Social choice with approximate interpersonal comparison of welfare gains

Pivato, Marcus

Department of Mathematics, Trent University

14 July 2011

Social choice with approximate interpersonal comparison of welfare gains

Marcus Pivato, Trent University, Canada

July 14, 2011

Abstract

Suppose it is possible to make approximate interpersonal comparisons of welfare gains and losses. Thus, if w, x, y, and z are personal psychophysical states (each encoding all ethically relevant information about the physical and mental state of a person), then it sometimes possible to say, "The welfare gain of the state change $w \rightsquigarrow x$ is greater than the welfare gain of the state change $y \rightsquigarrow z$." We can represent this by the formula "$(w \rightsquigarrow x) \succeq (y \rightsquigarrow z)$", where $(\succeq)$ is a *difference preorder*: an incomplete preorder on the space of all possible personal state changes. A *social state change* is a bundle of personal state changes. A *social difference preorder* (SDP) is an incomplete preorder on the space of social state changes, which satisfies Pareto and Anonymity axioms. The *minimal* SDP is the natural extension of the Suppes-Sen preorder to this setting; we show it is a subrelation of every other SDP. The *approximate utilitarian* SDP ranks social state changes by comparing the sum total utility gain they induce, with respect to all 'utility functions' compatible with $(\succeq)$. The *net gain* preorder ranks social state changes by comparing the aggregate welfare gain they induce upon various subpopulations. We show that, under certain conditions, all three of these preorders coincide.

Many rules for measuring social welfare or making collective choices rely on interpersonal comparisons of wellbeing. These interpersonal comparisons are fraught with difficulties, both philosophical and practical (Elster and Roemer, 1991; Fleurbaey and Hammond, 2004), and have sometimes been rejected as impossible or even meaningless (Robbins, 1935, 1938, for example). However, many of these problems arise from an insistence on 'precise' interpersonal comparisons. Such precision may be impossible, but it is also unnecessary. Sen (1970a, 1972 and Ch.7* of 1970b), Fine (1975), Blackorby (1975), Basu (1980, Ch.6), Baucells and Shapley (2006, 2008) and Pivato (2010a,b,c) have shown that it is often possible to make rough social evaluations using only 'approximate' interpersonal comparisons of utility. The present paper extends this approach.

Every person has both a 'physical' state (e.g. her health and wealth) and a 'psychological' state (e.g. her beliefs, desires, and personality). Like Pivato (2010a,b,c), this paper supposes that both physical states *and* the psychological states are mutable, and hence, potential targets of individual or collective choice. (For example: economic policies and

safety regulations influence people's physical states. Policies which subsidize or regulate education, arts and cultural industries, mental health care, and psychopharmaceuticals influence people's psychological states.) Interpersonal comparisons rank the welfare of different psychophysical states —either of different people, or of the same person at different moments in time. Not all interpersonal comparisons are possible, but some certainly are (otherwise even individual intertemporal choice would be impossible).

Formally, let $\mathcal{X}$ be a space of 'psychophysical states'. An element $x \in \mathcal{X}$ encodes all information about an individual's psychology (i.e. her personality, mood, knowledge, beliefs, memories, values, desires, etc.) and also all information about her personal physical state (i.e. her health, wealth, personal property, physical location, consumption bundle, sense-data, etc.).[1] Any person, at any moment in time, resides at some point in $\mathcal{X}$. Pivato (2010a,b,c) supposes that it is (sometimes) possible to compare the welfare levels of different psychophysical states: there is an (incomplete) preorder ($\succeq$) on $\mathcal{X}$, such that, for any $x, y \in \mathcal{X}$, the statement "$x \succeq y$" means that the welfare level of psychophysical state x is at least as high as that of y. [2]

The present paper, in contrast, supposes we can compare not absolute welfare *levels*, but rather, welfare *changes*. Thus, we can (sometimes) make sense of the statement:[3]

> "The welfare improvement in moving from psychophysical state x_1
> to state x_2 is greater than the welfare improvement in moving from   (1)
> state y_1 to y_2."

We can represent this with an (incomplete) preorder ($\succeq$) on the Cartesian product $\mathcal{X} \times \mathcal{X}$. We will write an ordered pair $(x_1, x_2) \in \mathcal{X} \times \mathcal{X}$ as "$x_1 \rightsquigarrow x_2$" to emphasize that it represents a *change* from x_1 to x_2. Then statement (1) is represented by the formula "$(x_1 \rightsquigarrow x_2) \succ (y_1 \rightsquigarrow y_2)$".

A social policy will change the psychophysical states of many people; some will gain in welfare, while others will lose. Using the preorder ($\succeq$), this paper shows how to compare and aggregate the welfare costs and benefits imposed upon different people, and identify the social policy which causes the greatest aggregate welfare enhancement. The paper is organized as follows. Section 1 introduces notation and terminology. Section 2 axiomatizes and discusses *difference preorders*: preorders on $\mathcal{X} \times \mathcal{X}$ which encode statements like (1). Section 3 then defines a *social difference preorder* (SDP) to be a preorder on the space of *social* state changes which satisfies weak versions of the Pareto and Anonymity axioms. A key examples is the family of *quasiutilitarian* SDPs, which rank two state changes by

---

[1]Unlike Pivato (2010a), this model does not assume it is possible to cleanly separate someone's 'psychological' state from her 'physical' state. Indeed, if the mind is a function of the brain, then her psychological state is simply one aspect of her physical state.

[2]For example, ($\succeq$) could represent the 'extended preferences' of Harsanyi (1955, fn.16 on p.316; 1977b, p.53 of §4.2), Sen (1970a, p.152 of §9*1) and Arrow (1977), or it could represent the 'fundamental preferences' of Kolm (1994a,b, 1995, 2002)

[3]For example, a 'preferencist' interpretation of this statement would be that a $(\frac{1}{2}, \frac{1}{2})$ lottery between outcomes y_1 and x_2 is preferable to a $(\frac{1}{2}, \frac{1}{2})$ lottery between x_1 and y_2. Note that it is not necessary to have a complete system of von Neumann-Morgenstern preferences over lotteries to make such judgements; it is only necessary to have reasonably consistent preferences over $(\frac{1}{2}, \frac{1}{2})$ lotteries —e.g. the 'quasicardinal' utility functions of (Basu, 1980, Ch.6).

2

comparing their utilitarian sums with respect to some list of utility functions compatible with ($\succeq$). Our first main result (Theorem 3.2) says that these are the *only* SDPs which can be represented by social welfare functions. Section 4 introduces the *minimal* SDP, which is a subrelation of every other SDP (Proposition 4.1). Our second major result (Theorem 4.2) says that, under certain conditions, the minimal SDP is the *approximate utilitarian* SDP —the quasiutilitarian SDP defined by the list of *all* utility functions compatible with ($\succeq$). Next, Section 5 introduces the *net gain* preorder, which is also a subrelation of every SDP (Proposition 5.2). The third main result, Theorem 5.1, says that, under certain hypotheses, the net gain preorder is the minimal SDP.

Section 6 applies the SDP concept to a simple model of redistributive wealth transfers. Section 7 discusses necessary and sufficient condition for an 'empathy' hypothesis which appears in Theorems 4.2 and 5.1. Appendix A contains the proofs of all results. Appendix B discusses complete extensions of difference preorders; it provides counterexamples to the analogues of Szpilrajn's Lemma and the Dushnik-Miller theorem.

Previous literature. Alt (1936, 1971), Suppes and Winet (1955), Scott and Suppes (1958), Debreu (1958), Pfanzagl (1968) and Krantz et al. (1971) used (complete) difference preorders to construct cardinal utility representations for individual preferences. Later, Dyer and Sarin (1978, 1979a,b), Harvey (1999) and Harvey and Østerdal (2010) studied the utilitarian aggregation of such individual difference preorders into a (complete) social difference preorder.[4] Theorem 3.2 of this paper is roughly comparable to these earlier results. However, the main goal of this paper is to grapple with imperfect interpersonal comparability, in the spirit of Sen (1970a,b, 1972), Fine (1975), Blackorby (1975), Basu (1980), Baucells and Shapley (2006, 2008) and Pivato (2010a,b,c). Thus, all the results are formulated in terms of *incomplete* difference preorders, and many do not assume the existence of a cardinal utility representation.

1 Preliminaries

Let $\mathcal{S}$ be a set. A *preorder* on $\mathcal{S}$ is a binary relation ($\succeq$) which is *transitive* (for all $r, s, t \in \mathcal{S}$, $(r \succeq s \succeq t) \implies (r \succeq t)$) and *reflexive* (for all $s \in \mathcal{S}$, $s \succeq s$), but not necessarily complete or antisymmetric. The *symmetric part* of ($\succeq$) is the relation ($\approx$) defined by $(s \approx t) \Leftrightarrow (s \succeq t$ and $t \succeq s)$. The *antisymmetric part* of ($\succeq$) is the relation ($\succ$) defined by $(s \succ t) \Leftrightarrow (s \succeq t$ and $t \not\succeq s)$. The preorder ($\succeq$) is *complete* if, for all $s, t \in \mathcal{S}$, either $s \succeq t$ or $t \succeq s$. Most of the preorders considered in this paper are incomplete.

Let ($\underset{1}{\trianglerighteq}$) and ($\underset{2}{\trianglerighteq}$) be two binary relations on $\mathcal{S}$. We say that ($\underset{2}{\trianglerighteq}$) *extends* ($\underset{1}{\trianglerighteq}$) if, for all $s, t \in \mathcal{S}$, we have $(s \underset{1}{\trianglerighteq} t) \implies (s \underset{2}{\trianglerighteq} t)$. (If we represent ($\underset{1}{\trianglerighteq}$) and ($\underset{2}{\trianglerighteq}$) as subsets of $\mathcal{S} \times \mathcal{S}$ in the standard way, this just means ($\underset{1}{\trianglerighteq}$) $\subseteq$ ($\underset{2}{\trianglerighteq}$).) Let ($\underset{1}{\widehat{=}}$) be the symmetric part of ($\underset{1}{\trianglerighteq}$), and let ($\underset{1}{\vartriangleright}$) be its antisymmetric part. We say that ($\underset{2}{\trianglerighteq}$) *refines* ($\underset{1}{\trianglerighteq}$) if, for all $s, t \in \mathcal{S}$, we have $(s \underset{1}{\vartriangleright} t) \implies (s \underset{2}{\vartriangleright} t)$, while $(s \underset{1}{\widehat{=}} t) \implies (s \underset{2}{\trianglerighteq} t$ or $s \underset{2}{\trianglelefteq} t)$. If ($\underset{1}{\vartriangleright}$) and ($\underset{2}{\vartriangleright}$) are *partial orders*

---

[4]I am grateful to Lars Peter Østerdal for making me aware of this prior literature, when I presented this paper at *New Directions in Welfare* (OECD, Paris, July, 2011).

(i.e. antisymmetric, transitive relations), then $(\underset{2}{\triangleright})$ extends $(\underset{1}{\triangleright})$ if and only if $(\underset{2}{\triangleright})$ refines $(\underset{1}{\triangleright})$. However, in general the two concepts do not coincide.

2 Difference preorders

Let $(\succeq)$ be a preorder on $\mathcal{X} \times \mathcal{X}$, intended to compare the welfare gains or losses imposed by different psychophysical state changes. Thus, the formula "$(x_1 \rightsquigarrow x_2) \succ (y_1 \rightsquigarrow y_2)$" translates into statement (1) above. The preorder $(\succeq)$ must satisfy four consistency conditions:

(DP0) For all $x, y \in \mathcal{X}$, we have $(x \rightsquigarrow x) \approx (y \rightsquigarrow y)$.

(DP1) For all $x_1, x_2, y_1, y_2 \in \mathcal{X}$, if $(x_1 \rightsquigarrow x_2) \succeq (y_1 \rightsquigarrow y_2)$, then $(x_2 \rightsquigarrow x_1) \preceq (y_2 \rightsquigarrow y_1)$.

(DP2) For all x_0, x_1, x_2 and $y_0, y_1, y_2 \in \mathcal{X}$, if $(x_0 \rightsquigarrow x_1) \succeq (y_0 \rightsquigarrow y_1)$ and $(x_1 \rightsquigarrow x_2) \succeq (y_1 \rightsquigarrow y_2)$, then $(x_0 \rightsquigarrow x_2) \succeq (y_0 \rightsquigarrow y_2)$.

(DP3) For all x_0, x_1, x_2 and $y_0, y_1, y_2 \in \mathcal{X}$, if $(x_0 \rightsquigarrow x_1) \succeq (y_1 \rightsquigarrow y_2)$ and $(x_1 \rightsquigarrow x_2) \succeq (y_0 \rightsquigarrow y_1)$, then $(x_0 \rightsquigarrow x_2) \succeq (y_0 \rightsquigarrow y_2)$.

A preorder on $\mathcal{X} \times \mathcal{X}$ satisfying conditions (DP0)-(DP3) will be called a *difference preorder* on $\mathcal{X}$. Condition (DP0) means that all 'null changes' are equally worthless. Condition (DP1) says that if one change is better than another, then the *reversal* of the first change is *worse* than the reversal of the second. Condition (DP2) prevents 'composition inconsistencies', where the composition of two apparently superior small changes yields an inferior large change. Condition (DP3) says that the logic of (DP2) is commutative: when aggregating the net gain of two state changes, the order doesn't matter.[5]

Example 2.1. Let $\mathcal{V}$ be a collection of real-valued ('utility') functions on $\mathcal{X}$. For any $x_1, x_2, y_1, y_2 \in \mathcal{X}$, define $(x_1 \rightsquigarrow x_2) \underset{\mathcal{V}}{\succeq} (y_1 \rightsquigarrow y_2)$ if and only if $v(x_2) - v(x_1) \geq v(y_2) - v(y_1)$ for all $v \in \mathcal{V}$. Then $(\underset{\mathcal{V}}{\succeq})$ is a difference preorder on $\mathcal{X}$. $\diamondsuit$

We will now generalize the construction of Example 2.1. A *linearly ordered abelian group* is a triple $(\mathcal{R}, +, >)$, where $\mathcal{R}$ is a set, $+$ is an abelian group operation on $\mathcal{R}$, and $>$ is a complete, antisymmetric, transitive binary relation on $\mathcal{R}$ such that, for all $r, s \in \mathcal{R}$, if $r > 0$, then $r + s > s$. (Here, 0 denotes the identity element of $\mathcal{R}$.) For example: the set $\mathbb{R}$ of real numbers is a linearly ordered abelian group (with the standard ordering and addition operator). So is any subgroup of $\mathbb{R}$ (e.g. the group $\mathbb{Q}$ of rational numbers). For

---

[5]This is *not* equivalent to having a 'zero discount rate'. When comparing the transition $(x_0 \rightsquigarrow x_2)$ to the transition $(y_0 \rightsquigarrow y_2)$, the preorder $(\succeq)$ treats them both as if they occur over a single time step; the fact that $(x_0 \rightsquigarrow x_2)$ *can* be decomposed into $(x_0 \rightsquigarrow x_1)$ followed by $(x_1 \rightsquigarrow x_2)$ does not imply that we *must* treat $(x_0 \rightsquigarrow x_2)$ as a 'two-step' transition. (Indeed, there may be many ways to decompose $(x_0 \rightsquigarrow x_2)$ into transition chains of various lengths.)

any $n \in \mathbb{N}$, the space $\mathbb{R}^n$ is a linearly ordered abelian group under vector addition and the lexicographic order.

A **weak utility function** is a function $u : \mathcal{X} \longrightarrow \mathcal{R}$ (for some linearly ordered abelian group $\mathcal{R}$) such that, for all $x_1, x_2, y_1, y_2 \in \mathcal{X}$,

$$\Big((x_1 \rightsquigarrow x_2) \ \succeq \ (y_1 \rightsquigarrow y_2) \Big) \quad \Longrightarrow \quad \Big(u(x_2) - u(x_1) \ \geq \ u(y_2) - u(y_1) \Big). \qquad (2)$$

(For example, let $\mathcal{V}$ and $(\underset{\mathcal{V}}{\succeq})$ be as in Example 2.1; then any element of $\mathcal{V}$ is a weak utility function for $(\underset{\mathcal{V}}{\succeq})$.) If $(\succeq)$ is a *complete* difference preorder, then the "$\Longrightarrow$" in (2) becomes "$\Longleftrightarrow$". In this case, Dyer and Sarin (1978, 1979a,b) call u a *measurable value function* for $(\succeq)$, while Harvey (1999) and Harvey and Østerdal (2010) call it a *worth function*.

There are three reasons for allowing utility functions to range over arbitrary linearly ordered abelian groups, rather than restricting them to the real numbers. First, at a technical level, this significantly extends the generality of our results, and simplifies many proofs. Second, at a philosophical level, it allows for 'non-Archimedean' or 'lexicographical' preferences, where some desires are given infinite priority over other desires. (We do not take a descriptive or normative stance on whether people can or should have such preferences, but nor do we wish to exclude them *a priori*.) Finally: non-real-valued utility functions arise naturally in the setting of infinite-horizon intertemporal choice and choice under uncertainty (Pivato, 2011).

Let $\mathcal{U}(\succeq)$ be the set of all weak utility functions for $(\succeq)$. We say $(\succeq)$ has a **multiutility representation** if there is some subset $\mathcal{U}' \subseteq \mathcal{U}(\succeq)$ such that, for all $x_1, x_2, y_1, y_2 \in \mathcal{X}$,

$$\Big((x_1 \rightsquigarrow x_2) \succeq (y_1 \rightsquigarrow y_2) \Big) \iff \Big(u(x_2) - u(x_1) \geq u(y_2) - u(y_1), \ \forall \ u \in \mathcal{U}' \Big). \qquad (3)$$

For example, the preorder $(\underset{\mathcal{V}}{\succeq})$ in Example 2.1 obviously admits a multiutility representation (set $\mathcal{U}' := \mathcal{V}$). Clearly, we can always assume $\mathcal{U}' = \mathcal{U}(\succeq)$ in (3); however, sometimes it will be convenient to use a smaller set of utility functions.

If $(\succeq)$ is a *complete* difference preorder, then any multiutility representation for $(\succeq)$ can be reduced to a *utility representation* for $(\succeq)$: a single function $u : \mathcal{X} \longrightarrow \mathcal{R}$ such that "$\Longleftrightarrow$" holds in formula (2). Sufficient conditions for the existence of (real-valued) utility representations of complete difference preorders have been given by Alt (1936, 1971),[6] Suppes and Winet (1955, §5), Scott and Suppes (1958, pp.121-122), Debreu (1958), Pfanzagl (1968, Ch.9) and Krantz et al. (1971). Suppose $\{ \underset{\ell}{\succeq} \}_{\ell \in \mathcal{L}}$ is a collection of such complete difference preorders on $\mathcal{X}$ (where $\mathcal{L}$ is some indexing set), and suppose, for all $\ell \in \mathcal{L}$, that u_ℓ is a utility representation for $(\underset{\ell}{\succeq})$ (perhaps obtained using the aforementioned literature). If $(\succeq)$ is the intersection of $\{ \underset{\ell}{\succeq} \}_{\ell \in \mathcal{L}}$, then $(\succeq)$ is an (incomplete) difference preorder, with a multiutility representation given by $\mathcal{U}' := \{u_\ell\}_{\ell \in \mathcal{L}}$.

A **strong utility function** for $(\succeq)$ is a function $u : \mathcal{X} \longrightarrow \mathcal{R}$ which satisfies condition (2), and also such that, for all $x_1, x_2, y_1, y_2 \in \mathcal{X}$, we have

$$\Big((x_1 \rightsquigarrow x_2) \ \succ \ (y_1 \rightsquigarrow y_2) \Big) \quad \Longrightarrow \quad \Big(u(x_2) - u(x_1) \ > \ u(y_2) - u(y_1) \Big).$$

---

[6]See also (Camacho, 1980, §3) for a summary of Alt's model.

Proposition 2.2 *If a difference preorder has a multiutility representation* (3), *then it has a strong utility function.*

Not all difference preorders admit a multiutility representation (3), or even a strong utility function. See Appendix B for details.

3   Social difference preorders

Let $\mathcal{I}$ be a finite or infinite set, indexing a population. A *social state* is an element $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$, which assigns a psychophysical state $x_i \in \mathcal{X}$ to each $i \in \mathcal{I}$.[7] Suppose the current social state is $\mathbf{x}^0$. Any policy will result in a change to some other social state; to decide on the best policy, the social planner must be able to compare the social value of one *social state change* $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1)$ with another social state change $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2)$. Or suppose the society splits into two subgroups of equal ethical importance (both indexed by $\mathcal{I}$). Call these groups Ex and Wy, and suppose they are initially in states $\mathbf{x}^0$ and $\mathbf{y}^0$, respectively. One policy will change Ex to state $\mathbf{x}^1$ and leave Wy unchanged. The other policy will change Wy to state $\mathbf{y}^1$ and leave Ex alone. Which policy is better? (Alternately, suppose there is only one population, but the initial state is unknown, so the planner faces a risky decision. Now let Ex and Wy represent two equally probable states of nature). To answer these questions, the social planner needs a difference preorder on the space $\mathcal{X}^{\mathcal{I}}$ of social states.

A *finitary permutation* of $\mathcal{I}$ is a bijection $\pi : \mathcal{I} \longrightarrow \mathcal{I}$ admitting some finite subset $\mathcal{J} \subseteq \mathcal{I}$ such that $\pi(i) = i$ for all $i \in \mathcal{I} \setminus \mathcal{J}$. Let Π be the group of all finitary permutations of $\mathcal{I}$. (If $\mathcal{I}$ is finite, then *every* permutation is finitary; then Π is simply the group of *all* permutations of $\mathcal{I}$.) For any $\pi \in \Pi$ and $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$, we define $\pi(\mathbf{x}) := [x_{\pi(i)}]_{i \in \mathcal{I}} \in \mathcal{X}^{\mathcal{I}}$. Given an interpersonal difference preorder $(\succeq)$ on $\mathcal{X}$, a $(\succeq)$-*social difference preorder* (SDP) is a preorder $(\trianglerighteq)$ on $\mathcal{X}^{\mathcal{I}}$ which satisfies the following axioms:[8]

(WPar) For any $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, if $(x_i^1 \rightsquigarrow x_i^2) \succeq (y_i^1 \rightsquigarrow y_i^2)$ for all $i \in \mathcal{I}$, then $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \trianglerighteq (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$.

(Anon) For any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ and $\pi \in \Pi$, $(\mathbf{x} \rightsquigarrow \mathbf{x}) \triangleq (\mathbf{x} \rightsquigarrow \pi(\mathbf{x}))$.

(DP0$^{\trianglerighteq}$) For all $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we have $(\mathbf{x} \rightsquigarrow \mathbf{x}) \triangleq (\mathbf{y} \rightsquigarrow \mathbf{y})$.

---

[7]Obviously, the living population of finite universe will always be finite. But we allow $\mathcal{I}$ to be infinite to accommodate variable populations, risk, and/or intergenerational justice. For example, we could set $\mathcal{I} := \mathcal{P} \times \mathcal{T}$, where $\mathcal{P}$ is a finite set of placeholders, and where $\mathcal{T}$ is an infinite set of time periods (to model nondiscounted, infinite-horizon intertemporal social choice), or where $\mathcal{T}$ is an infinite of equally probable 'states of nature' (to model risk). An element of $\mathcal{X}^{\mathcal{I}}$ thus assigns a psychophysical state $x_{p,t}$ to each placeholder p, in every time/state t. Allowing $\mathcal{I}$ to be infinite thus greatly extends the scope of the model. But it also increases the technical complexity to some of the definitions and proofs. It may be helpful to simply assume $\mathcal{I}$ is finite during a first reading.

[8]Not all social states or all social state changes are feasible, of course. But a normative theory can make ethical judgements even about non-feasible alternatives. Thus, we define the SDP $(\trianglerighteq)$ on *all* of $\mathcal{X}^{\mathcal{I}} \times \mathcal{X}^{\mathcal{I}}$, and not just on some feasible subset.

(DP1$^{\unrhd}$) For all $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, if $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \unrhd (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$, then $(\mathbf{x}^2 \rightsquigarrow \mathbf{x}^1) \unlhd (\mathbf{y}^2 \rightsquigarrow \mathbf{y}^1)$.

(DP2$^{\unrhd}$) For all $\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2$ and $\mathbf{y}^0, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, if $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \unrhd (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^1)$ and $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \unrhd (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$ then $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \unrhd (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2)$.

(DP3$^{\unrhd}$) For all $\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2$ and $\mathbf{y}^0, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, if $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \unrhd (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$ and $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \unrhd (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^1)$ then $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \unrhd (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2)$.

Axioms (DP0$^{\unrhd}$)-(DP3$^{\unrhd}$) are the analogs of (DP0)-(DP3), reflecting the fact that ($\unrhd$) compares the social value of social state *changes*, rather than the social states themselves. Axiom (WPar) is a weak Pareto axiom. We will sometimes consider SDPs which also satisfy the 'Strong Pareto' axiom:

(SPar) For any $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, if $(x_i^1 \rightsquigarrow x_i^2) \succeq (y_i^1 \rightsquigarrow y_i^2)$ for all $i \in \mathcal{I}$, and $(x_i^1 \rightsquigarrow x_i^2) \succ (y_i^1 \rightsquigarrow y_i^2)$ for some $i \in \mathcal{I}$, then $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \vartriangleright (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$.

Axiom (Anon) is a weak form of 'anonymity' or 'impartiality', which reflects the fact that the elements of $\mathcal{I}$ are merely 'placeholders', with no intrinsic psychological content. All information about the 'psychological identity' of individual i is encoded in x_i. Thus, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ and $i, j \in \mathcal{I}$, if $x_i = y_j$, then x_i and y_j are in every sense the *same* person (even though this person has different indices in the two social alternatives). Thus, $\mathbf{x}$ and $\pi(\mathbf{x})$ represent the 'same' social alternative: permuting the indices is ethically irrelevant. Thus, (Anon) asserts that a social state change which simply permutes indices is no different than no change at all.[9] If $\mathcal{I}$ is finite, then axiom (Anon) applies to *all* permutations of $\mathcal{I}$. However, if $\mathcal{I}$ is infinite, then (Anon) is restricted to 'finitary' permutations. This restriction is necessary: requiring ($\succeq$) to be invariant under *all* permutations of $\mathcal{I}$ leads to a contradiction with axiom (SPar).[10]

Quasiutilitarian SDPs. Let $\mathcal{V} \subseteq \mathcal{U}(\succeq)$ be a nonempty set. For any $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, we define $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \unrhd_{\mathcal{V}} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$ if, for all $v \in \mathcal{V}$ there exists some finite subset $\mathcal{J}_v \subseteq \mathcal{I}$ such that:

(AU1) $\sum_{j \in \mathcal{J}_v} \left(v(x_j^2) - v(x_j^1) \right) \geq \sum_{j \in \mathcal{J}_v} \left(v(y_j^2) - v(y_j^1) \right)$; and

(AU2) $(x_i^1 \rightsquigarrow x_i^2) \succeq (y_i^1 \rightsquigarrow y_i^2)$ for all $i \in \mathcal{I} \setminus \mathcal{J}_v$.

