Consistency in (Super-)Majority Decision

Xefteris, Dimitrios

University of Cyprus

1 May 2010

Consistency in (Super-)Majority Decision

Dimitrios Xefteris
University of Cyprus

February 2011

Abstract

This paper identifies necessary and sufficient single-profile conditions for consistent decision under all super-majority rules. It is demonstrated that if one begins by discarding any ordering and its inverse whenever they are both found in the preference profile, then the reduced profile will generate a transitive super-majority rule relation if and only if it is not *balanced enough* relative to the size of the super-majority.

Keywords: supermajority, transitivity

JEL classification: D70, D71

1 Introduction

Consistent decision making under supermajority rules has been neglected by economic theory in the most ostentatious way. To be fair in this judgement we should add that all the effort was rightfully directed in the understanding of the most interesting extreme of the supermajority spectrum; the simple majority rule. Simple majority has many good properties compared to other rules (including all other supermajority rules) and therefore conditions that guarantee its functionality have been sought for and subsequently presented in the literature (single-peakedness, value restrictedness etc.). This search though, for conditions that guarantee the well behaving of majority related Social Welfare Functions (SWF) has practically[1] stopped by Inada's (1969) seminal discourse on the simple majority rule for a seemingly obvious reason. Inada argues that, as far as the simple majority rule is concerned, Sen's (1966) value-restrictedness condition generates the widest list of individual preferences that, if a preference profile is formed by a fraction of them, then a transitive simple majority rule relation is always guaranteed. According to this interpretation value-restrictedness is both sufficient and necessary for consistency of simple majority decision; no wider condition can ever be obtained. On the other hand though, we know that a preference profile may yield a transitive simple majority rule relation and violate at the same time value-restrictedness.

[1] There are few recent studies though. See for example Dasgupta and Maskin (1998,2008).

This obvious fact points out to an incompatibility between the standard mathematical meaning of necessity and what Inada defines as such. This incompatibility is due to a very simple reason. The "input" of a SWF is a preference profile and not a vector of distinct individual preferences. As in any other class of functions, if we want to find necessary and sufficient conditions such that the "output" of a SWF has a certain property (that it yields a transitive relation in our case) then these conditions should refer to the "input" of the SWF, that is, on the domain of preference profiles and not on the domain of individual preferences from which a preference profile may be formed. Single-peakedness and value-restrictedness (and all conditions of such form) are conditions on the domain of individual preferences that may form a preference profile and, therefore, restrict the domain of preference profiles not directly but indirectly. For this reason they are very "strong" conditions and in such distance with the pure mathematical meaning of necessity. This simple observation allows us to revisit the issue of consistency of SWFs from the "preference profile domain" perspective and derive new results.

In particular, this paper will study the whole range of supermajority SWFs (from simple majority rule to unanimity) and will present necessary and sufficient conditions for the transitivity of any supermajority rule relation[2]. These conditions will describe a property that *a)* a certain preference

[2] A condition of equivalent nature regarding simple majority rule was proposed by Feld and Grofman (1983).

profile should posses in order to yield a transitive supermajority rule relation (sufficiency) and *b)* if a certain preference profile does not posses this property then it yields an intransitive supermajority rule relation (necessity).

Our conditions will, thus, describe a single-profile property and will therefore allow us to split the universal domain of preference profiles in two; the first domain of preference profiles consists of those who ensure transitivity and the second consists of those who violate it. We need to stress at this point that, obviously, these two domains will vary along with the exact supermajority rule under study. A preference profile may yield a transitive social preference relation under some supermajority rule and may yield an intransitive social preference relation under another one.

2 Analysis

Suppose that there exists a set of individuals $N = \{1, 2, ..., n\}$, $\#N \geq 3$ and odd and a finite set of alternatives X, $\#X = 3$. We consider three alternatives only for simplicity. We will argue later that our results apply to any arbitrary number of distinct alternatives. Each $i \in N$ has complete, transitive and strict (linear) preferences on X which can be represented by a linear order of the elements of X (any linear order on X is permitted).

Assume that a supermajority SWF is applied to aggregate social preferences.

