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MALLIAVIN DIFFERENTIABILITY OF THE HESTON VOLATILITY AND APPLICATIONS TO OPTION PRICING

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Abstract

We prove that the Heston volatility is Malliavin differentiable under the classical Novikov condition and give an explicit expression for the derivative. This result guarantees the applicability of Malliavin calculus in the framework of the Heston stochastic volatility model. Furthermore we derive conditions on the parameters which assure the existence of the second Malliavin derivative of the Heston volatility. This allows us to apply recent results of the first author [3] in order to derive approximate option pricing formulas in the context of the Heston model. Numerical results are given.

Keywords: Malliavin calculus; stochastic volatility models; Heston model; Cox-Ingersoll-Ross process; Hull and White formula; Option pricing

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1. Introduction

In recent years, Malliavin calculus has appeared as a major tool in both theoretical and computational mathematical finance. This fact is documented by the large number of published articles in this area. The assumptions on the possibly

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multidimensional diffusion process (X_t) which determines the factors of the model, in general require as a minimal condition that the coefficient functions β and σ in

$$dX_t = \beta(X_t, t)dt + \sigma(X_t, t)d\mathbb{W}_t$$

are continuously differentiable and satisfy a global Lipschitz condition. These assumptions work fine with the standard Black-Scholes model or more general models based on linear stochastic differential equations. Problems occur however when one uses more advanced models, like the Heston stochastic volatility model. In this model the stock price is given by the equation

$$dS_t = S_t(bdt + \sqrt{v_t}dB_t) \tag{1}$$

where (B_t) denotes a Brownian motion, but in contrast to the standard Black-Scholes model the volatility v_t is itself a diffusion process, satisfying the stochastic differential equation

$$dv_t = \kappa\left(\theta - v_t\right)dt + \nu\sqrt{v_t}dW_t \tag{2}$$

where W_t denotes a possibly correlated second Brownian motion. Obviously the coefficient functions of this model do not satisfy the standard assumptions. The square root function is neither differentiable in zero nor globally Lipschitz. In this article we present a direct proof of the Malliavin differentiability of the Heston volatility and its square root and give explicit expressions for their derivatives. Furthermore we discuss the existence of the second Malliavin derivative and derive conditions on the parameters κ,θ and ν which guarantee its existence. Recently in [3], Malliavin calculus techniques have been applied in order to obtain an extension of the classical Hull and White formula for the case of correlated stock and volatility. In order to apply the results to the Heston model, Malliavin differentiability as well as certain integrability conditions of the Malliavin derivative of the Heston volatility have to be verified. Our application includes an adaptation of the results from [3] to the case of the Heston volatility and a new approximative option pricing formula for the Heston model as well as a precise analysis of the goodness of this approximation.

The structure of the article is as follows. In Section 2 we give an explicit approximating sequence for the Heston volatility, while in Section 3 we provide some preliminaries on Malliavin calculus. We study the Malliavin differentiability of the Heston volatility in Section 4 and present our two main theoretical results. In Section 5 we include our application and the main practical results, while the main conclusions are summarized in Section 6.

2. The Heston volatility model and an approximating sequence

As mentioned in the introduction, the Heston stochastic volatility model consists of a money market account which we do not specify at the moment, a stock (S_t) and the volatility process (v_t) with dynamics specified in (1) and (2), where it is assumed that κ, θ and ν are positive constants, see [8]. In the following we consider one fixed probability space $(\Omega, \mathcal{G}, \mathbb{P})$ on which there is defined a Brownian motion (W_t) and which is filtered by the augmented and completed Brownian filtration which we denote with $(\mathcal{G}_t)$. We also fix an interval $[0, T]$. A standard assumption, when using the Heston model is $2\kappa\theta \geq \nu^2$. This is often called the Novikov condition. Given that $v_0 > 0$ this condition guarantees that the volatility process is always positive, i.e. $\mathbb{P}\left(\{v_t > 0 \ \forall \ t > 0\}\right) = 1$. We assume that $v_0 > 0$ and that the Novikov condition holds. It is then possible to consider the square root process $\sigma_t := \sqrt{v_t}$. It follows from the Itô formula that this process satisfies

$$d\sigma_t = \left[\left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right)\frac{1}{\sigma_t} - \frac{\kappa}{2}\sigma_t\right]dt + \frac{\nu}{2}dW_t. \tag{3}$$