---

[9]See (Pivato, 2010b, §3) for further discussion of the normative significance of (Anon).

[10]For example, suppose $\mathcal{X} = \mathbb{R}$ with the usual (complete) ordering, let $\mathcal{I} = \mathbb{Z}$, and define $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{X}^{\mathcal{I}}$ as follows: $x_i := i - 1$, $y_i := i$, and $z_i := i + 1$ for all $i \in \mathcal{X}^{\mathcal{I}}$. Thus, the transition $(\mathbf{y} \rightsquigarrow \mathbf{z})$ strictly improves every person's state, whereas $(\mathbf{y} \rightsquigarrow \mathbf{x})$ strictly worsens every person's state. Define $\pi : \mathbb{Z} \longrightarrow \mathbb{Z}$ by $\pi(i) := i + 1$. Then $\pi(\mathbf{x}) = \mathbf{y}$ and $\pi(\mathbf{y}) = \mathbf{z}$, so if ($\succeq$) was π-invariant, then we would have $(\mathbf{y} \rightsquigarrow \mathbf{x}) \cong (\mathbf{y} \rightsquigarrow \mathbf{y}) \cong (\mathbf{y} \rightsquigarrow \mathbf{z})$, which is both intuitively absurd, and logically inconsistent with axiom (SPar). See Basu and Mitra (2003, 2006) and Fleurbaey and Michel (2003; Theorem 1) for further analysis of the Pareto/anonymity conflict.

In particular, we define the *approximate utilitarian* SDP ($\underset{u}{\trianglerighteq}$) by setting $\mathcal{V} := \mathcal{U}(\succeq)$. (Note: If $\mathcal{J}' \subseteq \mathcal{I}$ is any finite set with $\mathcal{J}_v \subseteq \mathcal{J}'$, then (AU1) is also true if we replace $\mathcal{J}_v$ with $\mathcal{J}'$ (because (AU2) implies that $v(x_j^2) - v(x_j^1) \geq v(y_j^2) - v(y_j^1)$ for all $j \in \mathcal{J}' \setminus \mathcal{J}_v$). Thus, we can make $\mathcal{J}_v$ arbitrarily large in (AU1). In particular, if $\mathcal{I}$ is finite, then we can simply set $\mathcal{J}_v := \mathcal{I}$ for all $v \in \mathcal{V}$; then statement (AU2) becomes vacuous.)

Proposition 3.1 *Let* ($\succeq$) *be a difference preorder on* $\mathcal{X}$.

(a) *If* $\emptyset \neq \mathcal{V} \subseteq \mathcal{U}(\succeq)$, *then* ($\underset{\mathcal{V}}{\trianglerighteq}$) *is an* ($\succeq$)*-SDP on* $\mathcal{X}^{\mathcal{I}}$.

(b) *If* $\mathcal{V}$ *contains a strong utility function for* ($\succeq$), *or* $\mathcal{V}$ *yields a multiutility representation for* ($\succeq$), *then* ($\underset{\mathcal{V}}{\trianglerighteq}$) *satisfies axiom* (SPar).

(c) *If* $\emptyset \neq \mathcal{V} \subseteq \mathcal{W} \subseteq \mathcal{U}(\succeq)$, *then* ($\underset{\mathcal{V}}{\trianglerighteq}$) *extends and refines* ($\underset{\mathcal{W}}{\trianglerighteq}$).

(d) *In particular, every quasiutilitarian SDP extends and refines* ($\underset{u}{\trianglerighteq}$).

(e) *If* ($\succeq$) *has any strong utility functions, then* ($\underset{u}{\trianglerighteq}$) *satisfies axiom* (SPar).

In general, $\mathcal{U}(\succeq)$ will be large, and ($\underset{u}{\trianglerighteq}$) will be incomplete. By restricting to a smaller set $\mathcal{V} \subset \mathcal{U}(\succeq)$, we can obtain a more complete SDP ($\underset{\mathcal{V}}{\trianglerighteq}$). We might do this for technical reasons or normative reasons. At a technical level, perhaps we only wish to consider elements of $\mathcal{U}(\succeq)$ which satisfy certain 'regularity' conditions. (For example, if $\mathcal{X}$ is a topological space, we might only be interested in the continuous elements of $\mathcal{U}(\succeq)$.) At a normative level, perhaps some of the utility functions in $\mathcal{U}(\succeq)$ encode information which we think is 'ethically irrelevant' and should be ignored. Or perhaps we wish to give some information more 'weight' than other information. For example, let $u_1, u_2, \ldots, u_N : \mathcal{X} \longrightarrow \mathbb{R}$ be a set of functions measuring N components of 'quality of life', such as health, education, security, liberty, social participation, consumption of various commodities, etc. Let $\mathcal{U}'$ be the set of all positive linear combinations of $u_1, \ldots, u_N$. If $u_1, \ldots, u_N \in \mathcal{U}(\succeq)$, then $\mathcal{U}' \subseteq \mathcal{U}(\succeq)$. But perhaps we want to give component 1 twice the weight of component 2, and six times the weight of component 3, while excluding components 4 and 5 altogether. We could do this with the quasiutilitarian SDP ($\underset{\mathcal{V}}{\trianglerighteq}$), where $\mathcal{V} := \{6u_1 + 3u_2 + u_3, \ u_6, u_7, \ldots, u_N\}$.

When is an SDP quasiutilitarian? Let ($\trianglerighteq$) be a ($\succeq$)-SDP, and let $(\mathcal{R}, +, >)$ be a linearly ordered abelian group. An $\mathcal{R}$-valued *social welfare function* (SWF) for ($\trianglerighteq$) is a function $W : \mathcal{X}^{\mathcal{I}} \longrightarrow \mathcal{R}$ which is a weak utility function for ($\trianglerighteq$). That is: for any $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, we have

$$\Big((\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \trianglerighteq (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)\Big) \implies \Big(W(\mathbf{x}^2) - W(\mathbf{x}^1) \geq W(\mathbf{y}^2) - W(\mathbf{y}^1)\Big). \quad (4)$$

A collection $\mathcal{W}$ of SWFs yields a *multiwelfare representation* for ($\trianglerighteq$) if, for any $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, we have

$$\Big((\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \trianglerighteq (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)\Big) \iff \Big(W(\mathbf{x}^2) - W(\mathbf{x}^1) \geq W(\mathbf{y}^2) - W(\mathbf{y}^1), \ \forall \, W \in \mathcal{W}\Big). \quad (5)$$

We now come to our first major result.

Theorem 3.2 *Let $\mathcal{I}$ be finite. An $(\succeq)$-SDP on $\mathcal{X}^{\mathcal{I}}$ admits a multiwelfare representation if and only if it is quasiutilitarian.*

For example, suppose $(\succeq)$ and $(\rhd)$ are *complete* difference preorders, and can be represented by a single real-valued utility function u and a single real-valued SWF W, respectively. Then Theorem 3.2 says that $W(\mathbf{x}) = \sum_{i \in \mathcal{I}} u(x_i)$, so that $(\rhd)$ is equivalent to the classic utilitarian social welfare order. This conclusion is very similar to Theorem 1 of Dyer and Sarin (1979a), Theorem 6 of Harvey and Østerdal (2010), or the main result of Harvey (1999). However, Theorem 3.2 also applies to incomplete preorders and non-real-valued utility functions.

Theorem 3.2 is only applicable when $\mathcal{I}$ is finite. Also, not all SDPs admit a multiwelfare representation. The rest of this paper investigates the behaviour of SDPs when the hypotheses of Theorem 3.2 are not necessarily satisfied.

4 The minimal SDP

An $(\succeq)$-SDP is not necessarily a complete preorder on $\mathcal{X}^{\mathcal{I}} \times \mathcal{X}^{\mathcal{I}}$. Furthermore, there may be many different $(\succeq)$-SDPs, based on different ethical principles, which disagree on how to trade off between the interests of different individuals. It is thus desirable to find the common ground between these different SDPs. It is easy to see that the intersection of two or more SDPs is also an SDP. Let SDP be the set of all $(\succeq)$-social difference preorders on $\mathcal{X}^{\mathcal{I}}$ (we will see soon that this set is always nonempty). Define the *minimal* SDP: $(\underset{*}{\rhd}) := \bigcap_{(\rhd) \in \mathsf{SDP}} (\rhd)$. In other words, for any $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, we have

$$\left((\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{*}{\rhd} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2) \right) \iff \left((\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \rhd (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2) \text{ for every } (\rhd) \in \mathsf{SDP} \right). \quad (6)$$

Proposition 4.1 *Let $(\succeq)$ be a difference preorder on $\mathcal{X}$, and let $(\rhd)$ be an $(\succeq)$-SDP.*

    (a) $(\rhd)$ *extends* $(\underset{*}{\rhd})$.

    (b) $(\rhd)$ *satisfies* (SPar) *if and only if* $(\rhd)$ *refines* $(\underset{*}{\rhd})$ *and* $(\underset{*}{\rhd})$ *satisfies* (SPar).

    (c) *If* $(\succeq)$ *has a strong utility function, then* $(\underset{*}{\rhd})$ *satisfies* (SPar).

Unfortunately, definition (6) is nonconstructive, and thus, not very useful in practice. We now provide a more explicit and practical characterization of the minimal SDP $(\underset{*}{\rhd})$. Say $(\succeq)$ is *empathic* if, for any $x_1, x_2, y_1 \in \mathcal{X}$, there exists $y_2 \in \mathcal{X}$ such that $(x_1 \rightsquigarrow x_2) \approx (y_1 \rightsquigarrow y_2)$. In other words: for any possible state transition facing a person currently in state x_1, a person in state y_1 can imagine an exactly analogous transition for herself. (Necessary and sufficient conditions for empathy are given in §7.) Here is our second major result.

Theorem 4.2 *Suppose $(\succeq)$ is empathic, and has a multiutility representation* (3) *given by some subset $\mathcal{V} \subseteq \mathcal{U}(\succeq)$. If either $\mathcal{I}$ is finite or $\mathcal{V}$ is finite, then $(\underset{\mathcal{V}}{\rhd}) = (\underset{u}{\rhd}) = (\underset{*}{\rhd})$.*

Example 4.3. Suppose $\mathcal{X} = \mathbb{R}^N$, where the different coordinates represent different quantitative measures of well-being (e.g. health, education, etc.). For any $x_1, x_2, y_1, y_2 \in \mathcal{X}$, suppose $(x_1 \rightsquigarrow x_2) \succeq (y_1 \rightsquigarrow y_2)$ if and only if $(x_2 - x_1) \geq (y_2 - y_1)$ (where "$\geq$" is the coordinatewise dominance relation). Then $(\succeq)$ is empathic. Furthermore, the N coordinate projections on $\mathbb{R}^N$ provide a finite multiutility representation for $(\succeq)$; thus, Theorem 4.2 says that $(\underset{u}{\rhd})$ is the 'core' of every other $(\succeq)$-SDP on $\mathcal{X}^{\mathcal{I}}$.

In particular, if $N = 1$ (i.e. $\mathcal{X} = \mathbb{R}$), then $(\succeq)$ is a *complete* order on $\mathcal{X} \times \mathcal{X}$. In this case, $\mathcal{U}(\succeq)$ is simply the set of affine increasing functions from $\mathbb{R}$ to itself, so that $(\underset{u}{\rhd})$ is equivalent to the classic utilitarian social welfare order:

$$\left((\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{u}{\rhd} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2) \right) \iff \left(\sum_{i \in \mathcal{I}} \left(x_i^2 - x_i^1 \right) \geq \sum_{i \in \mathcal{I}} \left(y_i^2 - y_i^1 \right) \right)$$

(if $\mathcal{I}$ is finite). In this case, Theorem 4.2 implies that $(\underset{u}{\rhd})$ is the *unique* SDP on $\mathcal{X}^{\mathcal{I}}$ satisfying axiom (SPar).[11] $\diamondsuit$

For any $x \in \mathcal{X}$, $\mathbf{z} \in \mathcal{X}^{\mathcal{I}}$, and $j \in \mathcal{I}$, we define $\left(\begin{smallmatrix} x_j \\ \mathbf{z}_{-j} \end{smallmatrix} \right) \in \mathcal{X}^{\mathcal{I}}$ by setting $\left(\begin{smallmatrix} x_j \\ \mathbf{z}_{-j} \end{smallmatrix} \right)_j := x$, while $\left(\begin{smallmatrix} x_j \\ \mathbf{z}_{-j} \end{smallmatrix} \right)_i := z_i$ for all $i \in \mathcal{I} \setminus \{j\}$. Let $(\rhd)$ be a $(\succeq)$-SDP. We say that $(\rhd)$ exhibits *no extra hidden interpersonal comparisons* if the following holds:

(NEHIC) For all $x, x', y, y' \in \mathcal{X}$ and $\mathbf{z} \in \mathcal{X}^{\mathcal{I}}$,

$$\left((x \rightsquigarrow x') \succeq (y \rightsquigarrow x') \right) \iff \left(\left(\left(\begin{smallmatrix} x_j \\ \mathbf{z}_{-j} \end{smallmatrix} \right) \rightsquigarrow \left(\begin{smallmatrix} x'_j \\ \mathbf{z}_{-j} \end{smallmatrix} \right) \right) \rhd \left(\left(\begin{smallmatrix} y_j \\ \mathbf{z}_{-j} \end{smallmatrix} \right) \rightsquigarrow \left(\begin{smallmatrix} y'_j \\ \mathbf{z}_{-j} \end{smallmatrix} \right) \right) \right).$$

Note that the "$\Longrightarrow$" direction of (NEHIC) follows immediately from axiom (WPar). The real content of (NEHIC) lies in the "$\Longleftarrow$" direction. Intuitively, if $\left(\left(\begin{smallmatrix} x_j \\ \mathbf{z}_{-j} \end{smallmatrix} \right) \rightsquigarrow \left(\begin{smallmatrix} x'_j \\ \mathbf{z}_{-j} \end{smallmatrix} \right) \right) \rhd \left(\left(\begin{smallmatrix} y_j \\ \mathbf{z}_{-j} \end{smallmatrix} \right) \rightsquigarrow \left(\begin{smallmatrix} y'_j \\ \mathbf{z}_{-j} \end{smallmatrix} \right) \right)$, then $(\rhd)$ is implicitly making an interpersonal comparison that $(x \rightsquigarrow x')$ is a greater welfare gain than $(y \rightsquigarrow y')$. Axiom (NEHIC) says that $(\rhd)$ can only make such judgements when they are justified by the underlying difference preorder $(\succeq)$.

Theorem 4.4 *Suppose $\mathcal{I}$ is finite and $(\succeq)$ is empathic, and let $(\rhd)$ be an $(\succeq)$-SDP. If $(\rhd)$ has a multiwelfare representation* (5) *and satisfies* (NEHIC)*, then $(\rhd) = (\underset{u}{\rhd}) = (\underset{*}{\rhd})$.*

Not all difference preorders are empathic or admit a multiutility representation, so Theorems 4.2 and 4.4 are not always applicable. Indeed, if $\mathcal{U}(\succeq) = \emptyset$, then it is not even

---

[11] *Proof sketch.* $(\underset{u}{\rhd})$ is a complete preorder on $\mathbb{R}^{\mathcal{I}} \times \mathbb{R}^{\mathcal{I}}$. Theorem 4.2 says that any other SDP $(\rhd)$ on $\mathbb{R}^{\mathcal{I}}$ is an extension of $(\underset{u}{\rhd})$, which means $(\rhd)$ is obtained by 'thickening' some of the indifference curves of $(\underset{u}{\rhd})$. But if $(\rhd)$ satisfies (SPar), then it cannot have any 'thick' indifference curves; thus, $(\rhd) = (\underset{u}{\rhd})$.

clear that the set SDP is nonempty; hence it is not clear that the minimal SDP $(\underset{*}{\trianglerighteq})$ is well-defined. We will now provide an alternative, inductive definition of $(\underset{*}{\trianglerighteq})$. First, define an equivalence relation $(\underset{\mathrm{an}}{\widehat{=}})$ on $\mathcal{X}^{\mathcal{I}} \times \mathcal{X}^{\mathcal{I}}$ by:

$$\left((\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2)\underset{\mathrm{an}}{\widehat{=}}(\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)\right) \iff \left(\mathbf{y}^1 = \mathbf{x}^1, \text{ and } \exists \ \pi \in \Pi \text{ with } \mathbf{y}^2 = \pi(\mathbf{x}^2)\right).$$

Let $(\underset{\mathrm{par}}{\trianglerighteq})$ be the Pareto preorder. That is:

$$\left((\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2)\underset{\mathrm{par}}{\trianglerighteq}(\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)\right) \iff \left((x_i^1 \rightsquigarrow x_i^2) \succeq (y_i^1 \rightsquigarrow y_i^2) \text{ for all } i \in \mathcal{I}\right).$$

Let $(\underset{*}{\trianglerighteq})$ be the closure of the relation $(\underset{\mathrm{an}}{\widehat{=}}) \cup (\underset{\mathrm{par}}{\trianglerighteq})$ under transitivity, (DP2), and (DP3). That is: for any $\mathbf{x}^0, \mathbf{x}^2, \mathbf{z}^0, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, we recursively define $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2)\underset{*}{\trianglerighteq}(\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$ if *either*

  (∗1) $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2)\underset{\mathrm{an}}{\widehat{=}}(\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$; *or*

  (∗2) $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2)\underset{\mathrm{par}}{\trianglerighteq}(\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$; *or*

  (∗3) There exist $\mathbf{y}^0, \mathbf{y}^2$ with $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2)\underset{*}{\trianglerighteq}(\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2)$ and $(\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2)\underset{*}{\trianglerighteq}(\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$; *or*

  (∗4) There exist $\mathbf{x}^1, \mathbf{z}^1$ with $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1)\underset{*}{\trianglerighteq}(\mathbf{z}^0 \rightsquigarrow \mathbf{z}^1)$ and $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2)\underset{*}{\trianglerighteq}(\mathbf{z}^1 \rightsquigarrow \mathbf{z}^2)$; *or*

  (∗5) There exist $\mathbf{x}^1, \mathbf{z}^1$ with $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1)\underset{*}{\trianglerighteq}(\mathbf{z}^1 \rightsquigarrow \mathbf{z}^2)$ and $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2)\underset{*}{\trianglerighteq}(\mathbf{z}^0 \rightsquigarrow \mathbf{z}^1)$.

Conditions (∗1)-(∗3) correspond to a social preorder proposed by Suppes and Sen.[12] Conditions (∗4) and (∗5) ensure satisfaction of axioms (DP2$^{\trianglerighteq}$) and (DP3$^{\trianglerighteq}$).

Proposition 4.5 *The relation* $(\underset{*}{\trianglerighteq})$ *defined using rules* (∗1)-(∗5) *is the minimal SDP defined by formula* (6).

5 Net Gain

For any $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, and any finite subsets $\mathcal{J}, \mathcal{K} \subseteq \mathcal{I}$ with $J := |\mathcal{J}|$ and $K := |\mathcal{K}|$, we write "$(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2)\underset{\mathcal{J},\mathcal{K}}{\trianglerighteq}(\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$" if there exist $w_0, w_1, \ldots, w_J \in \mathcal{X}$ and $z_0, z_1, \ldots, z_K \in \mathcal{X}$ and bijections $\alpha : \mathcal{J} \longrightarrow [1 \ldots J]$ and $\beta : \mathcal{K} \longrightarrow [1 \ldots K]$ such that:

  (JK1) $(x_j^1 \rightsquigarrow x_j^2) \succeq (w_{\alpha(j)-1} \rightsquigarrow w_{\alpha(j)})$ for all $j \in \mathcal{J}$;

  (JK2) $(z_{\beta(k)-1} \rightsquigarrow z_{\beta(k)}) \succeq (y_k^1 \rightsquigarrow y_k^2)$, for all $k \in \mathcal{K}$; and

  (JK3) $(w_0 \rightsquigarrow w_J) \succeq (z_0 \rightsquigarrow z_K)$.

---

[12]See Suppes (1966), Sen (1970b, §9*1-§9*3, pp.150-156), Saposnik (1983) and Pivato (2010b, §3.1).

Intuitively, $w_0 \rightsquigarrow w_J$ aggregates the net welfare gain of the chain $w_0 \rightsquigarrow w_1 \rightsquigarrow w_2 \rightsquigarrow \cdots \rightsquigarrow w_J$. Thus, (JK1) implies that net welfare gain for the $\mathcal{J}$-population induced by the change $\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2$ is at least as large as the net welfare gain of $w_0 \rightsquigarrow w_J$. Meanwhile, (JK2) implies that the net welfare gain for the $\mathcal{K}$-population induced by $\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2$ is at *most* as large as $z_0 \rightsquigarrow z_K$. Thus, if (JK3) holds, then the $\mathcal{J}$-population, in aggregate, gains more welfare from $\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2$ than the $\mathcal{K}$-population gains from $\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2$

Let $\mathcal{I}_0 \subseteq \mathcal{I}$ be a finite subset. A **partition** of $\mathcal{I}_0$ is a collection $\{\mathcal{J}_\ell\}_{\ell \in \mathcal{L}}$ of disjoint subsets of $\mathcal{I}_0$ (where $\mathcal{L}$ is some indexing set), such that $\mathcal{I}_0 = \bigsqcup_{\ell \in \mathcal{L}} \mathcal{J}_\ell$. We define the **net gain** relation ($\underset{\mathrm{ng}}{\trianglerighteq}$) as follows. For any $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, define $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{ng}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$ if there exists some finite $\mathcal{I}_0 \subseteq \mathcal{I}$ and two partitions $\{\mathcal{J}_\ell\}_{\ell \in \mathcal{L}}$ and $\{\mathcal{K}_\ell\}_{\ell \in \mathcal{L}}$ of $\mathcal{I}_0$ (with the *same* indexing set $\mathcal{L}$), such that:

(NG1) $(x_i^1 \rightsquigarrow x_i^2) \succeq (y_i^1 \rightsquigarrow y_i^2)$ for all $i \in \mathcal{I} \setminus \mathcal{I}_0$.

(NG2) For all $\ell \in \mathcal{L}$, we have $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{J}_\ell, \mathcal{K}_\ell}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$.

Intuitively, condition (NG2) means we can split up $\mathcal{I}_0$ into disjoint subsets such that, for each $\ell \in \mathcal{L}$, the 'net welfare gain' induced by $\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2$ for $\mathcal{J}_\ell$ is demonstrably larger than the 'net welfare gain' induced by $\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2$ for $\mathcal{K}_\ell$ (as argued in the previous paragraph). Thus, if we aggregate over all $\ell \in \mathcal{L}$, then the 'net welfare gain' over all of $\mathcal{I}_0$ must be greater for $\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2$ than it is for $\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2$. Meanwhile, condition (NG1) ensures that the people in $\mathcal{I} \setminus \mathcal{I}_0$ unanimously prefer $\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2$ over $\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2$. (If $\mathcal{I}$ is finite, then we can simply set $\mathcal{I}_0 := \mathcal{I}$, in which case condition (NG1) becomes vacuous.) Our last major result characterizes the minimal SDP ($\underset{*}{\trianglerighteq}$) without assuming $\mathcal{I}$ is finite, or assuming the existence of any utility functions for ($\succeq$). Here is the last major result of the paper.

Theorem 5.1 *If* ($\succeq$) *is empathic, then* ($\underset{\mathrm{ng}}{\trianglerighteq}$) $=$ ($\underset{*}{\trianglerighteq}$)*, and satisfies* (SPar).

Note that Theorem 5.1 does not require ($\succeq$) to have any utility functions, much less a multiutility representation. (This is important, given the results of Appendix B.)

In general, if ($\succeq$) is not empathic, then ($\underset{\mathrm{ng}}{\trianglerighteq}$) itself might not even be an SDP. However, it will still be the case that every SDP extends ($\underset{\mathrm{ng}}{\trianglerighteq}$), as the next result describes.

Proposition 5.2   **(a)** *The relation* ($\underset{\mathrm{ng}}{\trianglerighteq}$) *is reflexive, and satisfies axioms* (WPar)*,* (Anon)*,* (DP0$^{\trianglerighteq}$)*, and* (DP1$^{\trianglerighteq}$)*.*

(b) *If* ($\trianglerighteq$) *is any* ($\succeq$)*-SDP on* $\mathcal{X}^{\mathcal{I}}$*, then* ($\trianglerighteq$) *extends* ($\underset{\mathrm{ng}}{\trianglerighteq}$)*. Furthermore, if* ($\trianglerighteq$) *also satisfies* (SPar)*, then* ($\trianglerighteq$) *also refines* ($\underset{\mathrm{ng}}{\trianglerighteq}$)*.*

6   Application: Redistributive transfers

Suppose $\mathcal{X} = \mathcal{P} \times \mathbb{R}_+$, where $\mathcal{P}$ is a set of 'personality types', and where the state $\langle p, r \rangle \in \mathcal{P} \times \mathbb{R}_+$ represents a p-type person holding r dollars. We suppose p encodes all psychological

or physical characteristics which influence the marginal welfare which money provides for a p-type person. The social planner can only approximately compare the marginal welfare of money for different personality types. However, we assume everyone obtains qualitatively similar benefits from money, which we model using a nondecreasing 'benefit function' $\beta : \mathbb{R}_+ \longrightarrow \mathbb{R}$. Formally, for any $p_1, p_2 \in \mathcal{P}$, we suppose there is some constant $C = C(p_1, p_2) \geq 1$ such that, for any $r_1 < s_1$ and $r_2 < s_2$ in $\mathbb{R}_+$, we have

$$\left(\frac{\beta(s_1) - \beta(r_1)}{\beta(s_2) - \beta(r_2)} \; > \; C \right) \Longrightarrow \Big((\langle p_1, r_1 \rangle \rightsquigarrow \langle p_1, s_1 \rangle) \succ (\langle p_2, r_2 \rangle \rightsquigarrow \langle p_2, s_2 \rangle) \Big). \qquad (7)$$

(Of course, $C(p, p) = 1$ for all $p \in \mathcal{P}$). We will use this simple model to investigate the social benefit of wealth redistribution. For simplicity, suppose $\mathcal{I}$ contains $\{1, 2\}$ ('Juan' and 'Sue'), and fix $\mathbf{p} \in \mathcal{P}^{\mathcal{I}}$. Let $C := C(p_1, p_2)$, and consider a social state $\langle \mathbf{p}, \mathbf{r} \rangle \in \mathcal{P}^{\mathcal{I}} \times \mathbb{R}_+^{\mathcal{I}}$, where $r_1 < r_2$ (so Juan is poorer than Sue). A *redistributive transfer* is a change $\langle \mathbf{p}, \mathbf{r} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{s} \rangle$, where $r_i = s_i$ for all $i \notin \{1, 2\}$, and $r_1 \leq s_1 \leq s_2 \leq r_2$, and where $s_1 + s_2 \leq r_1 + r_2$. (The gap $(r_1 + r_2) - (s_1 + s_2)$ represents the efficiency loss caused by the transfer —due to labour disincentive effects on Juan and Sue, the costs of managing and enforcing the necessary system of taxes and subsidies, and/or waste and corruption in the government.)[13] The 'status quo' option is simply the 'null' transfer $\langle \mathbf{p}, \mathbf{r} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{r} \rangle$. Under what conditions is redistribution socially superior to the status quo?