Define the supermajority SWF as follows:

$$x \succ y \text{ if } \#\{i \in N | x \succ_i y\} > \tfrac{\#N+1}{2} + a$$
$$y \succ x \text{ if } \#\{i \in N | y \succ_i x\} > \tfrac{\#N+1}{2} + a$$
and
$$x \sim y \text{ otherwise.}$$

where a is a non-negative non-integer number[3] such that $0 \leq a \leq \frac{\#N-1}{2}$. Obviously, a is a measure of the supermajority needed for an alternative to be preferred to another. We will therefore name as $a-$supermajority the supermajority rule that requires a votes in excess of simple majority for an alternative to be socially preferred to another. When $a = 0$ we are obviously in the simple majority case and when $a = \frac{\#N-1}{2}$ we are in the unanimity case.

We define as θ the preference profile of the society N on the set of alternatives X and $\theta(i)$ the linear order which represents the preferences of individual $i \in N$.

Given the environment that is described above, define the following.

Definition 1 (Mutual Exclusiveness) *Individuals i and j are mutually exclusive in X if and only if for every two distinct x and y from X either $x \succ_i y$ and $y \succ_j x$ or $y \succ_i x$ and $x \succ_j y$ (or, if and only if $\theta(i)$ is the inverse of $\theta(j)$).*

[3] Although we assume that a is generically a non-integer (for simplification of our analysis) we allow the value 0 (simple majority) and the value $\frac{\#N-1}{2}$ (unanimity rule). Our proof will perfectly work for these specific two integers.

Definition 2 (Reduced Population) *Given a set of alternatives X, a reduced population $\tilde{N}$ is a subset of N that a) does not include any mutually exclusive individuals in X and b) $N \backslash \tilde{N}$ is either the union of $\frac{\# N \backslash \tilde{N}}{2}$ disjoint pairs of mutually exclusive individuals or empty.*

Since $N \backslash \tilde{N}$ contains individuals that, in pairs, cancel out each other votes, it easily follows that if $\#\{i \in N | x \succ_i y\} > \frac{\# N+1}{2} + a$ then $\#\{i \in \tilde{N} | x \succ_i y\} > \frac{\# \tilde{N}+1}{2} + a$ and vice versa for any $\tilde{N}$. This allows us to re-phrase the definition of the majority rule in the following way:

$$x \succ y \text{ if } \#\{i \in \tilde{N} | x \succ_i y\} > \tfrac{\# \tilde{N}+1}{2} + a$$
$$y \succ x \text{ if } \#\{i \in \tilde{N} | y \succ_i x\} > \tfrac{\# \tilde{N}+1}{2} + a$$
$$\text{and}$$
$$x \sim y \text{ otherwise.}$$

Definition 3 (Reduced Preference Profile) *The reduced preference profile of θ, $r(\theta)$, is the preference profile of any reduced population $\tilde{N}$ (it is trivial to see that the preference profiles of all reduced populations are identical).*

Definition 4 (Relative Balancedness) *A preference profile θ is balanced relative to the $a-$supermajority rule if and only if:*

i) each $x \in X$ is ranked at the top and at the bottom of $\theta(i)$ for less than $\frac{\# \tilde{N}+1}{2} + a$ individuals $i \in \tilde{N}$ in the reduced population and *ii) at least one $x \in X$ is ranked at the top and at the bottom of $\theta(i)$ for less than $\frac{\#\tilde{N}-1}{2} - a$ individuals $i \in \tilde{N}$ in the reduced population.*

Before stating and proving the main result of this paper we will argue that the set of orderings in the reduced profile (we will call it $s(\theta)$) has at most three elements for any profile θ. Let L denote the set of all linear orderings on X. There will be six of them, and we can partition L into three two-element sets, each consisting of an ordering and its inverse. Let S be any subset of L consisting of four or more orderings. Then it excludes orderings from at most two of the members of the partition, leaving at least one ordering-inverse pair contained in S, which cannot therefore be the set of orderings for a reduced profile. Thus, for any profile θ we have $s(\theta) = \{R, Q, V\}$, possibly with $R = Q$, or even $R = Q = V$.

We can now state the theorem.

Theorem *An arbitrary preference profile θ yields a transitive $a-$supermajority rule relation if and only if it violates relative balancedness.*

Proof Let θ be an arbitrary profile. (1) Suppose that θ is balanced relative to the $a-$supermajority rule (relative balancedness satisfied).