We note that the Novikov condition implies in particular that the factor $\left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right)$ appearing in the drift term of σ_t is positive. This will play a significant role later. It is not a priori clear that the SDE (3) admits a unique strong solution, but the Yamada-Watanabe Lemma ([10], Chapter 5, Proposition 2.18) obviously implies uniqueness of the solution of SDE (2). For any solution σ_t of SDE (3) we find by applying the Itô formula, that σ_t^2 is a solution of SDE (2). As the latter one is unique, we conclude uniqueness of the solution for SDE (3) up to a sign. However if σ_t solves (3) it is obvious that $-\sigma_t$ does not and therefore we find uniqueness of the solution of SDE (3). In order to show in section 4, that σ_t is Malliavin differentiable we will now define an approximating sequence. Let $\varepsilon > 0$ and $\Phi_\varepsilon(x)$ be a continuously differentiable function satisfying $\Phi_\varepsilon(x) = 1$ if $x \geq 2\varepsilon$ and $\Phi_\varepsilon(x) = 0$ if $x < \varepsilon$, while $\Phi_\varepsilon(x) \leq 1$ for all $x \in \mathbb{R}$. We note that in this case $\Phi'_\varepsilon(x) = 0$ if $x < \varepsilon$ or $x \geq 2\varepsilon$. Furthermore we define the function $\Lambda_\varepsilon(x) = \Phi_\varepsilon(x)\frac{1}{x}$ with $\Lambda_\varepsilon(0) = 0$. The function $\Lambda_\varepsilon(x)$ is bounded and continuously differentiable satisfying $\Lambda'_\varepsilon(x) = \Phi'_\varepsilon(x)\frac{1}{x} - \Phi_\varepsilon(x)\frac{1}{x^2}$. In particular $\Lambda'_\varepsilon(x) = -\frac{1}{x^2}$ if $x \geq 2\varepsilon$ and $\Lambda'_\varepsilon(x) = 0$ if $x < \varepsilon$. Let us now define our approximations σ_t^ε as the solutions of the stochastic differential equations

$$d\sigma_t^\varepsilon = \left[\left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right)\Lambda_\varepsilon(\sigma_t^\varepsilon) - \frac{\kappa}{2}\sigma_t^\varepsilon\right]dt + \frac{\nu}{2}dW_t, \tag{4}$$

with $\sigma_0^\varepsilon = \sigma_0$ for all $\varepsilon > 0$.

Proposition 2.1. *For each $t \in [0,T]$ the sequence σ_t^ε converges to σ_t in $L^2(\Omega)$.*

Proof. We use the dominated convergence theorem in order to obtain this result. Let us first prove that σ_t^ε converges to σ_t point wise. This follows from a standard localization argument. For each $\varepsilon > 0$ define a stopping time τ_ε via $\tau_\varepsilon(\omega) := \inf\{t|\sigma_t(\omega) \leq \varepsilon\}$. Letting ε go to zero, the sequence of (τ_ε) defines an increasing sequence of stopping times, and it follows from the strict positivity of σ_t that