Proposition 6.1 *Suppose there exists $r_2' \geq r_2$ with $\dfrac{\beta(s_1) - \beta(r_1)}{\beta(r_2') - \beta(s_2)} > C$.[14] If $(\trianglerighteq)$ is any $(\succeq)$-SDP on $\mathcal{X}^{\mathcal{I}}$, then for all $\mathbf{q} \in \mathbb{R}_+^{\mathcal{I}}$, we have $(\langle \mathbf{p}, \mathbf{q} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{s} \rangle) \trianglerighteq (\langle \mathbf{p}, \mathbf{q} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{r} \rangle)$. In particular, $(\langle \mathbf{p}, \mathbf{r} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{s} \rangle) \trianglerighteq (\langle \mathbf{p}, \mathbf{r} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{r} \rangle)$.*

Furthermore, if $r_2' > r_2$, and $(\trianglerighteq)$ satisfies (SPar)*, then $(\langle \mathbf{p}, \mathbf{q} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{s} \rangle) \triangleright (\langle \mathbf{p}, \mathbf{q} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{r} \rangle)$ (and hence, $(\langle \mathbf{p}, \mathbf{r} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{s} \rangle) \triangleright (\langle \mathbf{p}, \mathbf{r} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{r} \rangle))$.*

Example 6.2. Suppose $\beta(r) = \log_2(r)$ for all $r \in \mathbb{R}_+$, and let $C := 2$ in statement (7). Let $r_1 := 128$ and $r_2 := 2047$. Let $s_1 := 513$ and $s_2 := 1024$. Thus, the transfer $\langle \mathbf{p}, \mathbf{r} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{s} \rangle$ taxes \$1023 from Sue, and gives \$385 to Juan (we suppose the other \$638 is lost due to inefficiencies). Let $r_2' := 2048$. Then $r_2' > r_2$, and

$$\frac{\log_2(s_1) - \log_2(r_1)}{\log_2(r_2') - \log_2(s_2)} \;=\; \frac{\log_2(513) - \log_2(128)}{\log_2(2048) - \log_2(1024)} \;>\; \frac{9 - 7}{11 - 10} \;=\; 2 \;=\; C.$$

Thus, *any* SDP will say that this wealth transfer is socially superior to the status quo, despite the large efficiency loss and the imprecise interpersonal utility comparisons. $\diamond$

---

[13] Some transfers, such as public education or public vaccination campaigns, subsidize activities with positive externalities, so that, in effect, $s_1 + s_2 \geq r_1 + r_2$. But we will ignore this possibility.

[14] Note: since $r_2' - s_2 \geq r_2 - s_2 \geq s_1 - r_1$, and $C \geq 1$, this inequality means that the average slope of β between r_2 and s_2 is smaller than its average slope between r_1 and s_1. This is consistent with the standard assumption that the marginal benefit of wealth is declining.

7 Empathy

Theorems 4.2, 4.4, and 5.1 illustrate the importance of empathy. What are necessary and sufficient conditions for a difference preorder to be empathic?

Let $\mathcal{J}$ be an indexing set (possibly infinite), let $\{\mathcal{R}_j\}_{j\in\mathcal{J}}$ be a collection of linearly ordered abelian groups, and for all $j \in \mathcal{J}$, let $u_j : \mathcal{X}\longrightarrow\mathcal{R}_j$. Let $\mathcal{U}' := \{u_j\}_{j\in\mathcal{J}}$, and suppose $(\succeq)$ has a multiutility representation (3). Define $\mathcal{R} := \prod_{j\in\mathcal{J}}\mathcal{R}_j$ (with the product group structure), and define $\mathbf{u} : \mathcal{X}\longrightarrow\mathcal{R}$ by $\mathbf{u}(x) := (u_j(x))_{j\in\mathcal{J}}$ for all $x \in \mathcal{X}$. Let $\mathbf{u}(\mathcal{X}) := \{\mathbf{u}(x); x \in \mathcal{X}\}$ (a subset of $\mathcal{R}$). Recall that $\mathbf{u}(\mathcal{X})$ is a *coset* in $\mathcal{R}$ if there is some subgroup $\mathcal{S} \subseteq \mathcal{R}$ and some $\mathbf{r} \in \mathcal{R}$ such that $\mathbf{u}(\mathcal{X}) := \mathbf{r} + \mathcal{S}$.

Proposition 7.1 *Suppose* $(\succeq)$ *has a multiutility representation* (3)*, and define* $\mathbf{u} : \mathcal{X}\longrightarrow\mathcal{R}$ *as above. Then* $(\succeq)$ *is empathic if and only if* $\mathbf{u}(\mathcal{X})$ *is a coset in* $\mathcal{R}$.

Example 7.2. (a) Suppose $(\succeq)$ is a *complete* difference preorder on $\mathcal{X}$, defined by a single utility function $u : \mathcal{X}\longrightarrow\mathbb{R}$ so that $(x_1 \rightsquigarrow x_2) \succeq (y_1 \rightsquigarrow y_2)$ if and only if $u(x_2) - u(x_1) \geq u(y_2)-u(y_1)$. Then $(\succeq)$ is empathic if $u(\mathcal{X})$ is a subgroup of $\mathbb{R}$ —in particular, if $u(\mathcal{X}) = \mathbb{R}$.

(b) Let $v_1, v_2, \ldots, v_N : \mathcal{X}\longrightarrow\mathbb{R}$ be real-valued functions. Let $\mathcal{V} := \{v_1, \ldots, v_N\}$ and define $(\underset{\mathcal{V}}{\succeq})$ as in Example 2.1. If the set $\{(v_1(x), \ldots, v_N(x)); \ x \in \mathcal{X}\}$ is an affine subspace of $\mathbb{R}^N$, then $(\underset{\mathcal{V}}{\succeq})$ is empathic. $\diamondsuit$

An *endomorphism* of $(\succeq)$ is a function $\alpha : \mathcal{X}\longrightarrow\mathcal{X}$ such that, for all $x_1, x_2 \in \mathcal{X}$, if $y_1 := \alpha(x_1)$ and $y_2 := \alpha(x_2)$, then $(x_1 \rightsquigarrow x_2)\approx(y_1 \rightsquigarrow y_2)$. Psychologically speaking, α defines a perfect analogy (in terms of welfare gains) between all state changes available to x_1 and those available to y_1.

The composition of two endomorphisms is also an endomorphism. Thus, if $\mathrm{End}(\succeq)$ is the set of all endomorphisms of $(\succeq)$, then $\mathrm{End}(\succeq)$ is a monoid.[15] We say that $\mathrm{End}(\succeq)$ acts *transitively* on $\mathcal{X}$ if, for any $x, y \in \mathcal{X}$, there exists $\alpha \in \mathrm{End}(\succeq)$ such that $\alpha(x) = y$.

Proposition 7.3 *Let* $(\succeq)$ *be a difference preorder. Then* $(\succeq)$ *is empathic if and only if* $\mathrm{End}(\succeq)$ *acts transitively on* $\mathcal{X}$.

Conclusion

It is reasonable to suppose that we can make at least approximate comparisons between the welfare gains and losses which different people experience under changes in the social state. Using even such an approximate interpersonal comparison scheme, it is possible to define a nontrivial (albeit incomplete) ranking of social state changes. This allows the social planner to judge that some state changes are clearly better than others (although there may be no unique optimum). In particular, even a very incomplete system of interpersonal

---

[15]An invertible endomorphism is called an *automorphism*; the set of automorphisms forms a group. But not all endomorphisms are necessarily invertible.

comparisons can be enough to show that some wealth transfers improve social welfare relative to the status quo (Proposition 6.1).

We have defined three social ranking schemes. One, $(\underset{u}{\trianglerighteq})$, is a straightforward generalization of the classic utilitarian social welfare order. Another, $(\underset{*}{\trianglerighteq})$, is a generalization of the Suppes-Sen ordering, and forms the logical core of every social ranking system compatible with our axioms (Proposition 4.1). The third, $(\underset{ng}{\trianglerighteq})$, ranks social state changes by comparing the aggregate costs/benefits they impose upon different sub-populations in a 'quasi-utilitarian' fashion; it is a sub-relation of $(\underset{*}{\trianglerighteq})$ (Proposition 5.2(b)). Under reasonable hypotheses, all three schemes are in fact equal (Theorems 4.2 and 4.4). This makes the approximate utilitarian scheme $(\underset{u}{\trianglerighteq})$ attractive as a basis for social choice. We end with some open questions.

- The 'empathy' hypothesis of Theorems 4.2, 4.4 and 5.1 is somewhat restrictive. Do the conclusions hold under a weaker condition?

- Aside from $(\underset{*}{\trianglerighteq})$ and $(\underset{u}{\trianglerighteq})$, are there any interesting and natural SDPs admitting axiomatic characterizations?

Appendix A: Proofs

Proof of Proposition 2.2. Let $\mathcal{J}$ be an indexing set (possibly infinite), let $\{\mathcal{R}_j\}_{j \in \mathcal{J}}$ be a collection of linearly ordered abelian groups, and for all $j \in \mathcal{J}$, let $u_j : \mathcal{X} \longrightarrow \mathcal{R}_j$. Let $\mathcal{U}' := \{u_j\}_{j \in \mathcal{J}}$, and suppose $(\succeq)$ has a multiutility representation (3).

Let $(\gg)$ be a well-ordering of $\mathcal{J}$. Let $\mathcal{R} := \prod_{j \in \mathcal{J}} \mathcal{R}_j$, and let $(+)$ be the componentwise addition operator on $\mathcal{R}$. Let $\mathbf{r}, \mathbf{s} \in \mathcal{R}$, with $\mathbf{r} \neq \mathbf{s}$. Since $(\mathcal{J}, \gg)$ is well-ordered, the set $\{j \in \mathcal{J};\ r_j \neq s_j\}$ has a minimal element; call this element $j^*(\mathbf{r}, \mathbf{s})$. Define the *lexicographical order* $(>)$ on $\mathcal{R}$ as follows: for any $\mathbf{r} \neq \mathbf{s} \in \mathcal{R}$, if $j = j^*(\mathbf{r}, \mathbf{s})$, then $\mathbf{r} > \mathbf{s}$ if and only if $r_j >_j s_j$ (where $(>_j)$ is the order on $\mathcal{R}_j$). It is easy to verify that $(\mathcal{R}, +, >)$ is a linearly ordered abelian group.

Now, define $\mathbf{u} : \mathcal{X} \longrightarrow \mathcal{R}$ by $\mathbf{u}(x) := (u_j(x))_{j \in \mathcal{J}}$ for all $x \in \mathcal{X}$. I claim that $\mathbf{u}$ is a strong utility function. If $(x_1 \rightsquigarrow x_2) \succeq (y_1 \rightsquigarrow y_2)$, then (3) says that $u(x_2) - u(x_1) \geq u(y_2) - u(y_1)$ for all $u \in \mathcal{U}'$. Thus, $\mathbf{u}(x_2) - \mathbf{u}(x_1) \geq \mathbf{u}(y_2) - \mathbf{u}(y_1)$.

Furthermore, if $(x_1 \rightsquigarrow x_2) \succ (y_1 \rightsquigarrow y_2)$, then $(x_1 \rightsquigarrow x_2) \not\preceq (y_1 \rightsquigarrow y_2)$, so the contrapositive of (3) says that it is false that $u(x_2) - u(x_1) \leq u(y_2) - u(y_1)$ for all $u \in \mathcal{U}'$. Thus,

$u(x_2) - u(x_1) > u(y_2) - u(y_1)$ for some $u \in \mathcal{U}'$. But then $\mathbf{u}(x_2) - \mathbf{u}(x_1) > \mathbf{u}(y_2) - \mathbf{u}(y_1)$, as desired. $\qquad\square$

Proof of Proposition 3.1. (a) The reflexive property follows from axiom (WPar), which, in turn, follows immediately by setting $\mathcal{J}_v := \emptyset$ for all $v \in \mathcal{V}$, and applying (AU2) to every element of $\mathcal{I}$.

(Anon) Let $\pi \in \Pi$. Let $\mathcal{J} \subseteq \mathcal{I}$ be a finite subset such that $\pi(i) = i$ for all $i \in \mathcal{I} \setminus \mathcal{J}$. Let $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$, and let $\mathbf{x}' = \pi(\mathbf{x})$. Then for all $v \in \mathcal{V}$, we have

$$
\sum_{j \in \mathcal{J}} \Big(v(x'_j) - v(x_j) \Big) \;=\; \sum_{j' \in \mathcal{J}} v(x'_{j'}) - \sum_{j \in \mathcal{J}} v(x_j)
$$
$$
\underset{(*)}{=\!=} \;\; \sum_{j \in \mathcal{J}} v(x_j) - \sum_{j \in \mathcal{J}} v(x_j) \;=\; \sum_{j \in \mathcal{J}} \Big(v(x_j) - v(x_j) \Big).
$$

Here, $(*)$ is by the change of variables $j' := \pi(j)$ (because $\pi : \mathcal{J} \longrightarrow \mathcal{J}$ bijectively). If we set $\mathcal{J}_v := \mathcal{J}$, then this verifies (AU1) in both directions. Meanwhile, we obviously have $(x_i \rightsquigarrow x'_i) = (x_i \rightsquigarrow x_i)$ for all $i \in \mathcal{I} \setminus \mathcal{J}$. This verifies (AU2) in both directions. Thus $(\mathbf{x} \rightsquigarrow \mathbf{x}') \widehat{\underset{\mathcal{V}}{=}} (\mathbf{x} \rightsquigarrow \mathbf{x})$, as desired.

Transitive. Suppose $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \underset{\mathcal{V}}{\trianglerighteq} (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^1)$ and $(\mathbf{y}^0 \rightsquigarrow \mathbf{y}^1) \underset{\mathcal{V}}{\trianglerighteq} (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^1)$; we must show that $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \underset{\mathcal{V}}{\trianglerighteq} (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^1)$. Let $v \in \mathcal{V}$, and suppose $v : \mathcal{X} \longrightarrow \mathcal{R}$, where $\mathcal{R}$ is some linearly ordered abelian group. By hypothesis, there exist finite subsets $\mathcal{J}'_v, \mathcal{J}''_v \subseteq \mathcal{I}$ such that

$$
\sum_{j \in \mathcal{J}'_v} \Big(v(x^1_j) - v(x^0_j) \Big) \;\geq\; \sum_{j \in \mathcal{J}'_v} \Big(v(y^1_j) - v(y^0_j) \Big), \tag{A1}
$$

$$
\sum_{j \in \mathcal{J}''_v} \Big(v(y^1_j) - v(y^0_j) \Big) \;\geq\; \sum_{j \in \mathcal{J}''_v} \Big(v(z^1_j) - v(z^0_j) \Big), \tag{A2}
$$

$$
(x^0_i \rightsquigarrow x^1_i) \;\succeq\; (y^0_i \rightsquigarrow y^1_i), \quad \text{for all } i \in \mathcal{I} \setminus \mathcal{J}'_v, \tag{A3}
$$
$$
\text{and} \quad (y^0_i \rightsquigarrow y^1_i) \;\succeq\; (z^0_i \rightsquigarrow z^1_i), \quad \text{for all } i \in \mathcal{I} \setminus \mathcal{J}''_v. \tag{A4}
$$

Let $\mathcal{J}_v := \mathcal{J}'_v \cup \mathcal{J}''_v$. Then

$$
\sum_{j \in \mathcal{J}_v} \Big(v(x^1_j) - v(x^0_j) \Big) \;\geq\; \sum_{j \in \mathcal{J}_v} \Big(v(y^1_j) - v(y^0_j) \Big), \tag{A5}
$$

$$
\sum_{j \in \mathcal{J}_v} \Big(v(y^1_j) - v(y^0_j) \Big) \;\geq\; \sum_{j \in \mathcal{J}_v} \Big(v(z^1_j) - v(z^0_j) \Big), \tag{A6}
$$

$$
(x^0_i \rightsquigarrow x^1_i) \;\succeq\; (y^0_i \rightsquigarrow y^1_i), \quad \text{for all } i \in \mathcal{I} \setminus \mathcal{J}_v, \tag{A7}
$$
$$
\text{and} \quad (y^0_i \rightsquigarrow y^1_i) \;\succeq\; (z^0_i \rightsquigarrow z^1_i), \quad \text{for all } i \in \mathcal{I} \setminus \mathcal{J}_v. \tag{A8}
$$

Here, (A5) is obtained by combining (A1), (A3), and (2). Likewise, (A6) is obtained by combining (A2), (A4), and (2). Next, (A7) follows from (A3), because $(\mathcal{I} \setminus \mathcal{J}_v) \subseteq (\mathcal{I} \setminus \mathcal{J}'_v)$

16

(because $\mathcal{J}_v \supseteq \mathcal{J}_v'$). Likewise, (A8) follows from (A4), because $(\mathcal{I} \setminus \mathcal{J}_v) \subseteq (\mathcal{I} \setminus \mathcal{J}_v'')$ (because $\mathcal{J}_v \supseteq \mathcal{J}_v''$). We conclude that

$$\sum_{j \in \mathcal{J}_v} \left(v(x_j^1) - v(x_j^0) \right) \geq \sum_{j \in \mathcal{J}_v} \left(v(z_j^1) - v(z_j^0) \right) \tag{A9}$$

$$\text{and} \quad (x_i^0 \rightsquigarrow x_i^1) \succeq (z_i^0 \rightsquigarrow z_i^1), \quad \text{for all } i \in \mathcal{I} \setminus \mathcal{J}_v. \tag{A10}$$

Here, (A9) is obtained be combining (A5) and (A6) using the transitivity of the ordering on $\mathcal{R}$. Meanwhile (A10) is obtained by combining (A7) and (A8), using the transitivity of $(\succeq)$.

Now, (A9) verifies (AU1), while (A10) verifies (AU2). We can do this for any $v \in \mathcal{V}$; thus, $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \underset{\mathcal{V}}{\trianglerighteq} (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^1)$, as desired.

(DP2$^{\trianglerighteq}$) The argument is closely analogous to the proof of *Transitivity*. Suppose $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \underset{\mathcal{V}}{\trianglerighteq} (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^1)$ and $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{V}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$; we must show that $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{V}}{\trianglerighteq} (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2)$. Let $v \in \mathcal{V}$, with $v : \mathcal{X} \longrightarrow \mathcal{R}$. By hypothesis, there exist finite subsets $\mathcal{J}_v', \mathcal{J}_v'' \subseteq \mathcal{I}$ such that

$$\sum_{j \in \mathcal{J}_v'} \left(v(x_j^1) - v(x_j^0) \right) \geq \sum_{j \in \mathcal{J}_v'} \left(v(y_j^1) - v(y_j^0) \right), \tag{A11}$$

$$\sum_{j \in \mathcal{J}_v''} \left(v(x_j^2) - v(x_j^1) \right) \geq \sum_{j \in \mathcal{J}_v''} \left(v(y_j^2) - v(y_j^1) \right), \tag{A12}$$

$$(x_i^0 \rightsquigarrow x_i^1) \succeq (y_i^0 \rightsquigarrow y_i^1), \quad \text{for all } i \in \mathcal{I} \setminus \mathcal{J}_v', \tag{A13}$$

$$\text{and} \quad (x_i^1 \rightsquigarrow x_i^2) \succeq (y_i^1 \rightsquigarrow y_i^2), \quad \text{for all } i \in \mathcal{I} \setminus \mathcal{J}_v''. \tag{A14}$$

Let $\mathcal{J}_v := \mathcal{J}_v' \cup \mathcal{J}_v''$. Then

$$\sum_{j \in \mathcal{J}_v} \left(v(x_j^1) - v(x_j^0) \right) \geq \sum_{j \in \mathcal{J}_v} \left(v(y_j^1) - v(y_j^0) \right), \tag{A15}$$

$$\sum_{j \in \mathcal{J}_v} \left(v(x_j^2) - v(x_j^1) \right) \geq \sum_{j \in \mathcal{J}_v} \left(v(y_j^2) - v(y_j^1) \right), \tag{A16}$$

$$(x_i^0 \rightsquigarrow x_i^1) \succeq (y_i^0 \rightsquigarrow y_i^1), \quad \text{for all } i \in \mathcal{I} \setminus \mathcal{J}_v, \tag{A17}$$

$$\text{and} \quad (x_i^1 \rightsquigarrow x_i^2) \succeq (y_i^1 \rightsquigarrow y_i^2), \quad \text{for all } i \in \mathcal{I} \setminus \mathcal{J}_v. \tag{A18}$$

Here, (A15) is obtained by combining (A11), (A13), and (2). Likewise, (A16) is obtained by combining (A12), (A14), and (2). Next, (A17) follows from (A13), because $(\mathcal{I} \setminus \mathcal{J}_v) \subseteq (\mathcal{I} \setminus \mathcal{J}_v')$ (because $\mathcal{J}_v \supseteq \mathcal{J}_v'$). Likewise, (A18) follows from (A14), because $(\mathcal{I} \setminus \mathcal{J}_v) \subseteq (\mathcal{I} \setminus \mathcal{J}_v'')$ (because $\mathcal{J}_v \supseteq \mathcal{J}_v''$). We conclude that

$$\sum_{j \in \mathcal{J}_v} \left(v(x_j^2) - v(x_j^0) \right) = \sum_{j \in \mathcal{J}} \left(v(x_j^2) - v(x_j^1) \right) + \sum_{j \in \mathcal{J}} \left(v(x_j^1) - v(x_j^0) \right)$$

$$\underset{(*)}{\geq} \sum_{j \in \mathcal{J}} \left(v(y_j^2) - v(y_j^1) \right) + \sum_{j \in \mathcal{J}} \left(v(y_j^1) - v(y_j^0) \right)$$

17

$$= \sum_{j \in \mathcal{J}_v} \left(v(y_j^2) - v(y_j^0) \right), \tag{A19}$$

$$\text{and} \quad (x_i^0 \rightsquigarrow x_i^2) \;\succeq\; (y_i^0 \rightsquigarrow y_i^2), \quad \text{for all } i \in \mathcal{I} \setminus \mathcal{J}_v. \tag{A20}$$

Here, $(*)$ comes from combining (A15) and (A16) and using the compatibility between the ordering and the addition operator on $\mathcal{R}$. Meanwhile (A20) comes from combining (A17) and (A18), and using axiom (DP2) for $(\succeq)$.

Now, (A19) verifies (AU1), while (A20) verifies (AU2). We can do this for any $v \in \mathcal{V}$; thus, $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \underset{\overline{v}}{\trianglerighteq} (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2)$, as desired.

(DP3$^{\trianglerighteq}$) The proof is very similar to (DP2$^{\trianglerighteq}$).

(DP0$^{\trianglerighteq}$) For any $v \in \mathcal{V}$, set $\mathcal{J}_v := \emptyset$, and observe that $(x_i \rightsquigarrow x_i) \approx (y_i \rightsquigarrow y_i)$ for all $i \in \mathcal{I} \setminus \mathcal{J}_v$, by (DP0). Thus, (AU2) is satisfied in both directions, so $(\mathbf{x} \rightsquigarrow \mathbf{x}) \underset{\overline{v}}{\widehat{=}} (\mathbf{y} \rightsquigarrow \mathbf{y})$, as desired.

(DP1$^{\trianglerighteq}$) Let $v \in \mathcal{V}$. If $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\overline{v}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$, then there exists some finite $\mathcal{J}_v \subseteq \mathcal{I}$ such that $\sum_{j \in \mathcal{J}_v} \left(v(x_j^2) - v(x_j^1) \right) \geq \sum_{j \in \mathcal{J}_v} \left(v(y_j^2) - v(y_j^1) \right)$, while $(x_i^1 \rightsquigarrow x_i^2) \succeq (y_i^1 \rightsquigarrow y_i^2)$ for all $i \in \mathcal{I} \setminus \mathcal{J}_v$. But then $\sum_{j \in \mathcal{J}_v} \left(v(x_j^1) - v(x_j^2) \right) \leq \sum_{j \in \mathcal{J}_v} \left(v(y_j^1) - v(y_j^2) \right)$. Also, since $(\succeq)$ itself satisfies (DP1), we get $(x_i^2 \rightsquigarrow x_i^1) \preceq (y_i^2 \rightsquigarrow y_i^1)$, for all $i \in \mathcal{I} \setminus \mathcal{J}_v$. Thus, (AU1) and (AU2) are satisfied. This holds for all $v \in \mathcal{V}$; we conclude that $(\mathbf{x}^2 \rightsquigarrow \mathbf{x}^1) \underset{\overline{v}}{\trianglelefteq} (\mathbf{y}^2 \rightsquigarrow \mathbf{y}^1)$.

(b) Suppose $(x_i^1 \rightsquigarrow x_i^2) \succeq (y_i^1 \rightsquigarrow y_i^2)$ for all $i \in \mathcal{I}$, and $(x_j^1 \rightsquigarrow x_j^2) \succ (y_j^1 \rightsquigarrow y_j^2)$ for some $j \in \mathcal{I}$. Axiom (WPar) implies that $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\overline{v}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$, To show that $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{u}{\triangleright} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$, we must show that $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\overline{v}}{\ntrianglelefteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$.

First suppose $\mathcal{V}$ contains a strong utility function v_0. Thus $v_0(x_i^2) - v_0(x_i^1) \geq v_0(y_i^2) - v_0(y_i^1)$ for all $i \in \mathcal{I}$, and $v_0(x_j^2) - v_0(x_j^1) > v_0(y_j^2) - v_0(y_j^1)$. We will show that there is no finite subset $\mathcal{J} \subseteq \mathcal{I}$ which satisfies (AU1) and (AU2) for v_0 in the way required to show that $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\overline{v}}{\trianglelefteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$.