Then $s(\theta)$ cannot consist out of a unique linear order. To prove that imagine that it does. Then we have two cases. Either $\#\tilde{N} > \frac{\#\tilde{N}+1}{2} + a$ or $\#\tilde{N} < \frac{\#\tilde{N}+1}{2} + a$. In the first case the condition (i) of relative balancedness is violated while in the second case the condition (ii) of relative balancedness is violated (this is because $\#\tilde{N} < \frac{\#\tilde{N}+1}{2} + a$ implies $\frac{\#\tilde{N}-1}{2} - a < 0$ and all elements of X appear at least zero times as best and worst in $r(\theta)$).

If $s(\theta)$ is composed out of two linear orders then $s(\theta)$ may take one of the three following forms:

a)

R	Q
x	y
y	x
z	z

b)

R	Q
x	z
y	x
z	y

c)

R	Q
x	x
y	z
z	y

d)

R	Q
x	y
y	z
z	x

Observe that in cases a) and c) non violation of (i) means $\#\tilde{N} < \frac{\#\tilde{N}+1}{2} + a$ and therefore $\frac{\#\tilde{N}-1}{2} - a < 0$; violation of (ii) since all elements of $s(\theta)$ appear first (and last) at least one time. Lets focus now on b) (the arguments for d) are equivalent). Non violation of (i) implies $x \sim z$ and $y \sim z$. If $\#\tilde{N} < \frac{\#\tilde{N}+1}{2} + a$ then $\frac{\#\tilde{N}-1}{2} - a < 0$; we have violation of (ii) since all elements of $s(\theta)$ appear first (and last) at least one time. Therefore, for both (i) and (ii) to be satisfied we should have $\#\tilde{N} > \frac{\#\tilde{N}+1}{2} + a$. This implies that $\frac{\#\tilde{N}-1}{2} - a > 0$ and, indeed, violation of (ii) becomes impossible since x (y) appears last (first) zero times. In this case $x \succ y$; intransitivity.

If $s(\theta)$ consists out of three linear orders, without loss of generality, it may take one of the two following forms[4]:

a)

R	Q	V
x	x	y
y	z	x
z	y	z

b)

R	Q	V
x	z	y
y	x	z
z	y	x

Lets focus on a) first. Condition (i) of relative balancedness implies that $\tilde{n}(R)+\tilde{n}(Q) < \frac{\#\tilde{N}+1}{2}+a$ and therefore that $\tilde{n}(V) > \frac{\#\tilde{N}-1}{2}-a$ and that $x \sim y$. Moreover, condition (i) of relative balancedness implies that $\tilde{n}(R)+\tilde{n}(V) < \frac{\#\tilde{N}+1}{2}+a$ and therefore that $\tilde{n}(Q) > \frac{\#\tilde{N}-1}{2}-a$ and that $y \sim z$. Thus, by condition (ii) we should have that $\tilde{n}(R) < \frac{\#\tilde{N}-1}{2}-a$ which implies that $\tilde{n}(Q)+\tilde{n}(V) > \frac{\#\tilde{N}+1}{2}+a$ and that $x \succ z$ (intransitivity).

Now lets focus on b). Assume without loss of generality that $\tilde{n}(R) < \frac{\#\tilde{N}-1}{2}-a$ (condition (ii)). Then $\tilde{n}(Q)+\tilde{n}(V) > \frac{\#\tilde{N}+1}{2}+a$ and $z \succ x$. We now have to distinguish between two cases; $\tilde{n}(Q) < \frac{\#\tilde{N}-1}{2}-a$ and $\tilde{n}(Q) > \frac{\#\tilde{N}-1}{2}-a$. We will solve only for the first case as a symmetric argument is valid for the second as well. If $\tilde{n}(Q) < \frac{\#\tilde{N}-1}{2}-a$ then $\tilde{n}(R)+\tilde{n}(V) > \frac{\#\tilde{N}+1}{2}+a$

[4]This is due to the fact that $s(\theta)$ cannot be composed of linear orders that have common intermediate elements.

and $y \succ z$. If $\tilde{n}(V) < \frac{\#\tilde{N}-1}{2} - a$ then $\tilde{n}(R) + \tilde{n}(Q) > \frac{\#\tilde{N}+1}{2} + a$ and $x \succ y$ (intransitivity). If $\tilde{n}(V) > \frac{\#\tilde{N}-1}{2} - a$ then $\tilde{n}(R) + \tilde{n}(Q) < \frac{\#\tilde{N}+1}{2} + a$ and $x \sim y$ (intransitivity).

