge of fundamental importance in the applications.

2 Preliminaries on normal-form C^1 games

We shall consider n-person games in normal-form. We give the definitions used in this work in order to ease the reader. The form of following definitions is particularly useful for our purpose.

Definition 1 (of game in normal-form). *Let $E = (E_i)_{i=1}^n$ be an ordered family of non-empty sets. We call n-**person game in normal-form upon the support** E each function*

$$f : {}^{\times}E \to \mathbb{R}^n,$$

*where ${}^{\times}E$ denotes the Cartesian product $\times_{i=1}^n E_i$ of the family E. The set E_i is called the **strategy set of player** i, for every index i of the family E, and the product ${}^{\times}E$ is called the **strategy profile space**, or the n-strategy space, of the game.*

Terminology. Together with the previous definition of game in normal form, we have to introduce some terminologies:

- the set $\{i\}_{i=1}^n$ of the first n positive integers is said *the set of the players* of the game;
- each element of the Cartesian product ${}^{\times}E$ is said a *strategy profile* of the game;
- the image of the function f, i.e., the set $f({}^{\times}E)$ of all real n-vectors of type $f(x)$, with x in the strategy profile space ${}^{\times}E$, is called the *n-payoff space*, or simply the *payoff space*, of the game f.

We recall, further, for completeness and, to ease the reader, the definition of Pareto boundary we shall use in the paper.

Definition 2 (of Pareto boundary). *The **Pareto maximal boundary of a game** f is the subset of the n-strategy space of those n-strategies x such that the corresponding payoff $f(x)$ is maximal in the n-payoff space, with respect* *to the usual order of the euclidean n-space $\mathbb{R}^n$. If S denote the strategy space $^{\times}E$, we shall denote the maximal boundary of the n-payoff space by $\overline{\partial} f(S)$ and the maximal boundary of the game by $\overline{\partial}_f(S)$ or by $\overline{\partial}(f)$. In other terms, the maximal boundary $\overline{\partial}_f(S)$ of the game is the reciprocal image (by the function f) of the maximal boundary of the payoff space $f(S)$. We shall use analogous terminologies and notations for the* ***minimal Pareto boundary****.*

The fundamental properties of Pareto boundaries have been presented in [6].

3 The method for C^1 games

The method we use to study a normal-form C^1 game is given and applied in [4, 5, 7].

The context. We deal with a type of normal-form game f defined on the product of n compact non-degenerate intervals of the real line, and such that f is the restriction to the n-strategy space of a C^1 function defined on an open set of $\mathbb{R}^n$ containing the n-strategy space S (which, in this case, is a compact non-degenerate n-interval of the n-space $\mathbb{R}^n$).

Before to give the main result of the method, we recall some basic notions.

3.1 Topological boundary

We recall that the topological boundary of a subset S of a topological space (X, τ) is the set defined by the following three equivalent propositions:

- it is the closure of S minus the interior of S:

$$\partial S = \mathrm{cl}(S) \backslash \mathrm{int}(S);$$

- it is the intersection of the closure of S with the closure of its complement

$$\partial S = \mathrm{cl}(S) \cap \mathrm{cl}(X \backslash S);$$

- it is the set of those points x of X such that every neighborhood of x contains at least one point of S and at least one point in the complement of S.

The key theorem of our method is the following one.

Theorem 1. *Let f be a C^1 function defined upon an open set O of the euclidean space $\mathbb{R}^n$ and with values in $\mathbb{R}^n$. Then, for every part S of the open O, the topological boundary of the image of S by the function f is contained in the union*

$$f(\partial S) \cup f(C),$$

where C is the critical set of f in S, that is the set of the points x of S such that the Jacobian matrix $J_f(x)$ is not invertible. If, more, the function f is not continuous over a part H of O and C^1 elsewhere in O, the topological boundary of the image of S by the function f is contained in the union

$$f(\partial S) \cup f(C) \cup f(H),$$

where C is (again) the critical set of f in S.

4 Two players parametric games

In this article we shall use the following definitions of parametric game.