If $j \in \mathcal{J}$, then $\sum_{j \in \mathcal{J}} \left(v_0(x_j^2) - v_0(x_j^1) \right) > \sum_{j \in \mathcal{J}} \left(v_0(y_j^2) - v_0(y_j^1) \right)$, so (AU1) is not satisfied. If $j \notin \mathcal{J}$, then $j \in \mathcal{I} \setminus \mathcal{J}$, and $(x_j^1 \rightsquigarrow x_j^2) \succ (y_j^1 \rightsquigarrow y_j^2)$, so (AU2) is not satisfied.

Thus, there exists at least one $v \in \mathcal{V}$ (namely v_0) such that (AU1) and (AU2) cannot both be satisfied. Thus, $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\overline{v}}{\ntrianglelefteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$, as desired.

The proof in the case when $\mathcal{V}$ provides a multiutility representation is similar.

(d) follows immediately from (c), which in turn follows from the definitions of $(\underset{\overline{v}}{\trianglerighteq})$ and $(\underset{\overline{w}}{\trianglerighteq})$. Finally, (e) follows from (b). $\qquad\square$

The Proof of Theorem 3.2 uses the following result, which is of independent interest.

Proposition A.1 *Let $\mathcal{I}$ be finite, and let $(\unrhd)$ be a $(\succeq)$-SDP on $\mathcal{X}^{\mathcal{I}}$. If $W : \mathcal{X}^{\mathcal{I}} \longrightarrow \mathcal{R}$ is a SWF for $(\unrhd)$, then there exists some $u \in \mathcal{U}(\succeq)$ and some constant $C \in \mathcal{R}$ such that*

$$W(\mathbf{x}) = C + \sum_{i \in \mathcal{I}} u(x_i) \text{ for all } \mathbf{x} \in \mathcal{X}^{\mathcal{I}}.$$

Proof: Let $\mathcal{J}, \mathcal{K} \subset \mathcal{I}$ be disjoint subsets, with $\mathcal{I} = \mathcal{J} \sqcup \mathcal{K}$. For $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we define $\binom{\mathbf{x}_{\mathcal{J}}}{\mathbf{y}_{\mathcal{K}}} \in \mathcal{X}^{\mathcal{I}}$ by setting $\binom{\mathbf{x}_{\mathcal{J}}}{\mathbf{y}_{\mathcal{K}}}_j := x_j$ for all $j \in \mathcal{J}$, while $\binom{\mathbf{x}_{\mathcal{J}}}{\mathbf{y}_{\mathcal{K}}}_k := y_k$ for all $k \in \mathcal{K}$. Fix some $o \in \mathcal{X}$, and define $\mathbf{o} \in \mathcal{X}^{\mathcal{I}}$ by $\mathbf{o}_i := o$ for all $i \in \mathcal{I}$. Now fix $k \in \mathcal{I}$, and define $u : \mathcal{X} \longrightarrow \mathcal{R}$ by setting $u(x) := W\binom{x_k}{\mathbf{o}_{-k}} - W(\mathbf{o})$ for all $x \in \mathcal{X}$. (It follows that $u(o) = 0$.)

Claim 1: For any $x, y \in \mathcal{X}$, $\mathbf{z} \in \mathcal{X}^{\mathcal{I}}$, and $j \in \mathcal{I}$, we have $W\binom{y_j}{\mathbf{z}_{-j}} - W\binom{x_j}{\mathbf{z}_{-j}} = u(y) - u(x)$.

Proof: We have

$$\begin{pmatrix} x_j & \rightsquigarrow & y_j \\ \mathbf{z}_{-j} & & \mathbf{z}_{-j} \end{pmatrix} \underset{(*)}{\widehat{=\!=}} \begin{pmatrix} x_j & \rightsquigarrow & y_j \\ \mathbf{o}_{-j} & & \mathbf{o}_{-j} \end{pmatrix} \underset{(\dagger)}{\widehat{=}} \begin{pmatrix} x_k & \rightsquigarrow & y_k \\ \mathbf{o}_{-k} & & \mathbf{o}_{-k} \end{pmatrix}. \tag{A21}$$

Here, $(\dagger)$ is by (Anon), and $(*)$ is by (WPar), along with the fact that (DP0) says $(z_i \rightsquigarrow z_i) \approx (o \rightsquigarrow o)$ for all $i \in \mathcal{I} \setminus \{j\}$. Combining (4) with (A21), we obtain

$$\begin{aligned} W\begin{pmatrix} y_j \\ \mathbf{z}_{-j} \end{pmatrix} - W\begin{pmatrix} x_j \\ \mathbf{z}_{-j} \end{pmatrix} &= W\begin{pmatrix} y_k \\ \mathbf{o}_{-k} \end{pmatrix} - W\begin{pmatrix} x_k \\ \mathbf{o}_{-k} \end{pmatrix} \\ &= W\begin{pmatrix} y_k \\ \mathbf{o}_{-k} \end{pmatrix} - W(\mathbf{o}) + W(\mathbf{o}) - W\begin{pmatrix} x_k \\ \mathbf{o}_{-k} \end{pmatrix} = u(y) - u(x), \end{aligned}$$

as desired $\diamondsuit$ `Claim 1`

Claim 2: $u \in \mathcal{U}(\succeq)$.

Proof: Let $x, x', y, y' \in \mathcal{X}$. Then

$$\begin{aligned} \Big((x \rightsquigarrow x') \succeq (y \rightsquigarrow y') \Big) &\underset{(*)}{\Longrightarrow} \left(\begin{pmatrix} x_k & \rightsquigarrow & x'_k \\ \mathbf{o}_{-k} & & \mathbf{o}_{-k} \end{pmatrix} \unrhd \begin{pmatrix} y_k & \rightsquigarrow & y'_k \\ \mathbf{o}_{-k} & & \mathbf{o}_{-k} \end{pmatrix} \right) \\ &\underset{(\dagger)}{\Longrightarrow} \left(W\begin{pmatrix} x'_k \\ \mathbf{o}_{-k} \end{pmatrix} - W\begin{pmatrix} x_k \\ \mathbf{o}_{-k} \end{pmatrix} \geq W\begin{pmatrix} y'_k \\ \mathbf{o}_{-k} \end{pmatrix} - W\begin{pmatrix} y_k \\ \mathbf{o}_{-k} \end{pmatrix} \right). \\ &\underset{(\diamond)}{\Longleftrightarrow} \Big(u(x') - u(x) \geq u(y') - u(y) \Big), \end{aligned}$$

as desired. Here, $(*)$ is by (WPar), $(\dagger)$ is by statement (4), and $(\diamond)$ is by Claim 1. $\diamondsuit$ `Claim 2`

Without loss of generality, suppose $\mathcal{I} = [1 \ldots I]$. Let $C := W(\mathbf{o})$. Then for all $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$,

$$\begin{aligned} W(\mathbf{x}) - C &= W(\mathbf{x}) - W(\mathbf{o}) \\ &= W(\mathbf{x}) - W\begin{pmatrix} o_1 \\ \mathbf{x}_{[2\ldots I]} \end{pmatrix} + W\begin{pmatrix} o_1 \\ \mathbf{x}_{[2\ldots I]} \end{pmatrix} - W\begin{pmatrix} \mathbf{o}_{\{1,2\}} \\ \mathbf{x}_{[3\ldots I]} \end{pmatrix} + \cdots + W\begin{pmatrix} \mathbf{o}_{[1\ldots I-1]} \\ x_I \end{pmatrix} - W(\mathbf{o}) \\ &\underset{(*)}{=} u(x_1) - u(o) + u(x_2) - u(o) + \cdots u(x_I) - u(o) = u(x_1) + u(x_2) + \cdots u(x_I). \end{aligned}$$

Here, $(*)$ is by Claim 1. Thus, $W(\mathbf{x}) = C + \sum_{i \in \mathcal{I}} u(x_i)$, as claimed. $\square$

19

Proof of Theorem 3.2. Let $(\underset{}{\trianglerighteq})$ be a $(\succeq)$-SDP on $\mathcal{X}^{\mathcal{I}}$.

"$\Longleftarrow$" Clearly, if $(\underset{}{\trianglerighteq})$ is quasiutilitarian, it admits a multiwelfare representation.

"$\Longrightarrow$" Let $\mathcal{W}$ be a collection of SWFs yielding a multiwelfare representation for $(\underset{}{\trianglerighteq})$. For all $W \in \mathcal{W}$, Proposition A.1 yields some $v_W \in \mathcal{U}(\succeq)$ such that, for all $\mathbf{x}^1, \mathbf{x}^2 \in \mathcal{X}^{\mathcal{I}}$, we have $W(\mathbf{x}^2) - W(\mathbf{x}^1) = \sum_{i \in \mathcal{I}} v_W(x_i^2) - \sum_{i \in \mathcal{I}} v_W(x_i^1)$. Let $\mathcal{V} := \{v_W; \ W \in \mathcal{W}\}$. Then for all $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, formula (5) implies that $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \trianglerighteq (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$ if and only if (AU1) holds (with $\mathcal{J}_v := \mathcal{I}$) for all $v \in \mathcal{V}$. Thus, $(\underset{}{\trianglerighteq}) = (\underset{\mathcal{V}}{\trianglerighteq})$. $\qquad\square$

The proofs of Proposition 4.1(b,c) use the next result.

Lemma A.2 *Let $(\underset{1}{\trianglerighteq})$ and $(\underset{2}{\trianglerighteq})$ be two preorders on $\mathcal{X}^{\mathcal{I}} \times \mathcal{X}^{\mathcal{I}}$. If $(\underset{2}{\trianglerighteq})$ satisfies* (SPar), *and $(\underset{2}{\trianglerighteq})$ extends $(\underset{1}{\trianglerighteq})$, then $(\underset{1}{\trianglerighteq})$ satisfies* (SPar) *also.*

Proof: Suppose $(x_i^1 \rightsquigarrow x_i^2) \preceq (y_i^1 \rightsquigarrow y_i^2)$ for all $i \in \mathcal{I}$, and $(x_i^1 \rightsquigarrow x_i^2) \prec (y_i^1 \rightsquigarrow y_i^2)$ for some $i \in \mathcal{I}$; we must show that $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{1}{\lhd} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$. Since $(\underset{1}{\trianglerighteq})$ satisfies (WPar), we have $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{1}{\unlhd} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$. We must show $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{1}{\ntrianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$.

By contradiction, suppose $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{1}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$. Then $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{2}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$, because $(\underset{2}{\trianglerighteq})$ extends $(\underset{1}{\trianglerighteq})$. But $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{2}{\lhd} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$ because $(\underset{2}{\trianglerighteq})$ satisfies (SPar). Contradiction. $\qquad\square$

The proofs of Proposition 4.1(b,c), Theorem 4.2 and Lemma A.5 (below) all depend on the next result.

Lemma A.3 *Let $(\succeq)$ be a preorder on $\mathcal{X} \times \mathcal{X}$ which satisfies* (DP1) *and* (DP2). *Let $x_0, x_1, \ldots, x_N, y_0, y_1, \ldots, y_N \in \mathcal{X}$.*

(a) *Suppose $(x_{n-1} \rightsquigarrow x_n) \succeq (y_{n-1} \rightsquigarrow y_n)$ for all $n \in [1 \ldots N]$. Then $(x_0 \rightsquigarrow x_N) \succeq (y_0 \rightsquigarrow y_N)$.*

(b) *Suppose $(x_{n-1} \rightsquigarrow x_n) \succeq (y_{n-1} \rightsquigarrow y_n)$ for all $n \in [1 \ldots N]$, and $(x_{n-1} \rightsquigarrow x_n) \succ (y_{n-1} \rightsquigarrow y_n)$ for some $n \in [1 \ldots N]$. Then $(x_0 \rightsquigarrow x_N) \succ (y_0 \rightsquigarrow y_N)$.*

Now let $\alpha : [1 \ldots N] \longrightarrow [1 \ldots N]$ be a permutation, and suppose $(\succeq)$ also satisfies (DP3).

(c) *Suppose $(x_{n-1} \rightsquigarrow x_n) \succeq (y_{\alpha(n)-1} \rightsquigarrow y_{\alpha(n)})$ for all $n \in [1 \ldots N]$. Then $(x_0 \rightsquigarrow x_N) \succeq (y_0 \rightsquigarrow y_N)$.*

(d) *Suppose $(x_{n-1} \rightsquigarrow x_n) \succeq (y_{\alpha(n)-1} \rightsquigarrow y_{\alpha(n)})$ for all $n \in [1 \ldots N]$, and $(x_{n-1} \rightsquigarrow x_n) \succ (y_{\alpha(n)-1} \rightsquigarrow y_{\alpha(n)})$ for some $n \in [1 \ldots N]$. Then $(x_0 \rightsquigarrow x_N) \succ (y_0 \rightsquigarrow y_N)$.*

Proof: Part (a) follows from inductive application of (DP2). Part (c) follows from a similar inductive application of both (DP3) and (DP2).

It suffices to prove part (b) in the case $N = 2$; the proof for longer chains can then be derived by applying the case $N = 2$ and performing induction on chain length. There are two cases.

First, suppose $(x_0 \rightsquigarrow x_1) \succeq (y_0 \rightsquigarrow y_1)$ and $(x_1 \rightsquigarrow x_2) \succ (y_1 \rightsquigarrow y_2)$. We must show that $(x_0 \rightsquigarrow x_2) \succ (y_0 \rightsquigarrow y_2)$. Axiom (DP2) implies that $(x_0 \rightsquigarrow x_2) \succeq (y_0 \rightsquigarrow y_2)$, so it suffices to show that $(x_0 \rightsquigarrow x_2) \not\preceq (y_0 \rightsquigarrow y_2)$.

By contradiction, suppose $(x_0 \rightsquigarrow x_2) \preceq (y_0 \rightsquigarrow y_2)$. Then (DP1) implies that $(x_2 \rightsquigarrow x_0) \succeq (y_2 \rightsquigarrow y_0)$. This, together with hypothesis $(x_0 \rightsquigarrow x_1) \succeq (y_0 \rightsquigarrow y_1)$ and (DP2), implies that $(x_2 \rightsquigarrow x_1) \succeq (y_2 \rightsquigarrow y_1)$. Then (DP1) implies that $(x_1 \rightsquigarrow x_2) \preceq (y_1 \rightsquigarrow y_2)$. But this contradicts the hypothesis that $(x_1 \rightsquigarrow x_2) \succ (y_1 \rightsquigarrow y_2)$.

A similar proof applies if $(x_0 \rightsquigarrow x_1) \succ (y_0 \rightsquigarrow y_1)$ and $(x_1 \rightsquigarrow x_2) \succeq (y_1 \rightsquigarrow y_2)$.

The proof of part (d) is similar to part (b), but using (DP3) as well as (DP2). $\qquad\square$

The proof of Proposition 4.1(a) follows immediately from defining formula (6). The proofs of Proposition 4.1(b,c) require Proposition 4.5, so we prove that first.

Proof of Proposition 4.5. Let $(\underset{*}{\trianglerighteq})$ be the SDP defined inductively by rules (∗1)-(∗5). Let $(\underset{\circ}{\trianglerighteq})$ be the SDP defined by formula (6). We must show that $(\underset{*}{\trianglerighteq}) = (\underset{\circ}{\trianglerighteq})$.

"$\supseteq$" By Proposition 4.1(a), it suffices to show that $(\underset{*}{\trianglerighteq})$ is an $(\succeq)$-SDP. It is easy to check that $(\underset{\text{an}}{\widehat{=}})$ is reflexive and satisfies axioms (Anon) and (DP1$^{\trianglerighteq}$). Likewise, $(\underset{\text{par}}{\trianglerighteq})$ is reflexive and satisfies axioms (WPar), (DP0$^{\trianglerighteq}$), and (DP1$^{\trianglerighteq}$). Thus, the binary relation $(\underset{\text{an}}{\widehat{=}}) \cup (\underset{\text{par}}{\trianglerighteq})$ is reflexive and satisfies axioms (Anon), (WPar), (DP0$^{\trianglerighteq}$), and (DP1$^{\trianglerighteq}$). Thus, $(\underset{*}{\trianglerighteq})$ will also satisfy these properties, and will also be transitive and satisfy (DP2$^{\trianglerighteq}$) and (DP3$^{\trianglerighteq}$) by definition. Thus, $(\underset{*}{\trianglerighteq})$ is an $(\succeq)$-SDP.

"$\subseteq$" We will show that $(\underset{*}{\trianglerighteq})$ is a subrelation of every other $(\succeq)$-SDP; in particular, this implies that $(\underset{*}{\trianglerighteq}) \subseteq (\underset{\circ}{\trianglerighteq})$.

Let $(\trianglerighteq)$ be another $(\succeq)$-SDP, and let $\mathbf{x}^0, \mathbf{x}^2, \mathbf{z}^0, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$. Suppose $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \underset{*}{\trianglerighteq} (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$; we must show that $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \trianglerighteq (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$. The proof is by induction, using the recursive definition of $(\underset{*}{\trianglerighteq})$.

(∗1) If $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \underset{\text{an}}{\widehat{=}} (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$, then $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \widehat{=} (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$ because $(\trianglerighteq)$ satisfies (Anon).

(∗2) If $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \underset{\text{par}}{\trianglerighteq} (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$, then $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \trianglerighteq (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$, because $(\trianglerighteq)$ satisfies (WPar).

(∗3) Suppose there exist $\mathbf{y}^0, \mathbf{y}^2$ with $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \underset{*}{\unrhd} (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2)$ and $(\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2) \underset{*}{\unrhd} (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$. By induction, suppose we have already shown that $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \unrhd (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2)$ and $(\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2) \unrhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$. Then $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \unrhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$, because $(\unrhd)$ is transitive.

(∗4) Suppose there exist $\mathbf{x}^1, \mathbf{z}^1$ with $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \underset{*}{\unrhd} (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^1)$ and $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{*}{\unrhd} (\mathbf{z}^1 \rightsquigarrow \mathbf{z}^2)$. By induction, suppose we have already shown $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \unrhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^1)$ and $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \unrhd (\mathbf{z}^1 \rightsquigarrow \mathbf{z}^2)$. Then $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \unrhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$, because $(\unrhd)$ satisfies $(\text{DP2}^{\unrhd})$.

(∗5) Suppose there exist $\mathbf{x}^1, \mathbf{z}^1$ with $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \underset{*}{\unrhd} (\mathbf{z}^1 \rightsquigarrow \mathbf{z}^2)$ and $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{*}{\unrhd} (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^1)$. By induction, suppose we have already shown $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \unrhd (\mathbf{z}^1 \rightsquigarrow \mathbf{z}^2)$ and $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \unrhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^1)$. Then $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \unrhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$, because $(\unrhd)$ satisfies $(\text{DP3}^{\unrhd})$.

By induction, we have $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \unrhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$. $\qquad\qquad\qquad\square$

Proof of Proposition 4.1(b). "$\Longleftarrow$" If $(\underset{*}{\unrhd})$ satisfies (SPar) and $(\unrhd)$ refines $(\underset{*}{\unrhd})$, then clearly $(\unrhd)$ also satisfies (SPar).

"$\Longrightarrow$" $(\unrhd)$ extends $(\underset{*}{\unrhd})$, by part (b). Thus, if $(\unrhd)$ satisfies (SPar), then Lemma A.2 says that $(\underset{*}{\unrhd})$ satisfies (SPar).

It remains to show that $(\unrhd)$ refines $(\underset{*}{\unrhd})$. Suppose $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \underset{*}{\rhd} (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$. Then $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \underset{*}{\unrhd} (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$ through some sequence of applications of Steps (∗1)-(∗5), but the same sequence does *not* yield $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \underset{*}{\unlhd} (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$. This means that in some application of Step (∗2), we must have a strict "$\underset{\text{par}}{\rhd}$" rather than "$\underset{\overline{\text{par}}}{\unrhd}$". Thus, we augment the proof of Proposition 4.5 "$\subseteq$" with the following additional observations:

(∗2) $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \underset{\text{par}}{\rhd} (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$, then $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \rhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$, because $(\succeq)$ satisfies (SPar).

(∗3) Suppose there exist $\mathbf{y}^0, \mathbf{y}^2$ with $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \rhd (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2)$ and $(\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2) \underset{*}{\unrhd} (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$. By induction, suppose we have already shown that $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \rhd (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2)$ and $(\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2) \unrhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$. Then $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \rhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$, because $(\unrhd)$ is transitive. Likewise, if $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \underset{*}{\unrhd} (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2)$ and $(\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2) \rhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$, then $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \rhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$.

(∗4) Suppose there exist $\mathbf{x}^1, \mathbf{z}^1$ with $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \rhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^1)$ and $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{*}{\unrhd} (\mathbf{z}^1 \rightsquigarrow \mathbf{z}^2)$. By induction, suppose we have already shown that $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \rhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^1)$ and $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \unrhd (\mathbf{z}^1 \rightsquigarrow \mathbf{z}^2)$. Then $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \rhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$ by Lemma A.3(b), because $(\unrhd)$ satisfies $(\text{DP2}^{\unrhd})$. Likewise, if $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \underset{*}{\unrhd} (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^1)$ and $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \rhd (\mathbf{z}^1 \rightsquigarrow \mathbf{z}^2)$, then $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \rhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$.

(∗5) Suppose there exist $\mathbf{x}^1, \mathbf{z}^1$ with $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \underset{*}{\rhd} (\mathbf{z}^1 \rightsquigarrow \mathbf{z}^2)$ and $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{*}{\unrhd} (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^1)$. By induction, suppose we have already shown that $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \rhd (\mathbf{z}^1 \rightsquigarrow \mathbf{z}^2)$ and $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \unrhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^1)$. Then $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \rhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$ by Lemma A.3(d), because $(\rhd)$ satisfies (DP3$^{\unrhd}$).

Likewise, if $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \underset{*}{\unrhd} (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^1)$ and $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{*}{\rhd} (\mathbf{z}^1 \rightsquigarrow \mathbf{z}^2)$, then $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \rhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$.

By induction, we conclude that $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \rhd (\mathbf{z}^0 \rightsquigarrow \mathbf{z}^2)$.

(c) If $(\succeq)$ has a strong utility function, then Proposition 3.1(e) says that $(\underset{u}{\unrhd})$ satisfies (SPar). Now part (b) implies that $(\underset{*}{\unrhd})$ also satisfies (SPar). $\qquad\square$

The proofs of Theorem 4.2 and Proposition 6.1 depend upon Proposition 5.2(b). Likewise, the proof of Theorem 5.1 depends on Proposition 5.2, so we will prove that first.

Proof of Proposition 5.2(a). *Reflexive* follows immediately, by setting $\mathbf{y}^1 = \mathbf{x}^1$ and $\mathbf{y}^2 = \mathbf{x}^2$ in (WPar).

(WPar) Let $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, and suppose that $(x_i^1 \rightsquigarrow x_i^2) \succeq (y_i^1 \rightsquigarrow y_i^2)$ for all $i \in \mathcal{I}$. Set $\mathcal{I}_0 := \emptyset$, so condition (NG2) is vacuous. Meanwhile, condition (NG1) is satisfied by hypothesis. Thus, $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{ng}}{\unrhd} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$.

(Anon) Let $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ and $\pi \in \Pi$; we must show that $(\mathbf{x} \rightsquigarrow \pi(\mathbf{x})) \underset{\mathrm{ng}}{\widehat{\triplexeq}} (\mathbf{x} \rightsquigarrow \mathbf{x})$. Since π is finitary, there exists some finite $\mathcal{I}_0 \subseteq \mathcal{I}$ such that $\pi(i) = i$ for all $i \in \mathcal{I} \setminus \mathcal{I}_0$.

The π-orbits of distinct points in $\mathcal{I}_0$ are either identical or disjoint; thus, they form a partition of $\mathcal{I}_0$. Thus, there is some finite indexing set $\mathcal{L}$ such that $\mathcal{I}_0 = \bigsqcup_{\ell \in \mathcal{L}} \mathcal{J}_\ell$, where for each $\ell \in \mathcal{L}$, we have $\mathcal{J}_\ell = \{i_\ell, \pi(i_\ell), \pi^2(i_\ell), \dots, \pi^{J_\ell - 1}(i_\ell)\}$ for some $i_\ell \in \mathcal{I}_0$ with $\pi^{J_\ell}(i_\ell) = i_\ell$. For all $\ell \in \mathcal{L}$, define $\mathcal{K}_\ell := \mathcal{J}_\ell$.

Claim 1: *For all $\ell \in \mathcal{L}$, we have* $(\mathbf{x} \rightsquigarrow \pi(\mathbf{x})) \widehat{\overline{\overline{\mathcal{J}_\ell, \mathcal{K}_\ell}}} (\mathbf{x} \rightsquigarrow \mathbf{x})$.

Proof: We will construct $w_0, w_1, \dots, w_{J_\ell}, z_0, z_1, \dots, z_{K_\ell} \in \mathcal{X}$ and bijections $\alpha_\ell : \mathcal{J}_\ell \longrightarrow [1 \dots J_\ell]$ and $\beta_\ell : \mathcal{K}_\ell \longrightarrow [1 \dots K_\ell]$ verifying properties (JK1)-(JK3) in the definition of $(\underset{\mathcal{J}_\ell, \mathcal{K}_\ell}{\unrhd})$.

For all $n \in [0 \dots J_\ell]$, let $w_n^\ell := x_{\pi^n(i_\ell)}$, and define $\alpha_\ell : \mathcal{J}_\ell \longrightarrow [1 \dots J_\ell]$ so that, for all $j \in \mathcal{J}_\ell$, if $j = \pi^n(i_\ell)$, then $\alpha_\ell(j) := n + 1$. Thus, $(x_j \rightsquigarrow x_{\pi(j)}) = (w_{\alpha_\ell(j)-1}^\ell \rightsquigarrow w_{\alpha_\ell(j)}^\ell)$ for all $j \in \mathcal{J}_\ell$, thereby verifying (JK1).

Fix $z \in \mathcal{X}$. For all $\ell \in \mathcal{L}$, define $\beta_\ell := \alpha_\ell : \mathcal{K}_\ell \longrightarrow [1 \dots K_\ell]$, and define $z_0^\ell = z_1^\ell = z_2^\ell = \cdots = z_{K_\ell}^\ell := z$. Then

$$(z_{\beta_\ell(k)-1}^\ell \rightsquigarrow z_{\beta_\ell(k)}^\ell) \quad = \quad (z \rightsquigarrow z) \underset{(DP0)}{\approx} (x_k \rightsquigarrow x_k),$$

for all $k \in \mathcal{K}_\ell$, thereby verifying (JK2).

Finally, note that $w^\ell_{J_\ell} = x_{\pi^{J_\ell}(i_\ell)} = x_{i_\ell} = w^\ell_0$ (because $\pi^{J_\ell}(i_\ell) = i_\ell$ by hypothesis). Thus

$$(w^\ell_0 \rightsquigarrow w^\ell_{J_\ell}) \quad = \quad (x_{i_\ell} \rightsquigarrow x_{i_\ell}) \underset{(DP0)}{\approx} (z \rightsquigarrow z) \quad = \quad (z^\ell_0 \rightsquigarrow z^\ell_{K_\ell}),$$

thereby verifying (JK3).