$\lim_{\varepsilon\to 0}\tau_\varepsilon = \infty$ a.s. Denoting with $\sigma^{\tau_\varepsilon}$ the process obtained from σ by stopping at τ_ε, then it follows from the choice of the function $\Lambda_\varepsilon(x)$ and equations (3) and (4), that $\sigma_t^{\tau_{2\varepsilon}} = \sigma_t^\varepsilon \;\; \forall \;\; t \leq \tau^\varepsilon$. Now, for fixed $t \in [0,T]$ letting ε go to zero one obtains that $\lim_{\varepsilon\to 0}\sigma_t^\varepsilon = \lim_{\varepsilon\to 0}\sigma_t^{\tau_{2\varepsilon}} = \sigma_t \quad$ a.s. Let us now prove that for each $t \in [0,T]$ σ_t^ε converges to σ_t in $L^2(\Omega)$. For this let us consider the Ornstein-Uhlenbeck process u_t satisfying $u_0 = \sigma_0$ and

$$du_t = -\frac{\kappa}{2}u_t dt + \frac{\nu}{2}dW_t.$$

We show that $u_t \leq \sigma_t^\varepsilon \leq \sigma_t$ for all t a.s. The first inequality follows directly from the Yamada-Watanabe comparison lemma. To prove the second inequality this lemma can not directly be applied as the drift term in the SDE for σ_t is not continuous. Since we know however, that under our assumptions on the coefficients $\sigma_t > 0$ a.s., the second inequality would indeed follow from $(\sigma_t^\varepsilon)^2 \leq \sigma_t^2$. In fact, applying Itô's formula to $v_t^\varepsilon = (\sigma_t^\varepsilon)^2$ gives

$$dv_t^\varepsilon = \left[\left(\kappa\theta - \frac{\nu^2}{4}\right)\sqrt{v_t^\varepsilon}\Lambda_\epsilon\left(\sqrt{v_t^\varepsilon}\right) - \kappa v_t^\varepsilon + \frac{\nu^2}{4}\right]dt + \nu\sqrt{v_t^\varepsilon}dW_t.$$

while $v_t = \sigma_t^2$ satisfies (2). For both v_t^ε and v_t the condition on the diffusion coefficient in [10] Chapter 5, Proposition 2.18. can easily be verified by choosing the function $h(x) = \nu\sqrt{x}$. Obviously the drift term in (2) is globally Lipschitz. In addition, it is not hard to verify that the drift term corresponding to v_t^ε is globally Lipschitz. We can therefore conclude the second inequality from

$$\begin{aligned} & \left(\kappa\theta - \tfrac{\nu^2}{4}\right)\sqrt{x}\Lambda_\epsilon\left(\sqrt{x}\right) - \kappa x + \tfrac{\nu^2}{4} \leq \kappa(\theta - x) \\ \Leftrightarrow \quad & \kappa\theta\left(\sqrt{x}\Lambda_\epsilon\left(\sqrt{x}\right) - 1\right) \leq \tfrac{\nu^2}{4}\left(\sqrt{x}\Lambda_\epsilon\left(\sqrt{x}\right) - 1\right) \\ \Leftrightarrow \quad & \kappa\theta \geq \tfrac{\nu^2}{4} \end{aligned}$$

the latter being true due to the Novikov condition. For the last equivalence we used the inequality $0 \leq \sqrt{x}\cdot\Lambda_\epsilon\left(\sqrt{x}\right) \leq 1$. Now it follows from $u_t \leq \sigma_t^\varepsilon \leq \sigma_t$

that $|\sigma_t^\varepsilon| \leq |u_t| + |\sigma_t|$. Since obviously u_t and σ_t belong to $L^2(\Omega)$ the dominated convergence theorem implies the desired convergence.