Definition 3. *Let $E = (E_t)_{t \in T}$ and $F = (F_t)_{t \in T}$ be two families of non empty sets and let*

$$f = (f_t)_{t \in T}$$

be a family of functions

$$f_t : E_t \times F_t \to \mathbb{R}^2.$$

We define parametric game over the strategy pair (E, F) and with family of payoff functions f the pair

$$G = (f, >),$$

where $>$ is the usual strict upper order of the Euclidean plane $\mathbb{R}^2$. We define payoff space of the parametric game G the union of all the payoff spaces of the game family

$$g = ((f_t, >))_{t \in T},$$

that is the union of the payoff family

$$P = (f_t(E_t \times F_t))_{t \in T}.$$

We note that the family P can be identified with the multi-valued path

$$p : T \to \mathbb{R}^2 : t \mapsto f_t(E_t \times F_t),$$

and that the graph of this path p is a subset of the Cartesian product $T \times \mathbb{R}^2$.

In particular we are concentrated on the following particular kind of parametric game:

- parametric games in which the families E and F consist of only one set, respectively.

In the latter case we can identify a parametric game with a pair $(f, >)$, where f is a function from a Cartesian product $T \times E \times F$ into the plane $\mathbb{R}^2$, where T, E and F are three non-empty sets.

Definition 4. *When the triple (T, E, F) is a triple of subsets of normed spaces, we define the parametric game $(f, >)$* ***of class*** *C^1 if the function f is of class C^1.*

5 Numerical Results

Consider a (loss) parametric game $(h, <)$, with strategy sets $E = F = [0, 1]$, parameter set $T = [0, 1]^2$ and biloss (disutility) function

$$h : \times(T, E, F) \to \mathbb{R}^2$$

whose section

$$h_{(a,b)} : \times(E, F) \to \mathbb{R}^2$$

is defined by

$$h_{(a,b)}\,(x, y) = (x - (1 - a)xy, y - (1 - b)xy)\,,$$

for all $(x, y) \in E \times F$ and $(a, b) \in [0, 1]^2$.

The above game is the von Neuman convexification of the finite game represented by the following array

$$\begin{array}{cc} (a, b) & (1, 0) \\ (0, 1) & (0, 0) \end{array}.$$

Assume, now, that the parameter points (a, b) belong also to the 1-sphere $\mathbf{S}^1_p$, with respect to the p-norm, in the Euclidean plane $\mathbb{R}^2$, for some positive real p; that is, let us assume

$$a^p + b^p = 1,$$

for some positive real p. Consider, then, the restriction

$$g : S \times E \times F \to \mathbb{R}^2$$

of the function h to the parameter set

$$S = \mathbf{S}^1_p \cap T.$$

By projecting on the first factor of the product $S \times E \times F$, we can consider, instead of the parametric game g, with parameter set $\mathbf{S}^1_p \cap T$, the equivalent parametric game $(f,<)$, with parameter set $[0,1]$ and a-payoff function f_a defined by

$$f_a(x,y) = \left(x-(1-a)xy, y-(1-(1-a^p)^{1/p})xy\right),$$

for all $(x,y) \in E \times F$ and $a \in [0,1]$. Here, by equivalent parametric game, we mean the existence of the bijection

$$j : S \to [0,1] : (a,b) \mapsto a$$

whose inverse is the bijection

$$j^{-1} : [0,1] \to S : a \mapsto (a,(1-a^p)^{1/p}).$$

In the following sections we shall consider the following sub-cases:

1. $p=1$:

$$f_a(x,y) = (x-(1-a)xy, y-axy),$$

for all x,y and a in $[0,1]$.

2. $p=0.1$:

$$f_a(x,y) = \left(x-(1-a)xy, y-(1-(1-a^{0.1})^{10})xy\right),$$

for all x,y and a in $[0,1]$.

3. $p=0.5$:

$$f_a(x,y) = \left(x-(1-a)xy, y-(1-(1-a^{0.5})^{2})xy\right),$$

for all x,y and a in $[0,1]$.