This construction shows that $(\mathbf{x} \rightsquigarrow \pi(\mathbf{x})) \underset{\mathcal{J}_\ell, \mathcal{K}_\ell}{\trianglerighteq} (\mathbf{x} \rightsquigarrow \mathbf{x})$. But the exact same construction can be read backwards to show that $(\mathbf{x} \rightsquigarrow \pi(\mathbf{x})) \underset{\mathcal{J}_\ell, \mathcal{K}_\ell}{\trianglelefteq} (\mathbf{x} \rightsquigarrow \mathbf{x})$. $\qquad \diamond$ Claim 1

Now, for all $i \in \mathcal{I} \setminus \mathcal{I}_0$, we have $\pi(\mathbf{x})_i = x_i$, so $(x_i \rightsquigarrow \pi(\mathbf{x})_i) \approx (x_i \rightsquigarrow x_i)$, verifying (NG1). Meanwhile, Claim 1 verifies (NG2). We conclude that $(\mathbf{x} \rightsquigarrow \pi(\mathbf{x})) \underset{\mathrm{ng}}{\widehat{=}} (\mathbf{x} \rightsquigarrow \mathbf{x})$.

(DP0$^{\trianglerighteq}$) Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$. We must show that $(\mathbf{x} \rightsquigarrow \mathbf{x}) \underset{\mathrm{ng}}{\widehat{=}} (\mathbf{y} \rightsquigarrow \mathbf{y})$. Let $\mathcal{I}_0 := \emptyset$, so that (NG2) is vacuous. To verify (NG1), observe that, for all $i \in \mathcal{I}$, we have $(x_i \rightsquigarrow x_i) \approx (y_i \rightsquigarrow y_i)$, because $(\succeq)$ satisfies (DP0). Thus, $(\mathbf{x} \rightsquigarrow \mathbf{x}) \underset{\mathrm{ng}}{\widehat{=}} (\mathbf{y} \rightsquigarrow \mathbf{y})$.

(DP1$^{\trianglerighteq}$) Let $\mathcal{I}_0 \subseteq \mathcal{I}$ be finite, and let $\{\mathcal{J}_\ell\}_{\ell \in \mathcal{L}}$ and $\{\mathcal{K}_\ell\}_{\ell \in \mathcal{L}}$ be partitions of $\mathcal{I}_0$, such that $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{ng}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$ via these partitions. I claim that $(\mathbf{x}^2 \rightsquigarrow \mathbf{x}^1) \underset{\mathrm{ng}}{\trianglelefteq} (\mathbf{y}^2 \rightsquigarrow \mathbf{y}^1)$ via the same partitions. To prove this, we must check (NG1) and (NG2).

(NG1) For all $i \in \mathcal{I} \setminus \mathcal{I}_0$, we have $(x^1_i \rightsquigarrow x^2_i) \succeq (y^1_i \rightsquigarrow y^2_i)$; thus, (DP1) implies that $(x^2_i \rightsquigarrow x^1_i) \preceq (y^2_i \rightsquigarrow y^1_i)$.

(NG2) Now fix $\ell \in \mathcal{L}$. Find $w_0, w_1, \ldots, w_{J_\ell}, z_0, z_1, \ldots, z_{K_\ell} \in \mathcal{X}$ and bijections $\alpha : \mathcal{J}_\ell \longrightarrow [1 \ldots J_\ell]$ and $\beta : \mathcal{K}_\ell \longrightarrow [1 \ldots K_\ell]$ satisfying conditions (JK1)-(JK3) to verify the relation $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{J}_\ell, \mathcal{K}_\ell}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$. By applying (DP1) to (JK1)-(JK3), we get:

(1KJ) $(x^2_j \rightsquigarrow x^1_j) \preceq (w_{\alpha(j)} \rightsquigarrow w_{\alpha(j)-1})$ for all $j \in \mathcal{J}_\ell$;

(2KJ) $(z_{\beta(k)} \rightsquigarrow z_{\beta(k)-1}) \preceq (y^2_k \rightsquigarrow y^1_k)$, for all $k \in \mathcal{K}_\ell$; and

(3KJ) $(w_{J_\ell} \rightsquigarrow w_0) \preceq (z_{K_\ell} \rightsquigarrow z_0)$.

Now define $w'_j := w_{J_\ell - j}$ for all $j \in [0 \ldots J_\ell]$ and and $\alpha'(j) := J_\ell - \alpha(j) + 1$ for all $j \in \mathcal{J}_\ell$. Likewise, define $z'_k := z_{K_\ell - k}$ for all $k \in [0 \ldots K_\ell]$ and and $\beta'(k) := K_\ell - \beta(k) + 1$ for all $k \in \mathcal{K}_\ell$. Then assertions (1KJ)-(3KJ) become:

(JK1″) $(x^2_j \rightsquigarrow x^1_j) \preceq (w'_{\alpha'(j)-1} \rightsquigarrow w'_{\alpha'(j)})$ for all $j \in \mathcal{J}_\ell$;

(JK2″) $(z'_{\beta'(k)-1} \rightsquigarrow z'_{\beta'(k)}) \preceq (y^2_k \rightsquigarrow y^1_k)$, for all $k \in \mathcal{K}_\ell$; and

(JK3″) $(w'_0 \rightsquigarrow w'_{J_\ell}) \preceq (z'_0 \rightsquigarrow z'_{K_\ell})$.

Thus, $(\mathbf{x}^2 \rightsquigarrow \mathbf{x}^1) \underset{\mathcal{J}_\ell, \mathcal{K}_\ell}{\trianglelefteq} (\mathbf{y}^2 \rightsquigarrow \mathbf{y}^1)$. We can do this for all $\ell \in \mathcal{L}$, thereby verifying (NG2).

We conclude that $(\mathbf{x}^2 \rightsquigarrow \mathbf{x}^1) \underset{\mathrm{ng}}{\trianglelefteq} (\mathbf{y}^2 \rightsquigarrow \mathbf{y}^1)$. $\qquad \qquad \square$

Proof of Proposition 5.2(b). Let $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^2, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$. Let $\mathcal{I}_0 \subseteq \mathcal{I}$ be a finite subset, and let $\{\mathcal{J}_\ell\}_{\ell \in \mathcal{L}}$ and $\{\mathcal{K}_\ell\}_{\ell \in \mathcal{L}}$ be partitions of $\mathcal{I}_0$. Suppose $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{ng}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$ via these partitions. We will show that $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \trianglerighteq (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$.

For all $\ell \in \mathcal{L}$ and $n = 1, 2$, let $\mathbf{x}^n_{\mathcal{J}_\ell} := (x^n_j)_{j \in \mathcal{J}_\ell} \in \mathcal{X}^{\mathcal{J}_\ell}$ and $\mathbf{y}^n_{\mathcal{K}_\ell} := (y^n_k)_{k \in \mathcal{K}_\ell} \in \mathcal{X}^{\mathcal{K}_\ell}$. Since $\mathcal{I}_0 = \bigsqcup_{\ell \in \mathcal{L}} \mathcal{J}_\ell$, we can write $(\mathbf{x}^1_{\mathcal{I}_0} \rightsquigarrow \mathbf{x}^2_{\mathcal{I}_0})$ as an $\mathcal{L}$-indexed structure $(\mathbf{x}^1_{\mathcal{J}_\ell} \rightsquigarrow \mathbf{x}^2_{\mathcal{J}_\ell})_{\ell \in \mathcal{L}}$. Likewise, since $\mathcal{I}_0 = \bigsqcup_{\ell \in \mathcal{L}} \mathcal{K}_\ell$, we can write $(\mathbf{y}^1_{\mathcal{I}_0} \rightsquigarrow \mathbf{y}^2_{\mathcal{I}_0})$ as an $\mathcal{L}$-indexed structure $(\mathbf{y}^1_{\mathcal{K}_\ell} \rightsquigarrow \mathbf{y}^2_{\mathcal{K}_\ell})_{\ell \in \mathcal{L}}$.

For all $\ell \in \mathcal{L}$, condition (NG2) says $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{J}_\ell, \mathcal{K}_\ell}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$. So, let $J_\ell := |\mathcal{J}_\ell|$ and $K_\ell := |\mathcal{K}_\ell|$, and let $w^\ell_0, w^\ell_1, \ldots, w^\ell_{J_\ell} \in \mathcal{X}$ and $z^\ell_0, z^\ell_1, \ldots, z^\ell_K \in \mathcal{X}$ and bijections $\alpha_\ell : \mathcal{J}_\ell \longrightarrow [1 \ldots J_\ell]$ and $\beta_\ell : \mathcal{K}_\ell \longrightarrow [1 \ldots K_\ell]$ satisfy conditions (JK1)-(JK3) in the definition of $(\underset{\mathcal{J}_\ell, \mathcal{K}_\ell}{\trianglerighteq})$. For all $\ell \in \mathcal{L}$, suppose we write $\mathcal{J}_\ell := \{j(\ell, 1), j(\ell, 2), \ldots, j(\ell, J_\ell)\}$, such that $\alpha_\ell[j(\ell, n)] = n$ for all $n \in [1 \ldots J_\ell]$. Then

$$
\begin{aligned}
(\mathbf{x}^1_{\mathcal{I}_0} \rightsquigarrow \mathbf{x}^2_{\mathcal{I}_0}) \quad &= \quad (\mathbf{x}^1_{\mathcal{J}_\ell} \rightsquigarrow \mathbf{x}^2_{\mathcal{J}_\ell})_{\ell \in \mathcal{L}} \\[4pt]
&= \quad \begin{bmatrix} x^1_{j(\ell,1)} & \rightsquigarrow & x^2_{j(\ell,1)} \\ x^1_{j(\ell,2)} & \rightsquigarrow & x^2_{j(\ell,2)} \\ x^1_{j(\ell,3)} & \rightsquigarrow & x^2_{j(\ell,3)} \\ \vdots & \vdots & \vdots \\ x^1_{j(\ell,J_\ell)} & \rightsquigarrow & x^2_{j(\ell,J_\ell)} \end{bmatrix}_{\ell \in \mathcal{L}} \underset{\mathrm{par}}{\trianglerighteq} \underbrace{\begin{bmatrix} w^\ell_0 & \rightsquigarrow & w^\ell_1 \\ w^\ell_1 & \rightsquigarrow & w^\ell_2 \\ w^\ell_2 & \rightsquigarrow & w^\ell_3 \\ \vdots & \vdots & \vdots \\ w^\ell_{J_\ell - 1} & \rightsquigarrow & w^\ell_{J_\ell} \end{bmatrix}_{\ell \in \mathcal{L}}}_{(\mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}'_{\mathcal{I}_0})} \underset{\mathrm{an}}{\widehat{=}} \underbrace{\begin{bmatrix} w^\ell_0 & \rightsquigarrow & w^\ell_{J_\ell} \\ w^\ell_1 & \rightsquigarrow & w^\ell_1 \\ w^\ell_2 & \rightsquigarrow & w^\ell_2 \\ \vdots & \vdots & \vdots \\ w^\ell_{J_\ell - 1} & \rightsquigarrow & w^\ell_{J_\ell - 1} \end{bmatrix}_{\ell \in \mathcal{L}}}_{(\mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}''_{\mathcal{I}_0})},
\end{aligned}
\tag{A22}
$$

where "$\underset{\mathrm{par}}{\trianglerighteq}$" is by (JK1), and "$\underset{\mathrm{an}}{\widehat{=}}$" is via the (finitary) permutation $\pi : \mathcal{I}_0 \longrightarrow \mathcal{I}_0$ defined by $\pi[j(\ell, 1)] := j(\ell, J_\ell)$ and $\pi[j(\ell, n)] := j(\ell, n - 1)$ for all $\ell \in \mathcal{L}$ and $n \in [2 \ldots J_\ell]$. If $\mathbf{w}_{\mathcal{I}_0}$, $\mathbf{w}'_{\mathcal{I}_0}$, and $\mathbf{w}''_{\mathcal{I}_0}$ are the elements of $\mathcal{X}^{\mathcal{I}_0}$ defined as indicated above, then formula (A22) can be rewritten: $(\mathbf{x}^1_{\mathcal{I}_0} \rightsquigarrow \mathbf{x}^2_{\mathcal{I}_0}) \underset{\mathrm{par}}{\trianglerighteq} (\mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}'_{\mathcal{I}_0}) \underset{\mathrm{an}}{\widehat{=}} (\mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}''_{\mathcal{I}_0})$. Thus, if $\mathcal{I}_0^{\complement} := \mathcal{I} \setminus \mathcal{I}_0$, then we have

$$
(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \quad = \quad \begin{pmatrix} \mathbf{x}^1_{\mathcal{I}_0} \rightsquigarrow \mathbf{x}^2_{\mathcal{I}_0} \\ \mathbf{x}^1_{\mathcal{I}_0^{\complement}} \rightsquigarrow \mathbf{x}^2_{\mathcal{I}_0^{\complement}} \end{pmatrix} \underset{\mathrm{par}}{\trianglerighteq} \begin{pmatrix} \mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}'_{\mathcal{I}_0} \\ \mathbf{x}^1_{\mathcal{I}_0^{\complement}} \rightsquigarrow \mathbf{x}^2_{\mathcal{I}_0^{\complement}} \end{pmatrix} \underset{\mathrm{an}}{\widehat{=}} \begin{pmatrix} \mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}''_{\mathcal{I}_0} \\ \mathbf{x}^1_{\mathcal{I}_0^{\complement}} \rightsquigarrow \mathbf{x}^2_{\mathcal{I}_0^{\complement}} \end{pmatrix}.
$$

Thus, since the SDP $(\trianglerighteq)$ satisfies axioms (WPar) and (Anon), we deduce that

$$
(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \quad = \quad \begin{pmatrix} \mathbf{x}^1_{\mathcal{I}_0} \rightsquigarrow \mathbf{x}^2_{\mathcal{I}_0} \\ \mathbf{x}^1_{\mathcal{I}_0^{\complement}} \rightsquigarrow \mathbf{x}^2_{\mathcal{I}_0^{\complement}} \end{pmatrix} \trianglerighteq \begin{pmatrix} \mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}'_{\mathcal{I}_0} \\ \mathbf{x}^1_{\mathcal{I}_0^{\complement}} \rightsquigarrow \mathbf{x}^2_{\mathcal{I}_0^{\complement}} \end{pmatrix} \widehat{=} \begin{pmatrix} \mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}''_{\mathcal{I}_0} \\ \mathbf{x}^1_{\mathcal{I}_0^{\complement}} \rightsquigarrow \mathbf{x}^2_{\mathcal{I}_0^{\complement}} \end{pmatrix}.
\tag{A23}
$$

Next, for all $\ell \in \mathcal{L}$, suppose we write $\mathcal{K}_\ell := \{k(\ell, 1), k(\ell, 2), \ldots, k(\ell, K_\ell)\}$, such that $\beta_\ell[k(\ell, n)] = n$ for all $n \in [1 \ldots K_\ell]$. Then

$$
(\mathbf{y}^1_{\mathcal{I}_0} \rightsquigarrow \mathbf{y}^2_{\mathcal{I}_0}) \quad = \quad (\mathbf{y}^1_{\mathcal{K}_\ell} \rightsquigarrow \mathbf{y}^2_{\mathcal{K}_\ell})_{\ell \in \mathcal{L}}
\tag{A24}
$$

$$= \begin{bmatrix} y^1_{k(\ell,1)} & \rightsquigarrow & y^2_{k(\ell,1)} \\ y^1_{k(\ell,2)} & \rightsquigarrow & y^2_{k(\ell,2)} \\ y^1_{k(\ell,3)} & \rightsquigarrow & y^2_{k(\ell,3)} \\ \vdots & \vdots & \vdots \\ y^1_{k(\ell,K_\ell)} & \rightsquigarrow & y^2_{k(\ell,K_\ell)} \end{bmatrix}_{\ell\in\mathcal{L}} \underset{\overline{\mathrm{par}}}{\trianglelefteq} \underbrace{\begin{bmatrix} z^\ell_0 & \rightsquigarrow & z^\ell_1 \\ z^\ell_1 & \rightsquigarrow & z^\ell_2 \\ z^\ell_2 & \rightsquigarrow & z^\ell_3 \\ \vdots & \vdots & \vdots \\ z^\ell_{K_\ell-1} & \rightsquigarrow & z^\ell_{K_\ell} \end{bmatrix}_{\ell\in\mathcal{L}}}_{(\mathbf{z}_{\mathcal{I}_0} \ \rightsquigarrow \ \mathbf{z}'_{\mathcal{I}_0})} \underset{\overline{\mathrm{an}}}{\widehat{=}} \underbrace{\begin{bmatrix} z^\ell_0 & \rightsquigarrow & z^\ell_{K_\ell} \\ z^\ell_1 & \rightsquigarrow & z^\ell_1 \\ z^\ell_2 & \rightsquigarrow & z^\ell_2 \\ \vdots & \vdots & \vdots \\ z^\ell_{K_\ell-1} & \rightsquigarrow & z^\ell_{K_\ell-1} \end{bmatrix}_{\ell\in\mathcal{L}}}_{(\mathbf{z}_{\mathcal{I}_0} \ \rightsquigarrow \ \mathbf{z}''_{\mathcal{I}_0})},$$

where "$\underset{\overline{\mathrm{par}}}{\trianglelefteq}$" is by (JK2), and "$\underset{\overline{\mathrm{an}}}{\widehat{=}}$" is via the (finitary) permutation $\pi : \mathcal{I}_0 \longrightarrow \mathcal{I}_0$ defined by $\pi[k(\ell,1)] := k(\ell,K_\ell)$ and $\pi[k(\ell,n)] := (\ell, n-1)$ for all $\ell \in \mathcal{L}$ and $n \in [2 \ldots K_\ell]$. If $\mathbf{z}_{\mathcal{I}_0}$, $\mathbf{z}'_{\mathcal{I}_0}$, and $\mathbf{z}''_{\mathcal{I}_0}$ are the elements of $\mathcal{X}^{\mathcal{I}_0}$ defined as indicated above, then formula (A24) can be rewritten: $(\mathbf{y}^1_{\mathcal{I}_0} \rightsquigarrow \mathbf{y}^2_{\mathcal{I}_0}) \underset{\overline{\mathrm{par}}}{\trianglelefteq} (\mathbf{z}_{\mathcal{I}_0} \ \rightsquigarrow \ \mathbf{z}'_{\mathcal{I}_0}) \underset{\overline{\mathrm{an}}}{\widehat{=}} (\mathbf{z}_{\mathcal{I}_0} \ \rightsquigarrow \ \mathbf{z}''_{\mathcal{I}_0})$. Combining this with the $\mathcal{I}_0^\complement$-coordinates, and recalling that $(\trianglerighteq)$ satisfies axioms (WPar) and (Anon), we obtain

$$(\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2) \ = \ \begin{pmatrix} \mathbf{y}^1_{\mathcal{I}_0} \rightsquigarrow \mathbf{y}^2_{\mathcal{I}_0} \\ \mathbf{y}^1_{\mathcal{I}_0^\complement} \rightsquigarrow \mathbf{y}^2_{\mathcal{I}_0^\complement} \end{pmatrix} \trianglelefteq \begin{pmatrix} \mathbf{z}_{\mathcal{I}_0} \ \rightsquigarrow \ \mathbf{z}'_{\mathcal{I}_0} \\ \mathbf{y}^1_{\mathcal{I}_0^\complement} \rightsquigarrow \mathbf{y}^2_{\mathcal{I}_0^\complement} \end{pmatrix} \ \widehat{=} \ \begin{pmatrix} \mathbf{z}_{\mathcal{I}_0} \ \rightsquigarrow \ \mathbf{z}''_{\mathcal{I}_0} \\ \mathbf{y}^1_{\mathcal{I}_0^\complement} \rightsquigarrow \mathbf{y}^2_{\mathcal{I}_0^\complement} \end{pmatrix}. \qquad \text{(A25)}$$

Now, $\mathcal{I}_0 = \mathcal{H}'_1 \sqcup \mathcal{H}'_2 = \mathcal{H}''_1 \sqcup \mathcal{H}''_2$, where

$$\begin{aligned} \mathcal{H}'_1 &:= \{j(\ell,1) \,;\, \ell \in \mathcal{L}\}; & \mathcal{H}'_2 &:= \{j(\ell,n) \,;\, \ell \in \mathcal{L} \text{ and } n \in [2 \ldots J_\ell]\}, \\ \mathcal{H}''_1 &:= \{k(\ell,1) \,;\, \ell \in \mathcal{L}\} & \text{and} \quad \mathcal{H}''_2 &:= \{k(\ell,n) \,;\, \ell \in \mathcal{L} \text{ and } n \in [2 \ldots K_\ell]\}. \end{aligned}$$

Clearly, $|\mathcal{H}'_1| = |\mathcal{L}| = |\mathcal{H}''_1|$, and $|\mathcal{H}'_2| = |\mathcal{I}_0| - |\mathcal{L}| = |\mathcal{H}''_2|$. Define bijection $\theta_1 : \mathcal{H}''_1 \longrightarrow \mathcal{H}'_1$ by $\theta_1[k(\ell,1)] := j(\ell,1)$ for all $\ell \in \mathcal{L}$. Let $\theta_2 : \mathcal{H}''_2 \longrightarrow \mathcal{H}'_2$ be any bijection. Let $\theta := \theta_1 \sqcup \theta_2 : \mathcal{I}_0 \longrightarrow \mathcal{I}_0$. Then θ is a (finitary) permutation of $\mathcal{I}_0$. For all $\ell \in \mathcal{L}$, we have

$$\begin{aligned} \theta(\mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}''_{\mathcal{I}_0})_{k(\ell,1)} &= (\mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}''_{\mathcal{I}_0})_{\theta(k(\ell,1))} = (\mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}''_{\mathcal{I}_0})_{j(\ell,1)} = (w^\ell_0 \rightsquigarrow w^\ell_{J_\ell}) \\ &\underset{(*)}{\succeq} (z^\ell_0 \rightsquigarrow z^\ell_{K_\ell}) = (\mathbf{z}_{\mathcal{I}_0} \rightsquigarrow \mathbf{z}''_{\mathcal{I}_0})_{k(\ell,1)}, \end{aligned} \qquad \text{(A26)}$$

where $(*)$ is by condition (JK3). Meanwhile, for any $\ell, \ell' \in \mathcal{L}$, and any $n \in [2 \ldots K_\ell]$, and $n' \in [2 \ldots J_{\ell'}]$, if $\theta(k(\ell,n)) = j(\ell',n')$, then

$$\begin{aligned} \theta(\mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}''_{\mathcal{I}_0})_{k(\ell,n)} &= (\mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}''_{\mathcal{I}_0})_{\theta(k(\ell,n))} = (\mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}''_{\mathcal{I}_0})_{j(\ell',n')} = (w^{\ell'}_{n'} \rightsquigarrow w^{\ell'}_{n'}) \\ &\underset{(*)}{\approx} (z^\ell_n \rightsquigarrow z^\ell_n) = (\mathbf{z}_{\mathcal{I}_0} \rightsquigarrow \mathbf{z}''_{\mathcal{I}_0})_{k(\ell,n)}, \end{aligned} \qquad \text{(A27)}$$

where $(*)$ is by (DP0). Combining (A26) and (A27) over all $\ell \in \mathcal{L}$ and $n \in [2 \ldots K_\ell]$, we conclude that $\theta(\mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}''_{\mathcal{I}_0}) \underset{\mathrm{par}}{\trianglerighteq} (\mathbf{z}_{\mathcal{I}_0} \rightsquigarrow \mathbf{z}''_{\mathcal{I}_0})$. Meanwhile, condition (NG1) says that $(\mathbf{x}^1_{\mathcal{I}_0^\complement} \rightsquigarrow \mathbf{x}^2_{\mathcal{I}_0^\complement}) \underset{\mathrm{par}}{\trianglerighteq} (\mathbf{y}^1_{\mathcal{I}_0^\complement} \rightsquigarrow \mathbf{y}^2_{\mathcal{I}_0^\complement})$. Thus,

$$\begin{pmatrix} \mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}''_{\mathcal{I}_0} \\ \mathbf{x}^1_{\mathcal{I}_0^\complement} \rightsquigarrow \mathbf{x}^2_{\mathcal{I}_0^\complement} \end{pmatrix} \ \underset{\overline{\mathrm{an}}}{\widehat{=}} \ \begin{pmatrix} \theta(\mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}''_{\mathcal{I}_0}) \\ \mathbf{x}^1_{\mathcal{I}_0^\complement} \rightsquigarrow \mathbf{x}^2_{\mathcal{I}_0^\complement} \end{pmatrix} \ \underset{\mathrm{par}}{\trianglerighteq} \ \begin{pmatrix} \mathbf{z}_{\mathcal{I}_0} \rightsquigarrow \mathbf{z}''_{\mathcal{I}_0} \\ \mathbf{y}^1_{\mathcal{I}_0^\complement} \rightsquigarrow \mathbf{y}^2_{\mathcal{I}_0^\complement} \end{pmatrix} \qquad \text{(A28)}$$

and thus, $\begin{pmatrix} \mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}''_{\mathcal{I}_0} \\ \mathbf{x}^1_{\mathcal{I}_0^\complement} \rightsquigarrow \mathbf{x}^2_{\mathcal{I}_0^\complement} \end{pmatrix} \ \widehat{=} \ \begin{pmatrix} \theta(\mathbf{w}_{\mathcal{I}_0} \rightsquigarrow \mathbf{w}''_{\mathcal{I}_0}) \\ \mathbf{x}^1_{\mathcal{I}_0^\complement} \rightsquigarrow \mathbf{x}^2_{\mathcal{I}_0^\complement} \end{pmatrix} \ \trianglerighteq \ \begin{pmatrix} \mathbf{z}_{\mathcal{I}_0} \rightsquigarrow \mathbf{z}''_{\mathcal{I}_0} \\ \mathbf{y}^1_{\mathcal{I}_0^\complement} \rightsquigarrow \mathbf{y}^2_{\mathcal{I}_0^\complement} \end{pmatrix}, \qquad \text{(A29)}$

because the SDP $(\trianglerighteq)$ satisfies axioms (WPar) and (Anon). Combining (A23), (A25) and (A29) through transitivity, we conclude that $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \trianglerighteq (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$, as desired.

Refinement. Now suppose $(\trianglerighteq)$ satisfies (SPar); we must show that $(\trianglerighteq)$ refines $(\underset{\mathrm{ng}}{\trianglerighteq})$.