We have proven that when θ is balanced relative to the $a-$supermajority rule then the $a-$supermajority rule relation is intransitive.

(2) Now suppose θ is not balanced relative to the $a-$supermajority rule (relative balancedness violated).

Note that violation of relative balancedness means violation of at least one of conditions (i) and (ii) and not necessarily both.

If (i) is violated then independently of the number of elements of $s(\theta)$ we must have that some $x \in X$ appears first (or last) more than $\frac{\#\tilde{N}+1}{2} + a$ times in $r(\theta)$. Therefore we have that $\#\{i \in \tilde{N} | x \succ_i y\} > \frac{\#\tilde{N}+1}{2} + a$ and $\#\{i \in \tilde{N} | x \succ_i z\} > \frac{\#\tilde{N}+1}{2} + a$ (or with $<$ if it appears last) which means that $x \succ y$ and that $x \succ z$ (or $y \succ x$, $z \succ x$ if it appears last). Transitivity is guaranteed independently of the social preference relation between y and z.

If (i) is not violated and (ii) is violated we observe that $s(\theta)$ should be composed out of two or three distinct linear orders. If (i) is not violated and $s(\theta)$ is composed out of one linear order then (by non violation of (i)) we have $\#\tilde{N} < \frac{\#\tilde{N}+1}{2} + a$ which implies that $\frac{\#\tilde{N}-1}{2} - a < 0$; non violation of (ii).

If (i) is not violated and $s(\theta)$ is composed out of two linear orders then $s(\theta)$ may take one of the three following forms:

a)

R	Q
x	y
y	x
z	z

b)

R	Q
x	z
y	x
z	y

c)

R	Q
x	x
y	z
z	y

d)

R	Q
x	y
y	z
z	x

Observe that in cases a) and c) non violation of (i) means $\#\tilde{N} < \frac{\#\tilde{N}+1}{2} + a$ and therefore $\frac{\#\tilde{N}-1}{2} - a < 0$; violation of (ii). We observe that in such a case we have $x \sim y \sim z \sim x$; transitivity. Lets focus on b) (as in the part (1) of the proof the arguments for d) are equivalent). Non violation of (i) implies $x \sim z$ and $y \sim z$. If $\#\tilde{N} < \frac{\#\tilde{N}+1}{2} + a$ and therefore $\frac{\#\tilde{N}-1}{2} - a < 0$ we have violation of (ii) since all elements of $s(\theta)$ appear first (and last) at least one time. In this case $x \sim y$ and transitivity is guaranteed. If $\#\tilde{N} > \frac{\#\tilde{N}+1}{2} + a$ then $\frac{\#\tilde{N}-1}{2} - a > 0$ and therefore violation of (ii) becomes impossible since x (y) appears last (first) zero times.

If (i) is not violated and $s(\theta)$ is composed out of three linear orders then $s(\theta)$, as in part (1) of the proof, may take one of the two following forms:

a)

R	Q	V
x	x	y
y	z	x
z	y	z

b)