4. $p=2$:

$$f_a(x,y) = \left(x-(1-a)xy, y-(1-(1-a^{2})^{0.5})xy\right),$$

for all x,y and a in $[0,1]$.

5. $p=10$:

$$f_a(x,y) = \left(x-(1-xy), y-(1-(1-a^{10})^{0.1})xy\right),$$

for all x,y and a in $[0,1]$.

Moreover, we shall present the following games:

6. $$f_a(x,y) = \left(x+y+a, x-y+a^2\right),$$

for all $x,y \in [0,2]$ and $a \in [0,1]$.

7. $$f_a(x,y) = (x+y+a, x-y+|a|),$$

for all $x,y \in [0,2]$ and $a \in [-1,1]$.

6 First game $p = 1$

Let $E = F = [0, 1]$ be the strategy sets and let a be a real number fixed in the interval $[0, 1]$. Consider the a-biloss (disutility) function of the parametric game $(f, <)$, defined by

$$f_a(x, y) = (x - (1 - a)xy, y - axy),$$

for all (x, y) in $[0, 1]$.

The critical zone of the function f_a (represented in Figure 1 for every a in $[0, 1]$) is the set

$$\mathcal{C}(f_a) = \left\{(x, y) \in [0, 1]^2 : 1 - ax - (1 - a)y = 0\right\}.$$

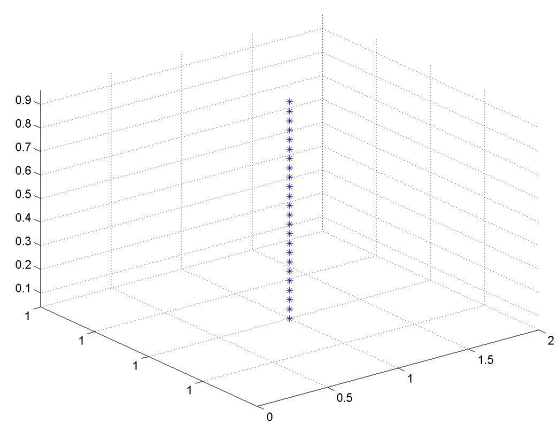

Figure 1: Disjoint union of the critical zones.

The disjoint union of the family

$$(f_a(\partial(E \times F)))_{a \in T},$$

that is the disjoint union of the transformations of the topological boundaries of the bistrategy space, with respect to the parameter set, is shown in Figure 2.

Recalling that the action of a family of functions (with common domain) on a subset of the common domain of the member-functions is the family of the images (transformations) of the subset, we can consider the above disjoint union as a faithful representation of the action of the entire family $(f_a)_{a \in T}$, on the boundary $\partial(E \times F)$.

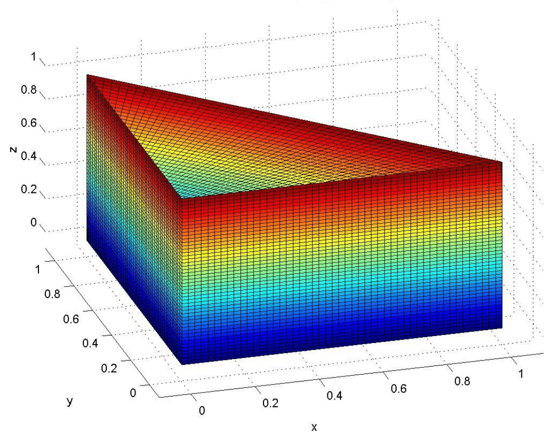

Figure 2: First Game: Disjoint union of transformations of the topological boundaries of the bistrategy space.

The disjoint union of transformations of the critical zones is shown in Figure 3.