Suppose $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{ng}}{\rhd} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$. In other words, $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{ng}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$, but $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{ng}}{\ntrianglelefteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$. Then there are three cases. *Either*

(i) $(\mathbf{x}^1_{\mathcal{I}_0^{\complement}} \rightsquigarrow \mathbf{x}^2_{\mathcal{I}_0^{\complement}}) \underset{\mathrm{par}}{\rhd} (\mathbf{y}^1_{\mathcal{I}_0^{\complement}} \rightsquigarrow \mathbf{y}^2_{\mathcal{I}_0^{\complement}})$, *or*

(ii) one of the "$\underset{\mathrm{par}}{\trianglerighteq}$" in formulae (A22) or (A24) is actually "$\underset{\mathrm{par}}{\rhd}$"; *or*

(iii) one of the preferences "$\underset{(*)}{\succeq}$" in formula (A26) is actually a "$\underset{(*)}{\succ}$".

(If none of these three cases are satisfied, then we would have $(\mathbf{x}^1_{\mathcal{I}_0^{\complement}} \rightsquigarrow \mathbf{x}^2_{\mathcal{I}_0^{\complement}}) \underset{\mathrm{par}}{\widehat{\widehat{=}}} (\mathbf{y}^1_{\mathcal{I}_0^{\complement}} \rightsquigarrow \mathbf{y}^2_{\mathcal{I}_0^{\complement}})$ and $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{J}_\ell, \mathcal{K}_\ell}{\widehat{\widehat{=}}} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$ for all $\ell \in \mathcal{L}$; which would imply that $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{ng}}{\widehat{\widehat{=}}} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$, contradicting our hypothesis.)

In case (ii), (SPar) yields a strict social preference in formulae (A23) or (A25). In case (i) or (iii), the "$\underset{\overline{\mathrm{par}}}{\trianglerighteq}$" in formula (A28) becomes a "$\underset{\mathrm{par}}{\rhd}$"; then (SPar) yields a strict social preference in formula (A29). Either way, when we combine (A23), (A25) and (A29) through transitivity, we conclude that $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \rhd (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$, as desired. $\square$

Proof of Theorem 4.2. We will show that $(\underset{\mathcal{V}}{\trianglerighteq}) \subseteq (\underset{\mathrm{ng}}{\trianglerighteq})$. Then Proposition 5.2(b) implies that every SDP is an extension of $(\underset{\mathcal{V}}{\trianglerighteq})$, which means that $(\underset{\mathcal{V}}{\trianglerighteq}) = (\underset{*}{\trianglerighteq})$. In particular, this will mean that $(\underset{\mathcal{V}}{\trianglerighteq}) \subseteq (\underset{\mathrm{u}}{\trianglerighteq})$ But Proposition 3.1(c) says that $(\underset{\mathcal{V}}{\trianglerighteq}) \supseteq (\underset{\mathrm{u}}{\trianglerighteq})$, so this will imply that $(\underset{\mathcal{V}}{\trianglerighteq}) = (\underset{\mathrm{u}}{\trianglerighteq})$.

Now, let $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, and suppose $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{V}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$. We must show that $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{ng}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$. For any $v \in \mathcal{V}$, there exists some $\mathcal{J}_v \subseteq \mathcal{I}$ satisfying (AU1) and (AU2) for v. Let $\mathcal{J} := \bigcup_{v \in \mathcal{V}} \mathcal{J}_v$; then $\mathcal{J} \subseteq \mathcal{I}$ satisfies (AU1) and (AU2) for all $v \in \mathcal{V}$. Note that $\mathcal{J}$ is finite, because either $\mathcal{V}$ is finite, or $\mathcal{I}$ itself is finite, by hypothesis.

Suppose $\mathcal{J} := \{j_1, j_2, \ldots, j_N\}$. Fix $w_0 \in \mathcal{X}$. Applying empathy repeatedly, find $w_1, w_2, \ldots, w_N \in \mathcal{X}$ such that

(JK1′) $(x^1_{j_n} \rightsquigarrow x^2_{j_n}) \approx (w_{n-1} \rightsquigarrow w_n)$ for all $n \in [1 \ldots N]$.

Fix $z_0 \in \mathcal{X}$. Applying empathy repeatedly, find $z_1, z_2, \ldots, z_N \in \mathcal{X}$ such that

(JK2′) $(z_{n-1} \rightsquigarrow z_n) \approx (y^1_{j_n} \rightsquigarrow y^2_{j_n})$, for all $n \in [1 \ldots N]$.

Claim 1: $(w_0 \rightsquigarrow w_N) \succeq (z_0 \rightsquigarrow z_N)$.

Proof: For any $v \in \mathcal{V}$, we have

$$
\begin{aligned}
v(w_N) - v(w_0) \quad &= \quad \sum_{n=1}^{N} \Big(v(w_n) - v(w_{n-1}) \Big) \quad \underset{(\diamond)}{=\!=} \quad \sum_{n=1}^{N} \Big(v(x_{j_n}^2) - v(x_{j_n}^1) \Big) \\
&= \quad \sum_{j \in \mathcal{J}} \Big(v(x_j^2) - v(x_j^1) \Big) \quad \underset{(*)}{\geq} \quad \sum_{j \in \mathcal{J}} \Big(v(y_j^2) - v(y_j^1) \Big) \\
&= \quad \sum_{n=1}^{N} \Big(v(y_{j_n}^2) - v(y_{j_n}^1) \Big) \quad \underset{(\dagger)}{=\!=} \quad \sum_{n=1}^{N} \Big(v(z_n) - v(z_{n-1}) \Big) \quad = \quad v(z_N) - v(z_0).
\end{aligned}
$$

Here, $(\diamond)$ is by (JK1$'$) and formula (2). Next, $(*)$ is by (AU1). Finally, $(\dagger)$ is by (JK2$'$) and formula (2).

Thus, $v(w_N) - v(w_0) \geq v(z_N) - v(z_0)$ for all $v \in \mathcal{V}$. Thus, the multiutility representation (3) yields $(w_0 \rightsquigarrow w_N) \succeq (z_0 \rightsquigarrow z_N)$. $\diamond$ Claim 1

Now, let $\mathcal{L} := \{1\}$ and set $\mathcal{I}_0 := \mathcal{J}_1 := \mathcal{K}_1 := \mathcal{J}$. Then observations (JK1$'$) and (JK2$'$) and Claim 1 together imply that $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{J}_1, \mathcal{K}_1}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$; this yields (NG2). Meanwhile, (AU2) implies that $(x_i^1 \rightsquigarrow x_i^2) \succeq (y_i^1 \rightsquigarrow y_i^2)$ for all $i \in \mathcal{I} \setminus \mathcal{I}_0$; this yields (NG1). Thus, $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{ng}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$. $\square$

The proof of Theorem 4.4 uses the next result.

Lemma A.4 *Let $\mathcal{V} \subseteq \mathcal{U}(\succeq)$. Then $(\underset{\mathcal{V}}{\trianglerighteq})$ satisfies* (NEHIC) *if and only if $\mathcal{V}$ yields a multiutility representation for $(\succeq)$.*

Proof: Let $w, w', z, z' \in \mathcal{X}$. Let $\mathbf{x} := \big(\begin{smallmatrix} w_j \\ \mathbf{o}_{-j} \end{smallmatrix} \big)$, $\mathbf{x}' := \big(\begin{smallmatrix} w'_j \\ \mathbf{o}_{-j} \end{smallmatrix} \big)$, $\mathbf{y} := \big(\begin{smallmatrix} z_j \\ \mathbf{o}_{-j} \end{smallmatrix} \big)$, and $\mathbf{y}' := \big(\begin{smallmatrix} z'_j \\ \mathbf{o}_{-j} \end{smallmatrix} \big)$. For all $v \in \mathcal{V}$, we have:

$$
\sum_{i \in \mathcal{I}} v(x_i') - \sum_{i \in \mathcal{I}} v(x_i) \; = \; v(w') - v(w) \quad \text{and} \quad \sum_{i \in \mathcal{I}} v(y_i') - \sum_{i \in \mathcal{I}} v(y_i) \; = \; v(z') - v(z). \tag{A30}
$$

"$\Longrightarrow$" Suppose $(\underset{\mathcal{V}}{\trianglerighteq})$ satisfies (NEHIC). Then:

$$
\begin{aligned}
&\Big(v(w') - v(w) \geq v(z') - v(z), \text{ for all } v \in \mathcal{V} \Big) \\
&\underset{(*)}{\Longleftrightarrow} \quad \Big(\sum_{i \in \mathcal{I}} v(x_i') - \sum_{i \in \mathcal{I}} v(x_i) \; \geq \; \sum_{i \in \mathcal{I}} v(y_i') - \sum_{i \in \mathcal{I}} v(y_i), \text{ for all } v \in \mathcal{V} \Big) \\
&\underset{(\dagger)}{\Longrightarrow} \quad \Big((\mathbf{x} \rightsquigarrow \mathbf{x}') \underset{\mathcal{V}}{\trianglerighteq} (\mathbf{y} \rightsquigarrow \mathbf{y}') \Big) \quad \underset{(\diamond)}{\Longleftrightarrow} \quad \Big((w \rightsquigarrow w') \succeq (z \rightsquigarrow z') \Big).
\end{aligned}
$$

Here, $(*)$ is by formula (A30), $(\dagger)$ is by definition of $(\underset{\mathcal{V}}{\trianglerighteq})$, and $(\diamond)$ is by (NEHIC).

This holds for all $w, w', z, z' \in \mathcal{X}$; Thus, $\mathcal{V}$ yields a multiutility representation for $(\succeq)$.

"$\Longleftarrow$" Suppose $\mathcal{V}$ yields a multiutility representation for $(\succeq)$. Then

$$\left((\mathbf{x} \rightsquigarrow \mathbf{x}') \underset{\mathcal{V}}{\trianglerighteq} (\mathbf{y} \rightsquigarrow \mathbf{y}') \right)$$

$$\underset{(\dagger)}{\Longleftrightarrow} \left(\sum_{i \in \mathcal{I}} v(x_i') - \sum_{i \in \mathcal{I}} v(x_i) \geq \sum_{i \in \mathcal{I}} v(y_i') - \sum_{i \in \mathcal{I}} v(y_i), \text{ for all } v \in \mathcal{V} \right)$$

$$\underset{(*)}{\Longleftrightarrow} \left(v(w') - v(w) \geq v(z') - v(z), \text{ for all } v \in \mathcal{V} \right) \underset{(\diamond)}{\Longleftrightarrow} \left((w \rightsquigarrow w') \succeq (z \rightsquigarrow z') \right).$$

Here, $(\dagger)$ is by definition of $(\underset{\mathcal{V}}{\trianglerighteq})$ (recall that $\mathcal{I}$ is finite). Meanwhile, $(*)$ is by formula (A30), and $(\diamond)$ is because $\mathcal{V}$ yields a multiutility representation for $(\succeq)$.

This holds for all $w, w', z, z' \in \mathcal{X}$. Thus, $(\underset{\mathcal{V}}{\trianglerighteq})$ satisfies (NEHIC). $\qquad\square$

Proof of Theorem 4.4. If $(\trianglerighteq)$ has a multiwelfare representation, and $\mathcal{I}$ is finite, then Theorem 3.2 says $(\trianglerighteq) = (\underset{\mathcal{V}}{\trianglerighteq})$ for some $\mathcal{V} \subseteq \mathcal{U}(\succeq)$. If $(\underset{\mathcal{V}}{\trianglerighteq})$ satisfies (NEHIC), then Lemma A.4 says $\mathcal{V}$ yields a multiutility representation for $(\succeq)$. Then Theorem 4.2 says that $(\underset{\mathcal{V}}{\trianglerighteq}) = (\underset{\mathrm{u}}{\trianglerighteq}) = (\underset{*}{\trianglerighteq})$. $\qquad\square$

Proof of Proposition 6.1. Set $\mathbf{x}^1 = \mathbf{y}^1 = \mathbf{y}^2 := \langle \mathbf{p}, \mathbf{r} \rangle$ and $\mathbf{x}^2 := \langle \mathbf{p}, \mathbf{s} \rangle$. Because of Proposition 5.2(b), it suffices to show that $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{ng}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$.

Let $\mathcal{I}_0 := \{1, 2\}$; then condition (NG1) holds because $r_i = s_i$ for all $i \in \mathcal{I} \setminus \{1, 2\}$. It remains to check (NG2). Let $\mathcal{L} := \{1\}$ and let $\mathcal{J}_1 := \mathcal{K}_1 := \mathcal{I}_0$; we will show that $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{J}_1, \mathcal{K}_1}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$, by verifying conditions (JK1)-(JK3).

Define $\alpha(1) := 2$ and $\alpha(2) := 1$. Define $w_0 := \langle p_2, r_2 \rangle$ ($= x_2^1$), $w_1 := \langle p_2, s_2 \rangle$ ($= x_2^2$) and $w_2 := \langle p_2, r_2' \rangle$. Then $(w_{\alpha(2)-1} \rightsquigarrow w_{\alpha(2)}) = (w_0 \rightsquigarrow w_1) = (x_2^1 \rightsquigarrow x_2^2)$. Meanwhile,

$$
\begin{aligned}
(w_{\alpha(1)-1} \rightsquigarrow w_{\alpha(1)}) &= (w_1 \rightsquigarrow w_2) = (\langle p_2, s_2 \rangle \rightsquigarrow \langle p_2, r_2' \rangle) \\
&\underset{(*)}{\prec} (\langle p_1, r_1 \rangle \rightsquigarrow \langle p_1, s_1 \rangle) = (x_1^1 \rightsquigarrow x_1^2),
\end{aligned}
$$

which verifies (JK1). Here $(*)$ is by formula (7), because $\dfrac{\beta(s_1) - \beta(r_1)}{\beta(r_2') - \beta(s_2)} > C$.

Now define $\beta(1) := 1$, $\beta(2) := 2$, and let $z_0 = z_1 = z_2 = \langle p_2, r_2 \rangle$ ($= y_2^1 = y_2^2$). Then

$$
\begin{aligned}
(z_{\beta(2)-1} \rightsquigarrow z_{\beta(2)}) &= (z_1 \rightsquigarrow z_2) = (\langle p_2, r_2 \rangle \rightsquigarrow \langle p_2, r_2 \rangle) = (y_2^1 \rightsquigarrow y_2^2), \\
\text{and} \quad (z_{\beta(1)-1} \rightsquigarrow z_{\beta(1)}) &= (z_0 \rightsquigarrow z_1) = (\langle p_2, r_2 \rangle \rightsquigarrow \langle p_2, r_2 \rangle) \\
&\underset{(*)}{\approx} (\langle p_1, r_1 \rangle \rightsquigarrow \langle p_1, r_1 \rangle) = (y_1^1 \rightsquigarrow y_1^2).
\end{aligned}
$$

which verifies (JK2). Here, $(*)$ is by (DP0). To check (JK3), observe that

$$(w_0 \rightsquigarrow w_2) = (\langle p_2, r_2 \rangle \rightsquigarrow \langle p_2, r_2' \rangle) \underset{(*)}{\succeq} (\langle p_2, r_2 \rangle \rightsquigarrow \langle p_2, r_2 \rangle) = (z_0 \rightsquigarrow z_2), \quad \text{(A31)}$$

where $(*)$ is because $r_2' \geq r_2$.

Thus, $(\langle \mathbf{p}, \mathbf{r} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{s} \rangle) \underset{\mathrm{ng}}{\trianglerighteq} (\langle \mathbf{p}, \mathbf{r} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{r} \rangle)$. Thus, if $(\succeq)$ is any social difference preorder then Proposition 5.2(b) says $(\langle \mathbf{p}, \mathbf{r} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{s} \rangle) \trianglerighteq (\langle \mathbf{p}, \mathbf{r} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{r} \rangle)$.

Furthermore, if $r_2' > r_2$, then the relation "$\underset{(*)}{\succeq}$" in eqn.(A31) becomes "$\underset{(*)}{\succ}$". Thus, $(\langle \mathbf{p}, \mathbf{r} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{s} \rangle) \underset{\mathrm{ng}}{\vartriangleright} (\langle \mathbf{p}, \mathbf{r} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{r} \rangle)$. Thus, if $(\trianglerighteq)$ satisfies (SPar), then Proposition 5.2(b) says that $(\langle \mathbf{p}, \mathbf{r} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{s} \rangle) \vartriangleright (\langle \mathbf{p}, \mathbf{r} \rangle \rightsquigarrow \langle \mathbf{p}, \mathbf{r} \rangle)$. $\qquad\square$

Let $(\underset{\mathrm{tng}}{\trianglerighteq})$ be the transitive closure of $(\underset{\mathrm{ng}}{\trianglerighteq})$. In other words: for any $\mathbf{a}^1, \mathbf{a}^2, \mathbf{z}^1, \mathbf{z}^2 \in \mathcal{X}^{\mathcal{I}}$, we have $(\mathbf{a}^1 \rightsquigarrow \mathbf{a}^2) \underset{\mathrm{tng}}{\trianglerighteq} (\mathbf{z}^1 \rightsquigarrow \mathbf{z}^2)$ if there exist $\mathbf{b}^1, \mathbf{b}^2, \mathbf{c}^1, \mathbf{c}^2, \ldots, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$ such that

$$(\mathbf{a}^1 \rightsquigarrow \mathbf{a}^2) \quad \underset{\mathrm{ng}}{\trianglerighteq} \quad (\mathbf{b}^1 \rightsquigarrow \mathbf{b}^2) \quad \underset{\mathrm{ng}}{\trianglerighteq} \quad (\mathbf{c}^1 \rightsquigarrow \mathbf{c}^2) \quad \underset{\mathrm{ng}}{\trianglerighteq} \quad \cdots \quad \underset{\mathrm{ng}}{\trianglerighteq} \quad (\mathbf{z}^1 \rightsquigarrow \mathbf{z}^2). \qquad (A32)$$

The proof of Theorem 5.1 uses the following result:

Lemma A.5 *Suppose* $(\succeq)$ *is empathic. For any* $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, *the following are equivalent:*

(a) $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{tng}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$.

(b) $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{ng}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$.

(c) *There exists a finite subset* $\mathcal{N}_0 \subseteq \mathcal{I}$ *such that, for any finite* $\mathcal{N} \subseteq \mathcal{I}$ *with* $\mathcal{N}_0 \subseteq \mathcal{N}$:

(NG1′) $(x_i^1 \rightsquigarrow x_i^2) \succeq (y_i^1 \rightsquigarrow y_i^2)$ *for all* $i \in \mathcal{I} \setminus \mathcal{N}$,

(NG2′) $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{N},\mathcal{N}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$.

Proof: "(b)$\Longrightarrow$(a)" is clear, because $(\underset{\mathrm{tng}}{\trianglerighteq})$ is the transitive closure of $(\underset{\mathrm{ng}}{\trianglerighteq})$.

"(c)$\Longrightarrow$(b)" If we set $\mathcal{I}_0 := \mathcal{N}$, set $\mathcal{L} := \{1\}$, and set $\mathcal{J}_1 := \mathcal{K}_1 := \mathcal{N}$; then (NG2′) implies (NG2), while (NG1′) implies (NG1). Thus, $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{ng}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$.

"(a)$\Longrightarrow$(c)" *Case 1.* Suppose $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{ng}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$ by some finite subset $\mathcal{I}_0 \subseteq \mathcal{I}$ and via the partitions $\{\mathcal{J}_\ell\}_{\ell \in \mathcal{L}'}$ and $\{\mathcal{K}_\ell\}_{\ell \in \mathcal{L}'}$ of $\mathcal{I}_0$, satisfying conditions (NG1) and (NG2) (for some indexing set $\mathcal{L}'$). Let $\mathcal{N}_0 := \mathcal{I}_0$.

Claim 1:   *For any finite* $\mathcal{N} \subseteq \mathcal{I}$, *if* $\mathcal{N}_0 \subseteq \mathcal{N}$, *then there exists a finite set* $\mathcal{L} \supseteq \mathcal{L}'$ *and partitions* $\{\mathcal{J}_\ell\}_{\ell \in \mathcal{L}}$ *and* $\{\mathcal{K}_\ell\}_{\ell \in \mathcal{L}}$ *of* $\mathcal{N}$ *which satisfies conditions* (NG1) *and* (NG2) *for* $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{ng}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$.

Proof: Let $\mathcal{M} := \mathcal{N} \setminus \mathcal{I}_0$, and suppose without loss of generality that $\mathcal{M}$ is disjoint from $\mathcal{L}'$. Let $\mathcal{L} := \mathcal{L}' \sqcup \mathcal{M}$. For all $\ell \in \mathcal{L}'$, define $\mathcal{J}_\ell$ and $\mathcal{K}_\ell$ as before. For all $m \in \mathcal{M}$, axiom (NG1) says that $(x_m^1 \rightsquigarrow x_m^2) \succeq (y_m^1 \rightsquigarrow y_m^2)$. Thus, if we introduce singleton partition elements $\mathcal{J}_m := \{m\}$ and $\mathcal{K}_m := \{m\}$, then we automatically have

30

$(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{J}_m, \mathcal{K}_m}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$. Do this for all $m \in \mathcal{M}$; then the collections $\{\mathcal{J}_\ell\}_{\ell \in \mathcal{L}}$ and $\{\mathcal{K}_\ell\}_{\ell \in \mathcal{L}}$ are partitions of $\mathcal{N}$, and satisfy (NG2).

Meanwhile, for all $i \in \mathcal{I} \backslash \mathcal{N}$, we have $(x_i^1 \rightsquigarrow x_i^2) \succeq (y_i^1 \rightsquigarrow y_i^2)$, because $(\mathcal{I} \backslash \mathcal{N}) \subseteq (\mathcal{I} \backslash \mathcal{I}_0)$, because $\mathcal{N} \supseteq \mathcal{I}_0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \diamondsuit$ `Claim 1`

(NG1$'$) follows from (NG1) in Claim 1. It remains to verify (NG2$'$). We must construct bijections $\alpha, \beta : \mathcal{N} \longrightarrow [1 \ldots N]$ and $w_0, \ldots, w_N, z_0, \ldots, z_N$ (where $N := |\mathcal{N}|$) such that:

(JK1$'$) $(x_n^1 \rightsquigarrow x_n^2) \succeq (w_{\alpha(n)-1} \rightsquigarrow w_{\alpha(n)})$ for all $n \in \mathcal{N}$;

(JK2$'$) $(z_{\beta(n)-1} \rightsquigarrow z_{\beta(n)}) \succeq (y_n^1 \rightsquigarrow y_n^2)$, for all $n \in \mathcal{N}$; and

(JK3$'$) $(w_0 \rightsquigarrow w_N) \succeq (z_0 \rightsquigarrow z_N)$.

Without loss of generality, suppose $\mathcal{L} = [1 \ldots L]$ for some $L \in \mathbb{N}$. For all $\ell \in [1 \ldots L]$, let $J_\ell' := |\mathcal{J}_\ell|$ and $K_\ell' := |\mathcal{K}_\ell|$. Let bijections $\alpha_\ell : \mathcal{J}_\ell \longrightarrow [1 \ldots J_\ell']$ and $\beta_\ell : \mathcal{K}_\ell \longrightarrow [1 \ldots K_\ell']$ and states $w_0^\ell, w_1^\ell, \ldots, w_{J_\ell'}^\ell, z_0^\ell, z_1^\ell, \ldots, z_{K_\ell'}^\ell \in \mathcal{X}$ satisfy the conditions (JK1)-(JK3) in the definition of $(\underset{\mathcal{J}_\ell, \mathcal{K}_\ell}{\trianglerighteq})$. Define $J_1 := K_1 := 0$, and for all $\ell \in [2 \ldots L+1]$, define $J_\ell := J_1' + \cdots + J_{\ell-1}'$ and $K_\ell := K_1' + \cdots + K_{\ell-1}'$. Then define $\alpha(j) := \alpha_\ell(j) + J_\ell$ for all $j \in \mathcal{J}_\ell$, and $\beta(k) := \beta_\ell(k) + K_\ell$ for all $k \in \mathcal{K}_\ell$. Then $\alpha, \beta : \mathcal{N} \longrightarrow [1 \ldots N]$ are bijections.

Note that $\alpha(j) = \alpha_1(j)$ for all $j \in \mathcal{J}_1$ (because $J_1 = 0$). Define $w_n := w_n^1$ for all $n \in [0 \ldots J_1']$. Then for all $j \in \mathcal{J}_1$, we have $(w_{\alpha_1(j)-1}^1 \rightsquigarrow w_{\alpha_1(j)}^1) = (w_{\alpha(j)-1} \rightsquigarrow w_{\alpha(j)})$. Thus, $(x_j^1 \rightsquigarrow x_j^2) \succeq (w_{\alpha(j)-1} \rightsquigarrow w_{\alpha(j)})$, as required by (JK1$'$).

Now, let $\ell \in [2 \ldots L]$, and suppose inductively that we have obtained $w_0, \ldots, w_{J_\ell} \in \mathcal{X}$ satisfying property (JK1$'$) for all $n \in \mathcal{J}_1 \sqcup \mathcal{J}_2 \sqcup \cdots \sqcup \mathcal{J}_{\ell-1}$. Let $j_1 \in \mathcal{J}_\ell$ be such that $\alpha_\ell(j_1) = 1$. Thus, $\alpha(j_1) = J_\ell + 1$. Empathy yields some $w_{J_\ell+1} \in \mathcal{X}$ such that $(w_0^\ell \rightsquigarrow w_1^\ell) \approx (w_{J_\ell} \rightsquigarrow w_{J_\ell+1})$. Thus, since $(x_{j_1}^1 \rightsquigarrow x_{j_1}^2) \succeq (w_{\alpha_\ell(j_1)-1}^\ell \rightsquigarrow w_{\alpha_\ell(j_1)}^\ell) = (w_0^\ell \rightsquigarrow w_1^\ell)$ (by (JK1)), and $(w_{J_\ell} \rightsquigarrow w_{J_\ell+1}) = (w_{\alpha(j_1)-1} \rightsquigarrow w_{\alpha(j_1)})$ (by definition of α), we obtain $(x_{j_1}^1 \rightsquigarrow x_{j_1}^2) \succeq (w_{\alpha(j_1)-1} \rightsquigarrow w_{\alpha(j_1)})$.