R	Q	V
x	z	y
y	x	z
z	y	x

Lets focus on a) first. Non violation of (i) implies $x \sim y$ and $y \sim z$. If $\#\tilde{N} < \frac{\#\tilde{N}+1}{2} + a$ and therefore $\frac{\#\tilde{N}-1}{2} - a < 0$ we have violation of (ii) since all elements of $s(\theta)$ appear first (and last) at least one time. In this case $x \sim z$; transitivity. If $\#\tilde{N} > \frac{\#\tilde{N}+1}{2} + a$ then $\frac{\#\tilde{N}-1}{2} - a > 0$ and therefore violation of (ii) becomes impossible since x (z) appears last (first) zero times. Finally, lets examine case b). Violation of (ii) means that $\tilde{n}(R) > \frac{\#\tilde{N}-1}{2} - a$, $\tilde{n}(Q) > \frac{\#\tilde{N}-1}{2} - a$ and $\tilde{n}(V) > \frac{\#\tilde{N}-1}{2} - a$. Therefore, $\tilde{n}(Q) + \tilde{n}(V) = \#\tilde{N} - \tilde{n}(R) < \#\tilde{N} - \frac{\#\tilde{N}-1}{2} + a = \frac{\#\tilde{N}+1}{2} + a$ and equivalently, $\tilde{n}(R) + \tilde{n}(V) < \frac{\#\tilde{N}+1}{2} + a$ and $\tilde{n}(R) + \tilde{n}(Q) < \frac{\#\tilde{N}+1}{2} + a$. This means that $x \sim z$, $y \sim z$ and $x \sim y$; transitivity. **QED**

Finally, we must stress that because transitivity requires focusing on triples of alternatives, the result presented here applies immediately to cases where X has four or more alternatives. Of course, it is essential to point out that the definition of *relative balancedness* for an arbitrary triple $A \subset X$ must apply to the restriction of the profile θ to A. So $\theta(i)$ and $\theta(j)$ need not be inverses of each other, but if the restriction of $\theta(i)$ to A is the inverse of the restriction of $\theta(j)$ to A, then both are eliminated when constructing the reduced profile $r(\theta, A)$ for the triple A.

3 Concluding remarks

Recent literature on majority decisions has been, mainly, focused in providing characterizations of the majority rule (e.g. Campbell and Kelly, 2000) and in identifying its "good" properties (Dasgupta and Maskin, 2008). The present paper, reviews the issue of transitivity of the majority rule and establishes a unique necessary and sufficient condition. By introducing the concept of the reduced population for every set of alternatives, we can now easily approach the problem, from an economically meaningful way.

The idea for the result is simple, and based on construction or elimination of Condorcet cycles. As far as triplets of alternatives (x, y, z) are concerned, the above condition states that one of the three alternatives in (x, y, z) is either strictly preferred to the other two, or strictly worse than the other two, for a majority of voters, after eliminating any pair of voters that have exactly offsetting preference orderings. It is easy to see that, if the identified condition holds for any triplet (x, y, z), majority voting implies transitivity almost by construction, because there is an alternative that is worst or best in any pairwise comparison with the two other alternatives. Therefore, it must be that pairwise majority voting generates transitive preferences.

For the counterpart, one can show that if there exists such a triplet for which none of the alternatives is ranked first or last by a majority of voters of the reduced population, then this triplet can be used to construct a Condorcet cycle. This argument follows from the observation that after eliminating off-setting (mutually exclusive) voters from considerations, at most three different profiles remain. If there are only one or two profiles, transitivity is guaranteed, so lack of transitivity requires three profiles, and they must form a Condorcet cycle. The result is then immediate.

4 References

References

Arrow, Kenneth J. (1951). *Social Choice and Individual Values.* John Wiley.

Black, David (1948). "On the Rationale of Group Decision-making." *Journal of Political Economy*, 56(1), 23–34.

Campbell, Donald, and Jerry Kelly (2000). "A Simple Characterization of Majority Rule." *Economic Theory*, 15, 689–700.

Condorcet, Marie Jean Antoine Nicolas Caritat (1785). *Essai sur l'application de l'analyse à la pluralité des voix.* Imprimerie Royale.

Dasgupta, Partha, and Eric Maskin (2008). "On the Robustness of Majority Rule." *Journal of the European Economic Association*, 6(5), 949-973.

Feld and Grofman (1983), "Partial Single-Peakdness: An extension and clarification", *Public Choice*, 51:71-80

Inada, Ken-Ichi (1969). "On the Simple Majority Rule Decision." *Econometrica*, 37(3), 490–506.

Sen, Amartya (1966). "A Possibility Theorem on Majority Decisions." *Econometrica*, 34(2), 491–499.

Sen, Amartya, and Prasanta Pattanaik (1969). "Necessary and Sufficient Conditions for Rational Choice under Majority Decision." *Journal of Economic Theory*, 1(2), 178–202.