3. A short review on Malliavin calculus

Let us review some of the basic features of Malliavin calculus. A standard reference for this is [11]. Let us consider the set $\mathcal{S}$ of cylindrical functionals $F : \Omega \to \mathbb{R}$, given by $F = f\left(W_{t_1}, ..., W_{t_l}\right)$ where $f \in C_b^\infty\left(\mathbb{R}^l\right)$ is a smooth function with bounded derivatives of all orders and (W_t) denotes a Brownian motion on Ω. We define the Malliavin derivative operator on $\mathcal{S}$ via

$$D_s F \;\; := \;\; \sum_{i=1}^{l} \frac{\partial f}{\partial x_i}\left(W_{t_1}(\omega), ..., W_{t_l}(\omega)\right) \cdot \mathbf{1}_{[0,t_i]}(s).$$

This operator and the iterated operators D^n are closable and unbounded from $L^p(\Omega)$ into $L^p\left(\Omega \times [0,T]^n\right)$, for all $n \geq 1$. Their respective domains are denoted by $\mathbb{D}^{n,p}$ and obtained as the closure of $\mathcal{S}$ with respect to the norms defined by $\|F\|_{n,p}^p = \|F\|_{L^p(\Omega)}^p + \sum_{k=1}^n \left\|D^k F\right\|_{L^p\left(\Omega\times[0,T]^k\right)}^p$. The adjoint of the Malliavin derivative operator $D : \mathbb{D}^{1,2} \to L^2(\Omega \times [0,T])$ is called the Skorohod integral and denoted with δ. This operator has the property that its domain contains the class $L_a^2(\Omega \times [0,T])$ of square integrable adapted stochastic processes and its restriction to this class coincides with the Itô-integral. We will make use of the notation $\delta(u) = \int_0^T u_t dW_t$ and recall that $\mathbb{L}^{n,2} := L^2([0,T], \mathbb{D}^{n,2})$ is included in the domain of δ for all $n \geq 1$. For more details we refer to [11]. We will later use the following anticipative Itô formula, see [5].

Proposition 3.1. *Let us consider the processes $X_t = X_0 + \int_0^t u_s dW_s + \int_0^t v_s ds$, where X_0 is $\mathcal{F}_0-$measurable and $u, v \in L_a^2\left([0,T] \times \Omega\right)$. Furthermore consider a process $Z_t = \int_t^T \theta_s ds$ for some $\theta \in \mathbb{L}^{1,2}$. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a twice continuously differentiable function for which there exists a positive constant C such that, for all $t \in [0,T]$, F and its derivatives evaluated in (t, X_t, Z_t) are bounded by C. Then it* *follows that*

$$\begin{aligned} F(t, X_t, Z_t) &= F(0, X_0, Z_0) + \int_0^t \frac{\partial F}{\partial s}(s, X_s, Z_s)\, ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s, Z_s)\, dX_s \\ &+ \int_0^t \frac{\partial F}{\partial z}(s, X_s, Z_s)\, dZ_s + \int_0^t \frac{\partial^2 F}{\partial x \partial z}(s, X_s, Z_s) \left(\int_s^T D_s \theta_r dr \right) u_s ds \\ &+ \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s, Z_s)\, u_s^2 ds \end{aligned}$$

4. Malliavin differentiability of the Heston volatility

In this section we will show that both the Heston volatility v_t as well as its square root σ_t belong to $\mathbb{D}^{1,2}$. We will also derive conditions under which the second Malliavin derivative of the Heston volatility exists.

Lemma 4.1. *We have $\sigma_t^\varepsilon \in \mathbb{D}^{1,2}$ and for $r < t$*

$$D_r \sigma_t^\varepsilon = \frac{\nu}{2} \exp \left\{ \int_r^t \left[-\frac{\kappa}{2} + \left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8} \right) \Lambda_\varepsilon' (\sigma_s^\varepsilon) \right] ds \right\}$$

Proof. This follows directly from [6], Theorem 2.1.

We are now ready to proof the following result.