Next, let $j_2 \in \mathcal{J}_\ell$ be such that $\alpha_\ell(j_2) = 2$. Thus, $\alpha(j_2) = J_\ell + 2$. Empathy yields some $w_{J_\ell+2} \in \mathcal{X}$ such that $(w_1^\ell \rightsquigarrow w_2^\ell) \approx (w_{J_\ell+1} \rightsquigarrow w_{J_\ell+2})$. Thus, since $(x_{j_2}^1 \rightsquigarrow x_{j_2}^2) \succeq (w_{\alpha_\ell(j_2)-1}^\ell \rightsquigarrow w_{\alpha_\ell(j_2)}^\ell) = (w_1^\ell \rightsquigarrow w_2^\ell)$ (by (JK1)), and $(w_{J_\ell+1} \rightsquigarrow w_{J_\ell+2}) = (w_{\alpha(j_2)-1} \rightsquigarrow w_{\alpha(j_2)})$ (by definition of α), we obtain $(x_{j_2}^1 \rightsquigarrow x_{j_2}^2) \succeq (w_{\alpha(j_2)-1} \rightsquigarrow w_{\alpha(j_2)})$.

Proceeding in this way, repeated application of empathy yields a collection $\{w_{\alpha(j)}; \ j \in \mathcal{J}_\ell\} \subset \mathcal{X}$ such that for all $j \in \mathcal{J}_\ell$, we have $(w_{\alpha_\ell(j)-1}^\ell \rightsquigarrow w_{\alpha_\ell(j)}^\ell) \approx (w_{\alpha(j)-1} \rightsquigarrow w_{\alpha(j)})$, and hence $(x_j^1 \rightsquigarrow x_j^2) \succeq (w_{\alpha(j)-1} \rightsquigarrow w_{\alpha(j)})$, as required by (JK1$'$).

By induction, we obtain $w_0, \ldots, w_N \in \mathcal{X}$ satisfying property (JK1$'$). An identical construction yields $z_0, \ldots, z_N \in \mathcal{X}$ satisfying property (JK2$'$).

For all $\ell \in [1 \ldots L]$, every link in the chain $w_{J_\ell} \rightsquigarrow w_{1+J_\ell} \rightsquigarrow w_{2+J_\ell} \rightsquigarrow \cdots \rightsquigarrow w_{J_{\ell+1}}$ is indifferent to the corresponding link in the chain $w_0^\ell \rightsquigarrow w_1^\ell \rightsquigarrow w_2^\ell \rightsquigarrow \cdots \rightsquigarrow w_{J_\ell'}^\ell$ (by construction), so Lemma A.3(a) yields $(w_{J_\ell} \rightsquigarrow w_{J_{\ell+1}}) \approx (w_0^\ell \rightsquigarrow w_{J_\ell'}^\ell)$. Likewise,

$(z_0^\ell \rightsquigarrow z_{K_\ell'}^\ell) \approx (z_{K_\ell} \rightsquigarrow z_{K_{\ell+1}})$. Thus,

$$(w_{J_\ell} \rightsquigarrow w_{J_{\ell+1}}) \quad \approx \quad (w_0^\ell \rightsquigarrow w_{J_\ell'}^\ell) \quad \underset{(*)}{\succeq} \quad (z_0^\ell \rightsquigarrow z_{K_\ell'}^\ell) \quad \approx \quad (z_{K_\ell} \rightsquigarrow z_{K_{\ell+1}}),$$

where $(*)$ is by (JK3) for the relation $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{J}_\ell, \mathcal{K}_\ell}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$. Thus, applying Lemma A.3(a) to the chains $w_0 = w_{J_1} \rightsquigarrow w_{J_2} \rightsquigarrow w_{J_3} \rightsquigarrow \cdots \rightsquigarrow w_{J_{L+1}} = w_N$ and $z_0 = z_{K_1} \rightsquigarrow z_{K_2} \rightsquigarrow z_{K_3} \rightsquigarrow \cdots \rightsquigarrow z_{K_{L+1}} = z_N$ yields (JK3′).

We conclude that $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{N}, \mathcal{N}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$, thus verifying (NG2′).

Case 2. Let $\mathbf{a}^1, \mathbf{a}^2, \mathbf{z}^1, \mathbf{z}^2 \in \mathcal{X}$, and suppose $(\mathbf{a}^1 \rightsquigarrow \mathbf{a}^2) \underset{\mathrm{tng}}{\trianglerighteq} (\mathbf{z}^1 \rightsquigarrow \mathbf{z}^2)$ via a chain like (A32). Let $\mathcal{I}_{\mathrm{a,b}} \subseteq \mathcal{I}$ be a finite subset satisfying (NG1) and (NG2) for the relation $(\mathbf{a}^1 \rightsquigarrow \mathbf{a}^2) \underset{\mathrm{ng}}{\trianglerighteq} (\mathbf{b}^1 \rightsquigarrow \mathbf{b}^2)$. Likewise define $\mathcal{I}_{\mathrm{b,c}}, \mathcal{I}_{\mathrm{c,d}}, \ldots, \mathcal{I}_{\mathrm{y,z}} \subseteq \mathcal{I}$. Finally, let $\mathcal{N}_0 := \mathcal{I}_{\mathrm{a,b}} \cup \mathcal{I}_{\mathrm{b,c}} \cup \cdots \cup \mathcal{I}_{\mathrm{y,z}}$. Then $\mathcal{N}_0$ is a finite subset of $\mathcal{I}$, and for any finite $\mathcal{N} \subseteq \mathcal{I}$ which contains $\mathcal{N}_0$, Claim 1 shows that $\mathcal{N}$ satisfies (NG1) and (NG2) for *all* of the relations in the chain (A32).

Applying the argument of *Case 1* to each link in the chain (A32), we get elements $A_0', A_1', \ldots, A_N', B_0, B_1, \ldots, B_N, B_0', B_1', \ldots, B_N', C_0, C_1, \ldots, C_N, C_0', C_1', \ldots, C_N', \ldots\ldots,$ $Y_0, Y_1, \ldots, Y_N, Y_0', Y_1', \ldots, Y_N',$ and $Z_0, Z_1, \ldots, Z_N$ in $\mathcal{X}$, and bijections $\alpha, \beta, \beta', \gamma, \gamma', \ldots,$ $\upsilon, \upsilon',$ and ζ from $\mathcal{N}$ into $[1 \ldots N]$, such that, for all $n \in \mathcal{N}$, we have:

$$
\begin{aligned}
(a_n^1 \rightsquigarrow a_n^2) &\succeq (A_{\alpha(n)-1}' \rightsquigarrow A_{\alpha(n)}'); \\
(B_{\beta(n)-1} \rightsquigarrow B_{\beta(n)}) \succeq (b_n^1 \rightsquigarrow b_n^2) &\succeq (B_{\beta'(n)-1}' \rightsquigarrow B_{\beta'(n)}'); \\
(C_{\gamma(n)-1} \rightsquigarrow C_{\gamma(n)}) \succeq (c_n^1 \rightsquigarrow c_n^2) &\succeq (C_{\gamma'(n)-1}' \rightsquigarrow C_{\gamma'(n)}'); \\
\vdots \qquad\qquad \vdots \qquad\quad &\qquad \vdots \\
(Y_{\upsilon(n)-1} \rightsquigarrow Y_{\upsilon(n)}) \succeq (y_n^1 \rightsquigarrow y_n^2) &\succeq (Y_{\upsilon'(n)-1}' \rightsquigarrow Y_{\upsilon'(n)}'); \\
(Z_{\zeta(n)-1} \rightsquigarrow Z_{\zeta(n)}) &\succeq (z_n^1 \rightsquigarrow z_n^2).
\end{aligned}
\tag{A33}
$$

The left-hand relations in (A33) are by (JK2′); the right-hand relations in (A33) are by (JK1′). Meanwhile, (JK3′) yields

$$
\begin{aligned}
(A_0' \rightsquigarrow A_N') &\succeq (B_0 \rightsquigarrow B_N); \\
(B_0' \rightsquigarrow B_N') &\succeq (C_0 \rightsquigarrow C_N); \\
\vdots \quad \vdots \quad &\vdots \\
\text{and} \quad (Y_0' \rightsquigarrow Y_N') &\succeq (Z_0 \rightsquigarrow Z_N).
\end{aligned}
\tag{A34}
$$

The transitivity of $(\succeq)$ collapses all but the first and last rows of (A33) into

$$
\begin{aligned}
(B_{\beta(n)-1} \rightsquigarrow B_{\beta(n)}) &\succeq (B_{\beta'(n)-1}' \rightsquigarrow B_{\beta'(n)}'), && \text{for all } n \in \mathcal{N}; \\
(C_{\gamma(n)-1} \rightsquigarrow C_{\gamma(n)}) &\succeq (C_{\gamma'(n)-1}' \rightsquigarrow C_{\gamma'(n)}'), && \text{for all } n \in \mathcal{N}; \\
\vdots \quad \vdots \quad &\vdots \\
\text{and} \quad (Y_{\upsilon(n)-1} \rightsquigarrow Y_{\upsilon(n)}) &\succeq (Y_{\upsilon'(n)-1}' \rightsquigarrow Y_{\upsilon'(n)}'), && \text{for all } n \in \mathcal{N}.
\end{aligned}
\tag{A35}
$$

Define $\beta'' := \beta' \circ \beta^{-1}$, $\gamma'' := \gamma' \circ \gamma^{-1}$, ..., $\upsilon'' := \upsilon' \circ \upsilon^{-1}$. Then $\beta'', \gamma'', \ldots, \upsilon''$ are all permutations of $[1 \ldots N]$. Applying a change of variables to each line in (A35) yields

$$
\begin{array}{rcll}
(B_{m-1} \rightsquigarrow B_m) & \succeq & (B'_{\beta''(m)-1} \rightsquigarrow B'_{\beta''(m)}), & \text{for all } m \in [1 \ldots N]; \\
(C_{m-1} \rightsquigarrow C_m) & \succeq & (C'_{\gamma''(m)-1} \rightsquigarrow C'_{\gamma''(m)}), & \text{for all } m \in [1 \ldots N]; \\
& \vdots \quad \vdots \quad \vdots & & \\
\text{and} \quad (Y_{m-1} \rightsquigarrow Y_m) & \succeq & (Y'_{\upsilon''(m)-1} \rightsquigarrow Y'_{\upsilon''(m)}), & \text{for all } m \in [1 \ldots N].
\end{array}
\tag{A36}
$$

Applying Lemma A.3(c) the relations in (A36) yields

$$
\begin{array}{rcl}
(B_0 \rightsquigarrow B_N) & \succeq & (B'_0 \rightsquigarrow B'_N); \\
(C_0 \rightsquigarrow C_N) & \succeq & (C'_0 \rightsquigarrow C'_N); \\
& \vdots \quad \vdots \quad \vdots & \\
\text{and} \quad (Y_0 \rightsquigarrow Y_N) & \succeq & (Y'_0 \rightsquigarrow Y'_N),
\end{array}
\tag{A37}
$$

Now, consider $A'_0, A'_1, \ldots, A'_N$ and $Z_0, Z_1, \ldots, Z_N$. The first row of (A33) says that $A'_0, A'_1, \ldots, A'_N$ satisfies (JK1′) with respect to $(\mathbf{a}^1 \rightsquigarrow \mathbf{a}^2)$. The last row of (A33) says that $Z_0, Z_1, \ldots, Z_N$ satisfies (JK2′) with respect to $(\mathbf{z}^1 \rightsquigarrow \mathbf{z}^2)$. Finally, combining (A34) and (A37) with transitivity, we get $(A'_0 \rightsquigarrow A'_N) \succeq (Z_0 \rightsquigarrow Z_N)$, thus verifying (JK3′).

We conclude that $(\mathbf{a}^1 \rightsquigarrow \mathbf{a}^2) \underset{\mathcal{N},\mathcal{N}}{\trianglerighteq} (\mathbf{z}^1 \rightsquigarrow \mathbf{z}^2)$, which verifies (NG2′). Meanwhile, for all $i \in \mathcal{I} \setminus \mathcal{N}$, we have $i \in \mathcal{I} \setminus \mathcal{I}_{a,b}$, $i \in \mathcal{I} \setminus \mathcal{I}_{b,c}$, ... and $i \in \mathcal{I} \setminus \mathcal{I}_{y,z}$. Thus, we deduce that $(a_i^1 \rightsquigarrow a_i^2) \succeq (z_i^1 \rightsquigarrow z_i^2)$, by applying transitivity and invoking the hypothesis (NG1) for each link in the chain (A32). This verifies (NG1′). $\qquad \square$

Proof of Theorem 5.1. $(\underset{\mathrm{ng}}{\trianglerighteq}) \subseteq (\underset{*}{\trianglerighteq})$ by Proposition 5.2(b). We must show that $(\underset{\mathrm{ng}}{\trianglerighteq}) \supseteq (\underset{*}{\trianglerighteq})$. It is equivalent to show that $(\underset{*}{\trianglerighteq}) \subseteq (\underset{\mathrm{tng}}{\trianglerighteq})$, because Lemma A.5 says $(\underset{\mathrm{ng}}{\trianglerighteq}) = (\underset{\mathrm{tng}}{\trianglerighteq})$, because $(\succeq)$ is empathic. To show that $(\underset{*}{\trianglerighteq}) \subseteq (\underset{\mathrm{tng}}{\trianglerighteq})$, it suffices to show that $(\underset{\mathrm{tng}}{\trianglerighteq})$ is an $(\succeq)$-SDP on $\mathcal{X}^{\mathcal{I}}$, and then apply Proposition 4.1(a).

Now, $(\underset{\mathrm{tng}}{\trianglerighteq})$ is transitive by definition. Also, $(\underset{\mathrm{tng}}{\trianglerighteq})$ is reflexive and satisfies axioms (DP0$^\trianglerighteq$), (DP1$^\trianglerighteq$), (WPar) and (Anon) because $(\underset{\mathrm{ng}}{\trianglerighteq})$ has these properties, by Proposition 5.2(a). It remains to verify axioms (DP2$^\trianglerighteq$), (DP3$^\trianglerighteq$), and (SPar).

(SPar) Suppose that

$$
\forall \, i \in \mathcal{I}, \quad (x_i^1 \rightsquigarrow x_i^2) \preceq (y_i^1 \rightsquigarrow y_i^2), \quad \text{and} \quad \exists \, i_0 \in \mathcal{I} : \quad (x_{i_0}^1 \rightsquigarrow x_{i_0}^2) \prec (y_{i_0}^1 \rightsquigarrow y_{i_0}^2). \tag{A38}
$$

We must show that $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{tng}}{\vartriangleleft} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$. Since $(\underset{\mathrm{tng}}{\trianglerighteq})$ satisfies (WPar), we have $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{tng}}{\trianglelefteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$. We must show $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{tng}}{\ntrianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$.

By contradiction, suppose $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{tng}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$. Then Lemma A.5 yields some finite subset $\mathcal{N} \subseteq \mathcal{I}$, with $i_0 \in \mathcal{N}$, satisfying (NG1′) and (NG2′). Let $N := |\mathcal{N}|$. Then (NG2′) yields some $\widetilde{w}_0, \widetilde{w}_1, \ldots, \widetilde{w}_N, \widetilde{z}_0, \widetilde{z}_1, \ldots, \widetilde{z}_N \in \mathcal{X}$ and bijections $\alpha, \beta : \mathcal{N} \longrightarrow [1 \ldots N]$ such that

($\widetilde{\text{JK}}$1) $(x_n^1 \rightsquigarrow x_n^2) \succeq (\widetilde{w}_{\alpha(n)-1} \rightsquigarrow \widetilde{w}_{\alpha(n)})$ for all $n \in \mathcal{N}$;

($\widetilde{\text{JK}}$2) $(\widetilde{z}_{\beta(n)-1} \rightsquigarrow \widetilde{z}_{\beta(n)}) \succeq (y_n^1 \rightsquigarrow y_n^2)$, for all $n \in \mathcal{N}$; and

($\widetilde{\text{JK}}$3) $(\widetilde{w}_0 \rightsquigarrow \widetilde{w}_N) \succeq (\widetilde{z}_0 \rightsquigarrow \widetilde{z}_N)$.

Without loss of generality, assume that $\mathcal{N} = [1 \dots N]$. Fix $w_0, z_0 \in \mathcal{X}$ arbitrarily. Repeated application of empathy yields $w_1, w_2, \dots, w_N, z_1, z_2, \dots, z_N \in \mathcal{X}$ such that

(JK1′) $(x_n^1 \rightsquigarrow x_n^2) \approx (w_{n-1} \rightsquigarrow w_n)$ for all $n \in [1 \dots N]$; and

(JK2′) $(z_{n-1} \rightsquigarrow z_n) \approx (y_n^1 \rightsquigarrow y_n^2)$, for all $n \in [1 \dots N]$.

Thus, transitivity and hypothesis (A38) imply that $(w_{n-1} \rightsquigarrow w_n) \preceq (z_{n-1} \rightsquigarrow z_n)$ for all $n \in [1 \dots N]$, while $(w_{n-1} \rightsquigarrow w_n) \prec (z_{n-1} \rightsquigarrow z_n)$ for some $n \in [1 \dots N]$ (because $i_0 \in \mathcal{N}$). Thus, applying Lemma A.3(b) to the chains $w_0 \rightsquigarrow w_1 \rightsquigarrow \cdots \rightsquigarrow w_N$ and $z_0 \rightsquigarrow z_1 \rightsquigarrow \cdots \rightsquigarrow z_N$, we get

(JK3′) $(w_0 \rightsquigarrow w_N) \prec (z_0 \rightsquigarrow z_N)$.

Meanwhile, combining (JK1′) and ($\widetilde{\text{JK}}$1) yields

($\widetilde{\text{JK}}$1′) $(w_{n-1} \rightsquigarrow w_n) \succeq (\widetilde{w}_{\alpha(n)-1} \rightsquigarrow \widetilde{w}_{\alpha(n)})$ for all $n \in [1 \dots N]$.

Likewise, combining (JK2′) and ($\widetilde{\text{JK}}$2) yields

($\widetilde{\text{JK}}$2′) $(\widetilde{z}_{\beta(n)-1} \rightsquigarrow \widetilde{z}_{\beta(n)}) \succeq (z_{n-1} \rightsquigarrow z_n)$, for all $n \in [1 \dots N]$.

Thus, we have

$$(w_0 \rightsquigarrow w_N) \underset{(*)}{\succeq} (\widetilde{w}_0 \rightsquigarrow \widetilde{w}_N) \underset{(\diamond)}{\succeq} (\widetilde{z}_0 \rightsquigarrow \widetilde{z}_N) \underset{(\dagger)}{\succeq} (z_0 \rightsquigarrow z_N). \tag{A39}$$

Here, $(*)$ is by applying Lemma A.3(c) to the relations in ($\widetilde{\text{JK}}$1′), and $(\dagger)$ is by applying Lemma A.3(c) to the relations in ($\widetilde{\text{JK}}$2′). Finally $(\diamond)$ is by ($\widetilde{\text{JK}}$3).

Equation (A39) implies that $(w_0 \rightsquigarrow w_N) \succeq (z_0 \rightsquigarrow z_N)$. But this contradicts (JK3′). By contradiction, $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\text{tng}}{\not\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$, as desired.

(DP2$^{\trianglerighteq}$) Let $\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^0, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, and suppose $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \underset{\text{tng}}{\trianglerighteq} (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^1)$ and $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\text{tng}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$. We must check that $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \underset{\text{tng}}{\trianglerighteq} (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2)$.

Since $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \underset{\text{tng}}{\trianglerighteq} (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^1)$, Lemma A.5 yields some finite subset $\mathcal{N}_0 \subseteq \mathcal{I}$ such that any superset of $\mathcal{N}_0$ satisfies (NG1′) and (NG2′) for $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1)$ and $(\mathbf{y}^0 \rightsquigarrow \mathbf{y}^1)$. Likewise, Lemma A.5 yields some finite subset $\mathcal{N}_1 \subseteq \mathcal{I}$ such that any finite superset of $\mathcal{N}_1$ satisfies (NG1′) and (NG2′) for $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2)$ and $(\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$. Thus, if $\mathcal{N} := \mathcal{N}_0 \cup \mathcal{N}_1$, then $\mathcal{N}$ satisfies (NG1′) and (NG2′) for both. This means that

(NG1′) $(x_i^0 \rightsquigarrow x_i^1) \succeq (y_i^0 \rightsquigarrow y_i^1)$ for all $i \in \mathcal{I} \setminus \mathcal{N}$; and

(NG1″) $(x_i^1 \rightsquigarrow x_i^2) \succeq (y_i^1 \rightsquigarrow y_i^2)$ for all $i \in \mathcal{I} \setminus \mathcal{N}$.

Furthermore, there exist elements $w_0', w_1', \ldots, w_N', z_0', z_1', \ldots, z_N' \in \mathcal{X}$ (where $N := |\mathcal{N}|$) and bijections $\alpha', \beta' : \mathcal{N} \longrightarrow [1 \ldots N]$ such that

(JK1′) $(x_n^0 \rightsquigarrow x_n^1) \succeq (w_{\alpha'(n)-1}' \rightsquigarrow w_{\alpha'(n)}')$, for all $n \in \mathcal{N}$;

(JK2′) $(z_{\beta'(n)-1}' \rightsquigarrow z_{\beta'(n)}') \succeq (y_n^0 \rightsquigarrow y_n^1)$, for all $n \in \mathcal{N}$; and

(JK3′) $(w_0' \rightsquigarrow w_N') \succeq (z_0' \rightsquigarrow z_N')$.

Likewise, there exist elements $w_0'', w_1'', \ldots, w_N'', z_0'', z_1'', \ldots, z_N'' \in \mathcal{X}$ and bijections $\alpha'', \beta'' : \mathcal{N} \longrightarrow [1 \ldots N]$ such that

(JK1″) $(x_n^1 \rightsquigarrow x_n^2) \succeq (w_{\alpha''(n)-1}'' \rightsquigarrow w_{\alpha''(n)}'')$, for all $n \in \mathcal{N}$;

(JK2″) $(z_{\beta''(n)-1}'' \rightsquigarrow z_{\beta''(n)}'') \succeq (y_n^1 \rightsquigarrow y_n^2)$, for all $n \in \mathcal{N}$; and

(JK3″) $(w_0'' \rightsquigarrow w_N'') \succeq (z_0'' \rightsquigarrow z_N'')$.

Finally, empathy means that we can assume without loss of generality that $w_0'' = w_N'$ and $z_0'' = z_N'$, so (JK3″) can be rewritten as $(w_N' \rightsquigarrow w_N'') \succeq (z_N' \rightsquigarrow z_N'')$. We can use (DP2) to combine this with (JK3′) and obtain:

$$(w_0' \rightsquigarrow w_N'') \quad \succeq \quad (z_0' \rightsquigarrow z_N''). \tag{A40}$$

Assume without loss of generality that $\mathcal{N} := [1 \ldots N]$. Let $w_0 := w_{\alpha'(1)-1}'$, and let $w_{\frac{1}{2}} := w_{\alpha'(1)}'$. Empathy yields some $w_1 \in \mathcal{X}$ such that $(w_{\frac{1}{2}} \rightsquigarrow w_1) \approx (w_{\alpha''(1)-1}'' \rightsquigarrow w_{\alpha''(1)}'')$. Then empathy yields some $w_{\frac{3}{2}} \in \mathcal{X}$ such that $(w_1 \rightsquigarrow w_{\frac{3}{2}}) \approx (w_{\alpha'(2)-1}' \rightsquigarrow w_{\alpha'(2)}')$, and some $w_2 \in \mathcal{X}$ such that $(w_{\frac{3}{2}} \rightsquigarrow w_2) \approx (w_{\alpha''(2)-1}'' \rightsquigarrow w_{\alpha''(2)}'')$.

Inductively, let $n \in [1 \ldots N]$, and suppose we have constructed $w_0, w_{\frac{1}{2}}, w_1, w_{\frac{3}{2}}, \ldots, w_{n-1} \in \mathcal{X}$. Empathy yields some $w_{n-\frac{1}{2}} \in \mathcal{X}$ such that $(w_{n-1} \rightsquigarrow w_{n-\frac{1}{2}}) \approx (w_{\alpha'(n)-1}' \rightsquigarrow w_{\alpha'(n)}')$; then empathy yields some $w_n \in \mathcal{X}$ such that $(w_{n-\frac{1}{2}} \rightsquigarrow w_n) \approx (w_{\alpha''(n)-1}'' \rightsquigarrow w_{\alpha''(n)}'')$. Thus, for all $n \in [1 \ldots N]$, we have:

$$\left. \begin{array}{rcccl}
(x_n^0 \rightsquigarrow x_n^1) & \underset{\text{(JK1′)}}{\succeq} & (w_{\alpha'(n)-1}' \rightsquigarrow w_{\alpha'(n)}') & \approx & (w_{n-1} \rightsquigarrow w_{n-\frac{1}{2}}) \\
\text{and} \quad (x_n^1 \rightsquigarrow x_n^2) & \underset{\text{(JK1″)}}{\succeq} & (w_{\alpha''(n)-1}'' \rightsquigarrow w_{\alpha''(n)}'') & \approx & (w_{n-\frac{1}{2}} \rightsquigarrow w_n), \\
\text{and thus,} \quad (x_n^0 \rightsquigarrow x_n^2) & \succeq & (w_{n-1} \rightsquigarrow w_n), & \text{by (DP2).} &
\end{array} \right\} \tag{A41}$$

Thus, if we let $\alpha : \mathcal{N} \longrightarrow [1 \ldots N]$ be the identity map, then $w_0, w_1, w_2, \ldots, w_N$ verify property (JK1) for transition $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2)$

In a similar way, empathy yields $z_0, z_{\frac{1}{2}}, z_1, z_{\frac{3}{2}}, \ldots, z_N \in \mathcal{X}$ such that $(z_{n-1} \rightsquigarrow z_{n-\frac{1}{2}}) \approx (z_{\beta'(n)-1}' \rightsquigarrow z_{\beta'(n)}')$, and $(z_{n-\frac{1}{2}} \rightsquigarrow z_n) \approx (z_{\beta''(n)-1}'' \rightsquigarrow z_{\beta''(n)}'')$ for all $n \in [1 \ldots N]$. Thus, for all $n \in [1 \ldots N]$, we have:

$$\left. \begin{array}{rcccl}
(y_n^0 \rightsquigarrow y_n^1) & \underset{\text{(JK2′)}}{\preceq} & (z_{\beta'(n)-1}' \rightsquigarrow z_{\beta'(n)}') & \approx & (z_{n-1} \rightsquigarrow z_{n-\frac{1}{2}}) \\
\text{and} \quad (y_n^1 \rightsquigarrow y_n^2) & \underset{\text{(JK2″)}}{\preceq} & (z_{\beta''(n)-1}'' \rightsquigarrow z_{\beta''(n)}'') & \approx & (z_{n-\frac{1}{2}} \rightsquigarrow z_n), \\
\text{and thus,} \quad (y_n^0 \rightsquigarrow y_n^2) & \preceq & (z_{n-1} \rightsquigarrow z_n), & \text{by (DP2).} &
\end{array} \right\} \tag{A42}$$

Thus, if we let $\beta : \mathcal{N} \longrightarrow [1 \dots N]$ be the identity map, then $z_0, z_1, z_2, \dots, z_N$ verify property (JK2) for transition $(\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2)$.