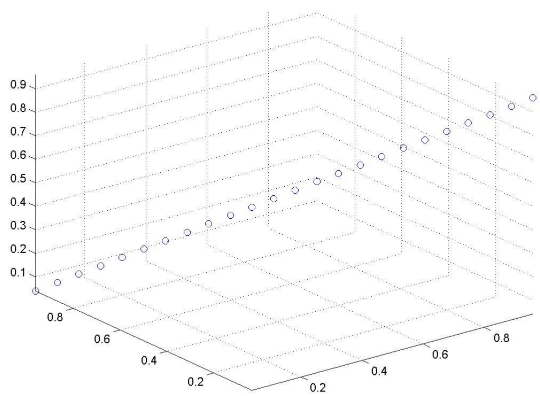

Figure 3: First Game: Disjoint union of transformations of the critical zones.

So, from the transformations of the topological boundaries and of the critical zones, we obtain the representation of the Payoff Space of the parametric game as disjoint union of the family

$$(f_a(E \times F))_{a \in T},$$

that is the disjoint union of the transformations of the payoff spaces, with respect to the parameter set, as shown in Figure 4. By the way, we observe that this last disjoint union is the graph of the multivalued curve

$$c : T \to \mathbb{R}^2 : a \mapsto f_a(E \times F),$$

but an irrelevant permutation

$$J : T \times \mathbb{R}^2 \to \mathbb{R}^2 \times T : (a, X, Y) \mapsto (X, Y, a).$$

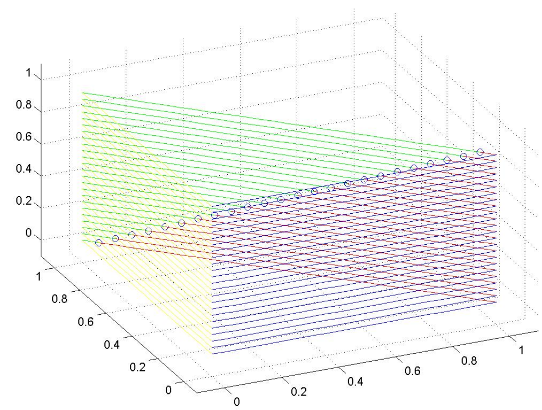

Figure 4: First Game: Disjoint union of payoff spaces.

7 Second game: $p = 0.1$

Let $E = F = [0, 1]$ be the strategy sets and let f_a be the a-biloss (disutility) function

$$f_a(x, y) = \left(x - (1 - a)xy, y - (1 - (1 - a^{0.1})^{10})xy\right),$$

for all x, y a in [0, 1]. The critical zone of the a-biloss function is

$$\mathcal{C}(f_a) = \left\{(x, y) \in [0, 1]^2 : 1 - (1 - (1 - a^{0.1})^{10})x - (1 - a)y = 0\right\}.$$

The disjoint union of critical zones is shown in Figure 5.

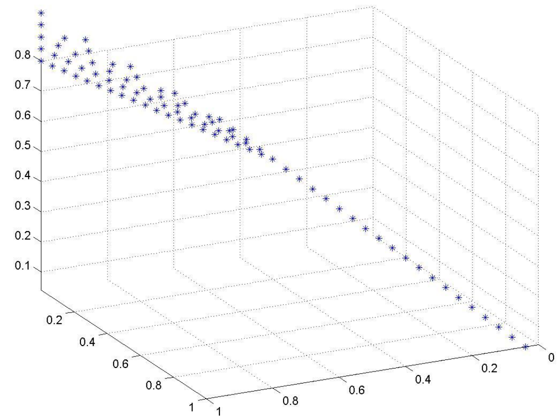

Figure 5: Second Game: Disjoint union of critical zones.

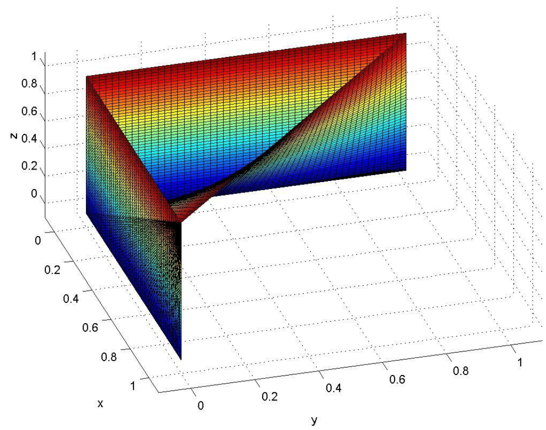

Figure 6: Second Game: Disjoint union of transformation of the topological boundary of the bistrategy space.