Finally, every step in the chain

$$w_0 \rightsquigarrow w_{\frac{1}{2}} \rightsquigarrow w_1 \rightsquigarrow w_{\frac{3}{2}} \rightsquigarrow \cdots \rightsquigarrow w_N$$

is indifferent (by construction) to a specific step in the chain

$$w_0' \rightsquigarrow w_1' \rightsquigarrow w_2' \rightsquigarrow \cdots \rightsquigarrow w_N' \;=\; w_0'' \rightsquigarrow w_1'' \rightsquigarrow w_2'' \rightsquigarrow \cdots \rightsquigarrow w_N'',$$

thus, applying Lemma A.3(c) in both directions implies that $(w_0 \rightsquigarrow w_N) \approx (w_0' \rightsquigarrow w_N'')$. Likewise, $(z_0 \rightsquigarrow z_N) \approx (z_0' \rightsquigarrow z_N'')$. Combining these observations with equation (A40), we have $(w_0 \rightsquigarrow w_N) \succeq (z_0 \rightsquigarrow z_N)$, thereby verifying (JK3).

Setting $\mathcal{L} := \{1\}$, $\mathcal{J}_1 := \mathcal{K}_1 := \mathcal{N}$ and $\alpha := \beta :=$ identity map, we have $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{J}_1, \mathcal{K}_1}{\succeq} (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2)$, verifying (NG2). Meanwhile, (NG1) follows from observations (NG1$'$) and (NG1$''$), and axiom (DP2). Thus $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \underset{\overline{\text{ng}}}{\succeq} (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2)$, and thus $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \underset{\text{tng}}{\succeq} (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2)$.

(DP3$^{\trianglerighteq}$) Let $\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^0, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, and suppose $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1) \underset{\text{tng}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$ and $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\text{tng}}{\trianglerighteq} (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^1)$. We must check that $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2) \underset{\text{tng}}{\trianglerighteq} (\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2)$.

The argument is very similar to the proof of (DP2$^{\trianglerighteq}$); we will simply point out the differences. Formulae (JK1$'$), (JK3$'$), (JK1$''$) and (JK3$''$) remain exactly as before, but formulae (NG1$'$), (NG1$''$), (JK2$'$), and (JK2$''$) now become:

(NG1$'$) $(x_i^0 \rightsquigarrow x_i^1) \succeq (y_i^1 \rightsquigarrow y_i^2)$ for all $i \in \mathcal{I} \setminus \mathcal{N}$;

(NG1$''$) $(x_i^1 \rightsquigarrow x_i^2) \succeq (y_i^0 \rightsquigarrow y_i^1)$ for all $i \in \mathcal{I} \setminus \mathcal{N}$;

(JK2$'$) $(z_{\beta'(n)-1}' \rightsquigarrow z_{\beta'(n)}') \succeq (y_n^1 \rightsquigarrow y_n^2)$, for all $n \in \mathcal{N}$; and

(JK2$''$) $(z_{\beta''(n)-1}'' \rightsquigarrow z_{\beta''(n)}'') \succeq (y_n^0 \rightsquigarrow y_n^1)$, for all $n \in \mathcal{N}$.

We use empathy to construct $w_0, w_{\frac{1}{2}}, w_1, w_{\frac{3}{2}}, \dots, w_N \in \mathcal{X}$ verifying statements (A41) exactly as before. However, the construction of $z_0, z_{\frac{1}{2}}, z_1, z_{\frac{3}{2}}, \dots, z_N \in \mathcal{X}$ now changes so that the statements in (A42) become:

$$(y_n^0 \rightsquigarrow y_n^1) \underset{(\text{JK2}'')}{\preceq} (z_{\beta''(n)-1}'' \rightsquigarrow z_{\beta''(n)}'') \approx (z_{n-1} \rightsquigarrow z_{n-\frac{1}{2}})$$

$$\text{and} \quad (y_n^1 \rightsquigarrow y_n^2) \underset{(\text{JK2}')}{\preceq} (z_{\beta'(n)-1}' \rightsquigarrow z_{\beta'(n)}') \approx (z_{n-\frac{1}{2}} \rightsquigarrow z_n),$$

$$\text{and thus,} \quad (y_n^0 \rightsquigarrow y_n^2) \quad \preceq \quad (z_{n-1} \rightsquigarrow z_n), \qquad \text{by (DP2)}.$$

Thus, if we let $\alpha, \beta : \mathcal{N} \longrightarrow [1 \dots N]$ be the identity maps, then $w_0, w_1, w_2, \dots, w_N$ verify property (JK1) for transition $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2)$, while $z_0, z_1, z_2, \dots, z_N$ verify property (JK2) for transition $(\mathbf{y}^0 \rightsquigarrow \mathbf{y}^2)$. The rest of the proof proceeds as before. $\qquad \square$

36

Let $(\mathcal{R}, +)$ be an abelian group. Recall that a subset $\mathcal{C} \subseteq \mathcal{R}$ is a *coset* if there is some subgroup $\mathcal{S} \subseteq \mathcal{R}$ and some $r \in \mathcal{R}$ such that $\mathcal{C} := r + \mathcal{S} := \{r + s; \ s \in \mathcal{S}\}$.

Lemma A.6 *Let $(\mathcal{R}, +)$ be an abelian group, and let $\mathcal{C} \subseteq \mathcal{R}$. Then $\mathcal{C}$ is a coset if and only if, for all $c_1, c_2, c_3 \in \mathcal{C}$, we have $(c_1 - c_2 + c_3) \in \mathcal{C}$.*

Proof: "$\Longrightarrow$" Suppose $\mathcal{C} := r + \mathcal{S}$ for some subgroup $\mathcal{S} \subseteq \mathcal{R}$ and some $r \in \mathcal{R}$. Then for any $c_1, c_2, c_3 \in \mathcal{C}$, there exist $s_1, s_2, s_3 \in \mathcal{S}$ such that $c_1 = r + s_1$, $c_2 = r + s_2$, and $c_3 = r + s_3$. Thus, $c_1 - c_2 + c_3 = (r + s_1) - (r + s_2) + (r + s_3) = r + (s_1 - s_2 + s_3) = r + s$, where $s := s_1 - s_2 + s_3$ is an element of $\mathcal{S}$ (because $\mathcal{S}$ is a group). Thus, $(c_1 - c_2 + c_3) \in r + \mathcal{S} = \mathcal{C}$, as desired.

"$\Longleftarrow$" Fix $r \in \mathcal{C}$, and define $\mathcal{S} := \{c - r; \ c \in \mathcal{C}\}$.

Claim 1: $\mathcal{S}$ *is a subgroup of $\mathcal{R}$.*

Proof: Let $s_1, s_2 \in \mathcal{S}$; we must show that $s_1 - s_2 \in \mathcal{S}$. By definition, there exist $c_1, c_2 \in \mathcal{C}$ such that $s_1 = c_1 - r$ and $s_2 = c_2 - r$. But then $s_1 - s_2 = (c_1 - r) - (c_2 - r) = c_1 - c_2$. Now, if $c := c_1 - c_2 + r$, then $c \in \mathcal{C}$ by hypothesis. Thus, $c_1 - c_2 = c - r \in \mathcal{S}$, as desired. $\diamond$ Claim 1

Claim 2: $\mathcal{C} = r + \mathcal{S}$.

Proof: "$\subseteq$" Let $c \in \mathcal{C}$; we must show that $c \in r + \mathcal{S}$. If $s := c - r$, then $s \in \mathcal{S}$. Thus, $c = r + s \in r + \mathcal{S}$, as desired.

"$\supseteq$" Let $s \in \mathcal{S}$; we must show that $r + s \in \mathcal{C}$. By definition, there exists $c \in \mathcal{C}$ such that $s = c - r$. But then $r + s = c$. $\diamond$ Claim 2

It follows that $\mathcal{C}$ is a coset in $\mathcal{R}$. $\square$

Proof of Proposition 7.1. Let $\{\mathcal{R}_j\}_{j \in \mathcal{J}}$ and $\mathcal{U}' := \{u_j\}_{j \in \mathcal{J}}$ and $\mathbf{u} : \mathcal{X} \longrightarrow \mathcal{R}$ be defined as prior to Proposition 7.1.

"$\Longrightarrow$" Let $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3 \in \mathbf{u}(\mathcal{X})$. By Lemma A.6, it suffices to show that $\mathbf{c}_1 - \mathbf{c}_2 + \mathbf{c}_3 \in \mathbf{u}(\mathcal{X})$ also. By definition, there exist $x_1, x_2, x_3 \in \mathcal{X}$ such that $\mathbf{c}_n = \mathbf{u}(x_n)$ for $n \in \{1, 2, 3\}$. Since $(\succeq)$ is empathic, there exists some x_4 such that $(x_1 \rightsquigarrow x_2) \approx (x_3 \rightsquigarrow x_4)$. But then the multiutility representation (3) implies that $u(x_2) - u(x_1) = u(x_4) - u(x_3)$ for all $u \in \mathcal{U}'$. In other words, $\mathbf{u}(x_2) - \mathbf{u}(x_1) = \mathbf{u}(x_4) - \mathbf{u}(x_3)$. Thus, if $\mathbf{c}_4 := \mathbf{u}(x_4)$, then we have $\mathbf{c}_4 \in \mathbf{u}(\mathcal{X})$, and $\mathbf{c}_4 = \mathbf{c}_1 - \mathbf{c}_2 + \mathbf{c}_3$, as desired.

"$\Longleftarrow$" Let $x_1, x_2, x_3 \in \mathcal{X}$; we must construct $x_4 \in \mathcal{X}$ such that $(x_1 \rightsquigarrow x_2) \approx (x_3 \rightsquigarrow x_4)$. Let $\mathbf{c}_n = \mathbf{u}(x_n)$ for $n \in \{1, 2, 3\}$, and let $\mathbf{c}_4 := \mathbf{c}_1 - \mathbf{c}_2 + \mathbf{c}_3$. Now, $\mathbf{c}_n \in \mathbf{u}(\mathcal{X})$ for $n \in \{1, 2, 3\}$, and $\mathbf{u}(\mathcal{X})$ is a coset by hypothesis, so Lemma A.6 says that $\mathbf{c}_4 \in \mathbf{u}(\mathcal{X})$ also. Thus, there exists some $x_4 \in \mathcal{X}$ such that $\mathbf{c}_4 = \mathbf{u}(x_4)$. This means $\mathbf{u}(x_2) - \mathbf{u}(x_1) = \mathbf{u}(x_4) - \mathbf{u}(x_3)$. In other words, $u(x_2) - u(x_1) = u(x_4) - u(x_3)$ for all $u \in \mathcal{U}'$. But then the multiutility representation (3) implies that $(x_1 \rightsquigarrow x_2) \approx (x_3 \rightsquigarrow x_4)$, as desired. $\square$

Proof of Proposition 7.3. "⟸" Let $x_1, x_2, y_1 \in \mathcal{X}$. Since $\text{End}(\succeq)$ acts transitively on $\mathcal{X}$, there is some $\alpha \in \text{End}(\succeq)$ such that $\alpha(x_1) = y_1$. Let $y_2 := \alpha(x_2)$. Then $(x_1 \rightsquigarrow x_2) \approx (y_1 \rightsquigarrow y_2)$, as desired.

"⟹" Fix $x_0, y_0 \in \mathcal{X}$. We must construct some $\alpha \in \text{End}(\succeq)$ such that $\alpha(x_0) = y_0$. For all $x_1 \in \mathcal{X}$, empathy yields some $y_1 \in \mathcal{X}$ (not necessarily unique) such that $(x_0 \rightsquigarrow x_1) \approx (y_0 \rightsquigarrow y_1)$. So, define $\alpha(x_1) := y_1$.

It remains to show that $\alpha \in \text{End}(\succeq)$. Let $x_1, x_2 \in \mathcal{X}$. Let $y_1 := \alpha(x_1)$ and $y_2 := \alpha(x_2)$; we must show that $(x_1 \rightsquigarrow x_2) \approx (y_1 \rightsquigarrow y_2)$. By definition of α, we have $(x_0 \rightsquigarrow x_1) \approx (y_0 \rightsquigarrow y_1)$. Thus, axiom (DP1) says

$$(x_1 \rightsquigarrow x_0) \quad \approx \quad (y_1 \rightsquigarrow y_0). \tag{A43}$$

By definition of α, we also have

$$(x_0 \rightsquigarrow x_2) \quad \approx \quad (y_0 \rightsquigarrow y_2). \tag{A44}$$

Combining statements (A43) and (A44) via (DP2), we get $(x_1 \rightsquigarrow x_2) \approx (y_1 \rightsquigarrow y_2)$, as desired. $\square$

Appendix B: Complete extensions

Szpilrajn's Lemma (1930) says that every partial order on a set can be extended to a *linear order* (a complete, antisymmetric, transitive relation). A result of Dushnik and Miller (1941) says that every partial order is the intersection of all its linear extensions. By analogy, we will say that a difference preorder $(\succeq)$ is *Szpilrajn* if it is extended and refined by some complete difference preorder. We will say that $(\succeq)$ is *Dushnik-Miller* if it is the intersection of all the complete difference preorders which extend it.

These properties are closely related to the existence of strong utility functions and multiutility representations. To see this, let $\mathcal{R}$ be a linearly ordered abelian group, and let $u : \mathcal{X} \longrightarrow \mathcal{R}$ be any function. We can define a complete difference preorder $(\underset{u}{\succeq})$ on $\mathcal{X}$ as follows. For all $x_1, x_2, y_1, y_2 \in \mathcal{X}$,

$$\left((x_1 \rightsquigarrow x_2) \underset{u}{\succeq} (y_1 \rightsquigarrow y_2) \right) \quad \Longleftrightarrow \quad \left(u(x_2) - u(x_1) \geq u(y_2) - u(y_1) \right). \tag{B1}$$

If $(\succeq)$ is another difference preorder on $\mathcal{X}$, then u is a (strong) utility function for $(\succeq)$ if and only if $(\underset{u}{\succeq})$ extends (and refines) $(\succeq)$. Thus, the existence of a strong utility function implies that $(\succeq)$ is Szpilrajn. Furthermore, if $(\succeq)$ has a multiutility representation (3), then $(\succeq)$ is Dushnik-Miller.

Not all difference preorders are Dushnik-Miller. For example, let $\mathcal{X} := \{x_0, x_1, x_2, y_0, y_1, y_2\}$, and define the preorder $(\succeq)$ on $\mathcal{X} \times \mathcal{X}$ as follows. Begin with all the 36 'trivial' relations implied by (DP0) (e.g. "$(x_0 \rightsquigarrow x_0) \approx (y_1 \rightsquigarrow y_1)$", etc.). To this set, add the three relations "$(x_0 \rightsquigarrow x_1) \approx (x_1 \rightsquigarrow x_2)$", "$(y_0 \rightsquigarrow y_1) \approx (y_1 \rightsquigarrow y_2)$", and "$(x_0 \rightsquigarrow x_2) \succ (y_0 \rightsquigarrow y_2)$", along

with their 'reversals' under (DP1). This yields a system of 42 relations, which is closed under the application of (DP2) and (DP3). Thus, it is a difference preorder on $\mathcal{X}$. Note that $(\succeq)$ cannot compare $(x_0 \rightsquigarrow x_1)$ with $(y_0 \rightsquigarrow y_1)$. However, if $(\underset{c}{\succeq})$ is any complete difference preorder which extends $(\succeq)$, then (DP2) implies that $(x_0 \rightsquigarrow x_1) \underset{c}{\succeq} (y_0 \rightsquigarrow y_1)$. Thus, if $(\underset{DM}{\succeq})$ is the intersection of all the complete difference preorder extensions of $(\succeq)$, then we must have $(x_0 \rightsquigarrow x_1) \underset{DM}{\succeq} (y_0 \rightsquigarrow y_1)$. Thus, $(\underset{DM}{\succeq}) \neq (\succeq)$, so $(\succeq)$ is *not* Dushnik-Miller. It follows that $(\succeq)$ does *not* have a multiutility representation.

Likewise, not all difference preorders are Szpilrajn. For example, let $\mathcal{X} := \{x_0, x_1, \ldots, x_7, y_0, y_1, \ldots, y_7, z_0, z_1, \ldots, z_7\}$, and define the preorder $(\succeq)$ on $\mathcal{X} \times \mathcal{X}$ as follows. Begin with all $|\mathcal{X}|^2 = 576$ 'trivial' relations implied by (DP0). To this set, add the following relations, for all $n, m, n', m' \in [0 \ldots 7]$:

(a) $(x_n \rightsquigarrow x_m) \approx (x_{n'} \rightsquigarrow x_{m'})$ and $(y_n \rightsquigarrow y_m) \approx (y_{n'} \rightsquigarrow y_{m'})$ and $(z_n \rightsquigarrow z_m) \approx (z_{n'} \rightsquigarrow z_{m'})$ if and only if $n - m = n' - m'$.

(b) $(x_n \rightsquigarrow x_m) \succ (y_{n'} \rightsquigarrow y_{m'})$ if and only if $n - m = n' - m' > 0$ and is divisible by 3.

(c) $(y_n \rightsquigarrow y_m) \succ (z_{n'} \rightsquigarrow z_{m'})$ if and only if $n - m = n' - m' > 0$ and is divisible by 5.

(d) $(z_0 \rightsquigarrow z_7) \succ (x_0 \rightsquigarrow x_7)$.

Also add the (DP1)-reversals of the sets of relations described in (b), (c) and (d) (the set (a) is already closed under (DP1)). Observe that the four relation sets described in (a)-(d) are each separately closed under the application of (DP2) and (DP3). Also, there is no way to combine a relation from one of these sets (e.g. (b)) with one from another (e.g. (c)) using (DP2) or (DP3). Thus, the entire system is closed under (DP2) and (DP3); thus, it is a difference preorder. We claim it is not Szpilrajn.

By contradiction, suppose that $(\underset{c}{\succeq})$ is a complete difference preorder on $\mathcal{X}$ which extends and refines $(\succeq)$. Then (a), (b) and (DP2) imply that $(x_0 \rightsquigarrow x_1) \succ (y_0 \rightsquigarrow y_1)$. Likewise, (a), (c) and (DP2) imply that $(y_0 \rightsquigarrow y_1) \succ (z_0 \rightsquigarrow z_1)$. Finally, (a), (d) and (DP2) imply that $(z_0 \rightsquigarrow z_1) \succ (x_0 \rightsquigarrow x_1)$. Thus, we have an cycle of strict preferences, yielding a contradiction. It follows that $(\succeq)$ is not Szpilrajn. Thus, $(\succeq)$ cannot have any strong utility functions.

The interpretation of these counterexamples depends upon whether we believe the incompleteness of $(\succeq)$ to be *epistemic* or *metaphysical* in origin. According to the 'epistemic' account, precise interpersonal comparisons are meaningful in principle; we simply lack the necessary information to make these comparisons in practice. The incomplete difference preorder $(\succeq)$ reflects our incomplete knowledge of some unknown, *complete* difference preorder $(\underset{*}{\succeq})$, which encodes the 'true' interpersonal comparisons. Thus, $(\succeq)$ should be Szpilrajn. Furthermore, if $(\succeq)$ is not Dushnik-Miller, then it can and should be extended to its 'Dushnik-Miller completion', because the extra interpersonal comparisons encoded in this completion must be part of $(\underset{*}{\succeq})$.

According to the 'metaphysical' account, however, certain interpersonal comparisons are not meaningful, even in principle. Thus, there is no reason to expect $(\succeq)$ to be Szpilrajn.

If $(\succeq)$ is *not* Szpilrajn, and we have good reason to regard $(\succeq)$ as our 'best possible model' of interpersonal comparisons, then this provides evidence for the metaphysical account.

Nevertheless, we suspect that most 'natural' difference preorders are both Szpilrajn and Dushnik-Miller. Are there simple sufficient conditions for these properties? If $(\,\rhd\,)$ is an $(\succeq)$-SDP, then what is the relationship between the Szpilrajn/Dushnik-Miller properties of $(\succeq)$ and those of $(\,\rhd\,)$? Finally, we have:

Conjecture. *Let $(\succeq)$ be a complete difference preorder on $\mathcal{X}$. Then there exists a linearly ordered abelian group $\mathcal{R}$ and utility function $u : \mathcal{X} \longrightarrow \mathcal{R}$, such that $(\succeq) = (\underset{u}{\succeq})$, as defined by formula* (B1).

If true, this conjecture would imply that any Dushnik-Miller difference preorder has a multiutility representation (3). In particular, any Dushnik-Miller SDP would have a multiwelfare representation (5), and thus would be quasiutilitarian (by Theorem 3.2(b)).

References

Alt, F., 1936. Über die messbarkeit des nutzens. Zeitschrift für Nationalökonomie VII (2), 161–169, reprinted as Alt (1971).

Alt, F., 1971. On the measurability of utility. In: Chipman, J. S., Hurwicz, L., Richter, M. K., Sonnenschein, H. F. (Eds.), Preferences, Utility, Demand: A Minnesota symposium. Harcourt Brace Jovanovich, reprint of Alt (1936), translated by Sigfried Schach.

Arrow, K. J., 1977. Extended sympathy and the possibility of social choice. American Economic Review: Papers and Proceedings 67, 219–225.

Basu, K., 1980. Revealed preference of government. Cambridge UP, Cambridge, UK.

Basu, K., Mitra, T., 2003. Aggregating infinite utility streams with intergenerational equity: the impossibility of being Paretian. Econometrica 71 (5), 1557–1563.

Basu, K., Mitra, T., 2006. On the existence of paretian social welfare relations for infinite utility streams with extended anonymity. In: Roemer, J., Suzumura, K. (Eds.), Intergenerational equity and sustainability. Palgrave Macmillan.

Baucells, M., Shapley, L. S., 2006. Multiperson utility: the linearly independent case. (preprint) http://webprofesores.iese.edu/mbaucells.

Baucells, M., Shapley, L. S., 2008. Multiperson utility. Games Econom. Behav. 62 (2), 329–347.

Blackorby, C., 1975. Degrees of cardinality and aggregate partial orderings. Econometrica 43 (5–6), 845–852.

Camacho, A., 1980. Approaches to cardinal utility. Theory and Decision 12 (4), 359–379.

Debreu, G., 1958. Stochastic choice and cardinal utility. Econometrica 26, 440–444.

Dushnik, B., Miller, E. W., 1941. Partially ordered sets. Amer. J. Math. 63, 600–610.

Dyer, J. S., Sarin, R. K., 1978. Cardinal preference aggregation rules for the case of certainty. In: Zioats, S. (Ed.), Proceedings of the conference on multiple criteria problem solving. Springer-Verlag.

Dyer, J. S., Sarin, R. K., 1979a. Group preference aggregation rules based on strength of preference. Management Sci. 25 (9), 822–832 (1980).

Dyer, J. S., Sarin, R. K., 1979b. Measurable multiattribute value functions. Oper. Res. 27 (4), 810–822.

Elster, J., Roemer, J. E. (Eds.), 1991. Interpersonal comparisons of well-being. Cambridge UP, Cambridge, UK.

Fine, B., 1975. A note on "Interpersonal aggregation and partial comparability". Econometrica 43, 169–172.

Fleurbaey, M., Hammond, P. J., 2004. Interpersonally comparable utility. In: Barberá, S., Hammond, P. J., Seidl, C. (Eds.), Handbook of Utility Theory. Vol. II. Kluwer, Boston, pp. 1179–1285.

Fleurbaey, M., Michel, P., 2003. Intertemporal equity and the extension of the Ramsey criterion. J. Math. Econom. 39 (7), 777–802.

Harsanyi, J. C., 1955. Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility. Journal of Political Economy 63, 309–321.

Harsanyi, J. C., 1977. Rational behaviour and bargaining equilibrium in games and social situations. Cambridge UP, Cambridge, UK.

Harvey, C. M., 1999. Aggregation of individuals' preference intensities into social preference intensity. Soc. Choice Welf. 16 (1), 65–79.

Harvey, C. M., Østerdal, L. P., 2010. Cardinal scales for health evaluation. Decision Analysis 7 (3), 256–281.

Kolm, S., 2002. Justice and Equity. MIT Press, Cambridge, MA, translated by Harold F. See, originally *Justice et Equité* (1972).

Kolm, S.-C., 1994a. The meaning of "fundamental preferences". Social Choice and Welfare 11, 193–198.

Kolm, S.-C., 1994b. Rejoinder to John Broome. Social Choice and Welfare 11, 203–204.

Kolm, S.-C., 1995. Erratum: The meaning of "fundamental preferences". Social Choice and Welfare, 311.

Krantz, D. H., Luce, R. D., Suppes, P., Tversky, A., 1971. Foundations of measurement. Academic Press, New York, vol. 1: Additive and polynomial representations.

Pfanzagl, J., 1968. Theory of measurement. Physica-Verlag, Würzburg.

Pivato, M., 2010a. Approximate interpersonal comparisons of well-being. (preprint) http://mpra.ub.uni-muenchen.de/25224.

Pivato, M., 2010b. Aggregation of incomplete ordinal preferences with approximate interpersonal comparisons. (preprint) http://mpra.ub.uni-muenchen.de/25271.

Pivato, M., 2010c. Risky social choice with incomplete or noisy interpersonal comparisons of well-being. (preprint) http://mpra.ub.uni-muenchen.de/25222.

Pivato, M., 2011. Additive representation of separable preferences over infinite products. (preprint) http://mpra.ub.uni-muenchen.de/28262.

Robbins, L., 1935. An essay on the nature and significance of economic science, 2nd Edition. MacMillon, London, Ch. 6.

Robbins, L., 1938. Interpersonal comparisons of utility. Economic Journal 48, 635–641.

Saposnik, R., 1983. On evaluating income distributions: Rank dominance, the Suppes-Sen grading principle of justice, and Pareto optimality. Public Choice 40 (3), 329–336.

Scott, D., Suppes, P., 1958. Foundational aspects of theories of measurement. J. Symb. Logic 23, 113–128.

Sen, A., 1970a. Interpersonal aggregation and partial comparability. Econometrica 38, 393–409.

Sen, A., 1972. Interpersonal comparison and partial comparability: A correction. Econometrica 40 (5), 959.

Sen, A. K., 1970b. Collective choice and social welfare. Holden Day, San Francisco.

Suppes, P., 1966. Some formal models of the grading principle. Synthese 6, 284–306.

Suppes, P., Winet, M., 1955. An axiomatization of utility based on the notion of utility differences. Management Sci. 1, 259–270.

Szpilrajn, E., 1930. Sur l'extension de l'ordre partiel. Fundamenta Matematicae 16, 386–389.