The disjoint union of transformations of the topological boundary of the bistrategy space is presented in Figure 6.

The disjoint union of transformation of critical zones is illustrated in Figure 7.

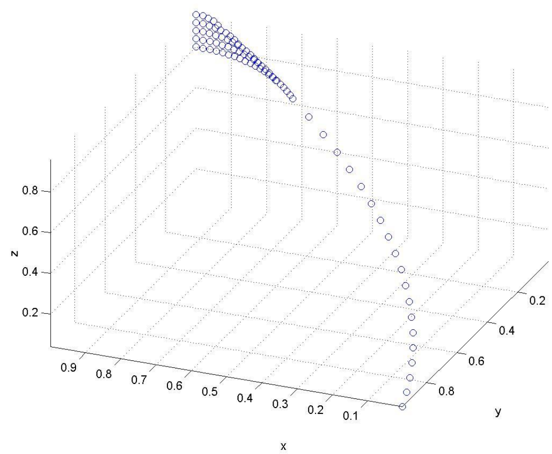

Figure 7: Second Game: Disjoint union of transformations of the critical zones.

The Payoff Space (represented in Figure 8 for every a in T), is obtained by the union of transformations of the critical zone and of the topological boundary of the bistrategy space.

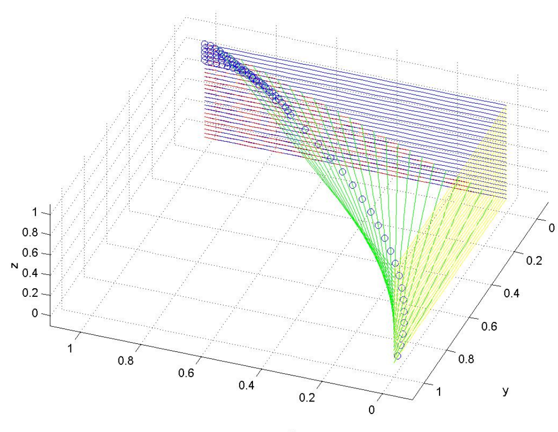

Figure 8: Second Game: Payoff space of the parametric game as disjoint union of partial payoff spaces.

8 Third Game: $p = 0.5$

Let the strategy sets of the parametric game $G = (f, <)$ be $E = F = [0, 1]$ and let the a-biloss (disutility) function of G be defined by

$$f_a(x, y) = \left(x - (1-a)xy, y - (1 - (1 - a^{0.5})^2)xy\right),$$

for all x, y and a in $[0, 1]$.

The critical zones, in Figure 9, are the sets

$$\mathcal{C}(f_a) = \left\{(x, y) \in [0, 1]^2 : 1 - (1 - (1 - a^{0.5})^2)x - (1 - a)y = 0\right\},$$

with a varying in T.

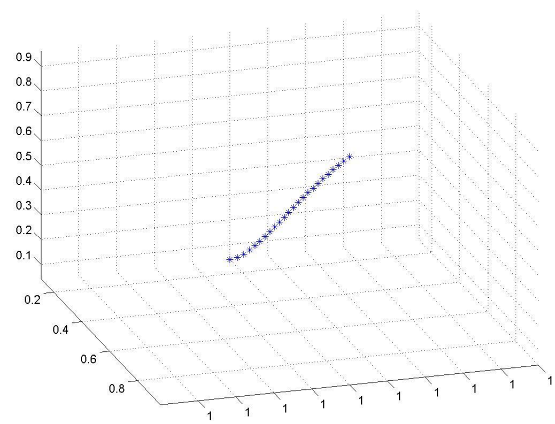

Figure 9: Second Game: Critical zones.

The transformations of the topological boundary of the bistrategy space are shown in Figure 10.

The transformations of critical zones are presented in Figure 11.

We obtain the payoff space as before, shown in Figure 12 in form of disjoint union.

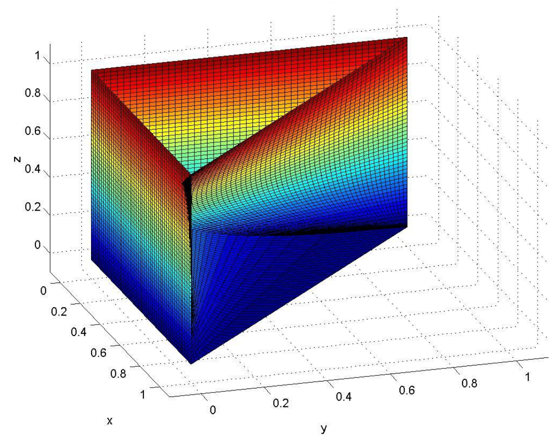

Figure 10: Third Game: Transformations of the topological boundary of the bistrategy space.

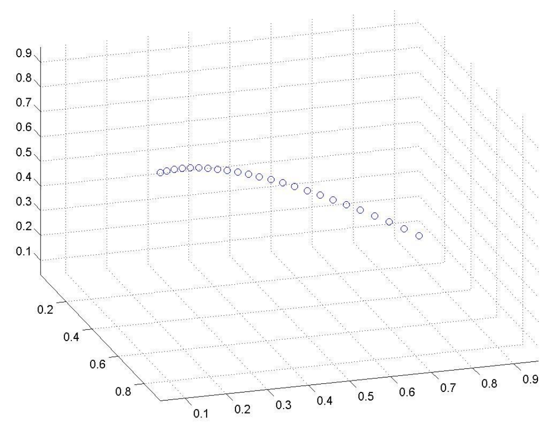

Figure 11: Third Game: Transformations of critical zones.

9 Forth Game: $p = 2$

Let the strategy sets of the parametric game $G = (f, <)$ be $E = F = [0, 1]$ and let the a-biloss (disutility) function of G be defined by

$$f_a(x, y) = \left(x - (1 - a)xy, y - (1 - (1 - a^2)^{0.5})xy\right),$$

for all x, y and a in [0, 1].

The a-critical zone is

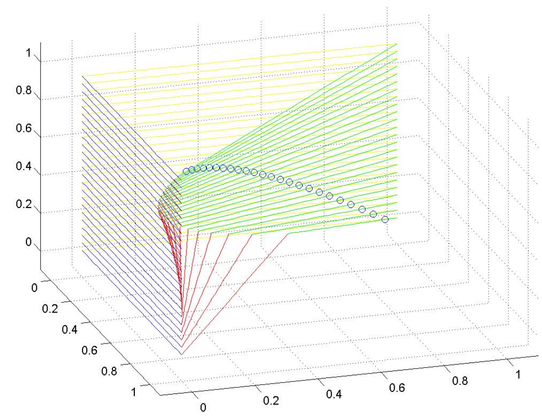

Figure 12: Third Game: Payoff space in form of disjoin union.

$$\mathcal{C}(f_a) = \left\{(x, y) \in [0, 1]^2 : 1 - (1 - (1 - a^2)^{0.5})x - (1 - a)y = 0\right\}.$$

So the Payoff Space overlap the transformation of the topological boundary, as shown in Figure 13 for every a.

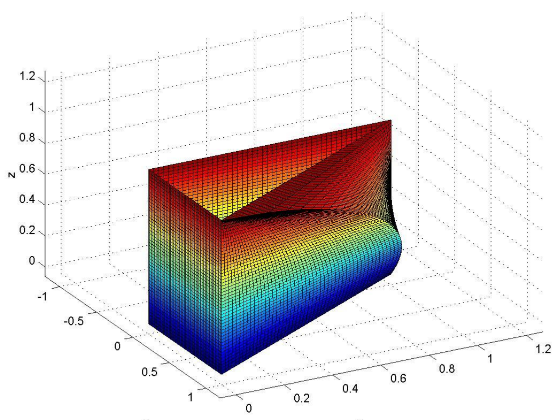

Figure 13: Forth Game: Transformations of the topological boundary.

10 Fifth game: p=10

Let strategy sets be $E = F = [0,1]$ and biloss (disutility) function be

$$f_a(x,y) = \left(x - (1-a)xy, y - (1-(1-a^{10})^{0.1})xy\right),$$

for all (x,y) and a in $[0,1]$.

The critical zones, in Figure 14, are

$$\mathcal{C}(f_a) = \left\{(x,y) \in [0,1]^2 : 1 - (1-(1-a^{10})^{0.1})x - (1-a)y = 0\right\},$$

with a varying in T.

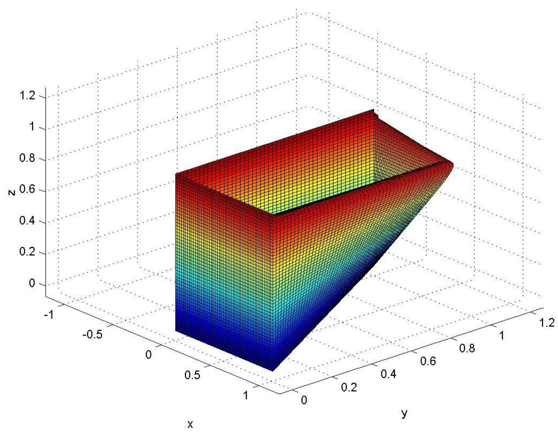

Figure 14: Fifth Game: Transformations of the topological boundary.

11 Sixth game

In this section we present a new game, where strategy sets are $E = F = [0,2]$, the parameter set is $T = [0,1]$ and the a-biloss (disutility) function is defined by

$$f_a(x,y) = \left(x + y + a, x - y + a^2\right),$$

for all x, y in $[0,2]$ and a in $[0,1]$.

The critical zone is void, so the payoff spaces overlap the transformations of the topological boundary, in Figure 15.

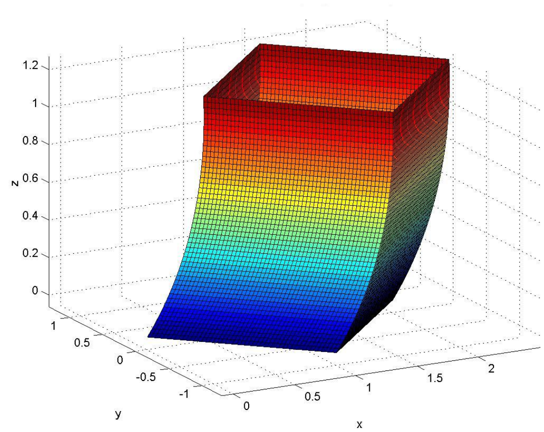

Figure 15: Sixth Game: Transformation of the topological boundary

12 Seventh game

In this section we present a new game, where strategy sets are $E = F = [0, 2]$, the parameter set is $T = [-1, 1]$ and the a-biloss (disutility) function is

$$f_a(x, y) = (x + y + a, x - y + |a|),$$

for all x, y, a in $[-1, 1]$.

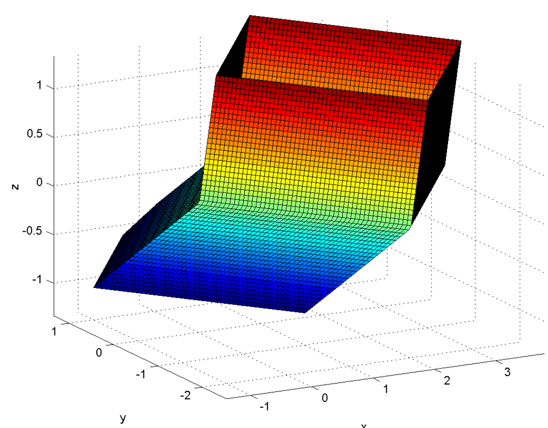

Figure 16: Seventh Game: Transformations of the topological boundary

The critical zone is empty, so the payoff spaces overlap the transformations of the topological boundary, in Figure 15.

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