Proposition 4.1. *Assuming $2\kappa\theta \geq \nu^2$ we have $\sigma \in \mathbb{D}^{1,2}$ and for $r < t$*

$$D_r \sigma_t = \frac{\nu}{2} \exp \left\{ \int_r^t \left[-\frac{\kappa}{2} - \left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8} \right) \frac{1}{\sigma_t^2} \right] ds \right\}.$$

Proof. We know from Proposition 2.1 that for each $t \in [0, T]$ the sequence σ_t^ε converges to σ_t in $L^2(\Omega)$. Since this convergence is also point wise, we conclude by using the properties of the function $\Lambda_\varepsilon(x)$ that

$$D_r \sigma_t^\varepsilon = \frac{\nu}{2} \exp \left\{ \int_r^t \left[-\frac{\kappa}{2} - \left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8} \right) \Lambda_\varepsilon' (\sigma_t^\varepsilon) \right] ds \right\}$$

converges point wise to $G := \frac{\nu}{2} \exp \left\{ \int_r^t \left[-\frac{\kappa}{2} - \left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8} \right) \frac{1}{\sigma_t^2} \right] ds \right\}$. It follows from the Novikov condition, that the exponent in $D_r \sigma_t^\varepsilon$ is negative for all choices of ε

and therefore that $|D_r\sigma_t^\varepsilon| \leq \frac{\nu}{2}$ for all ε. From the bounded convergence theorem we conclude that $D_r\sigma_t^\varepsilon$ converges to G in $L^2(\Omega)$. Finally, Lemma 1.2.3 in [11] implies that $\sigma_t \in \mathbb{D}^{1,2}$ and $D_r\sigma_t = G$.

Corollary 4.1. *$|D_r\sigma_t| \leq \frac{\nu}{2}\exp\left(-\frac{\kappa}{2}(t-r)\right)$ and $\sigma_t \in \mathbb{L}^{1,2}$.*

Proof. Follows directly from Proposition 4.1.

Corollary 4.2. *$v_t \in \mathbb{L}^{1,2}$ and for $r < t$ $D_r v_t = \nu \exp\left\{\int_r^t \left[-\frac{\kappa}{2} - \left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right)\frac{1}{v_t}\right] ds\right\}\sqrt{v_t}$.*

Proof. For fixed $t \in [0,T]$ we have $v_t \in L^2(\Omega)$ and

$$\nu \exp\left\{\int_r^t \left[-\frac{\kappa}{2} - \left(\frac{\kappa\theta}{2} - \frac{\nu^2}{8}\right)\frac{1}{v_t}\right] ds\right\}\sqrt{v_t} \in L^2(\Omega)$$

follows again from the boundedness of the exponential. It then follows from Exercise 1.2.13 in [11] that $v_t \in \mathbb{D}^{1,2}$. As in Corollary 4.1 one concludes from the explicit expression, that $v_t \in \mathbb{L}^{1,2}$.

Let us now discuss the existence of the second Malliavin derivative of the Heston volatility. As indicated before, in order to guarantee the existence of the second Malliavin derivative we have to strengthen the conditions on the coefficients slightly. The following lemma will be used in the proofs of Proposition 4.2 and Proposition 5.2.

Lemma 4.2. *Let $n > 1$ and $\delta := \frac{4\kappa\theta}{\nu^2} > n$ and denoting $L(t) = \left(1 - e^{-kt}\right)$ there exists a positive constant $C(n)$ such that, for all $t \in [0,T]$*

$$\mathbb{E}\left(\frac{1}{\sigma_t^n}\right) \leq \frac{C(n)}{L(t)}\left(\frac{e^{kt}}{\sigma_0}\right)^{\frac{n}{2}-1}.$$

Proof. From the proof of Lemma A.1 in [2] we deduce that

$$\begin{aligned}\mathbb{E}\left(\frac{1}{\sigma_t^n}\right) &= \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)L(t)^{\frac{n}{2}}}\int_0^1 u^{\frac{n}{2}-1}(1-u)^{\frac{2k\theta}{\nu^2}-\frac{n}{2}-1}\exp\left(-\frac{\sigma_0 e^{-kt}u}{2L(t)}\right)du \\ &= \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)L(t)}\int_0^1 u^{\frac{n}{2}-1}(1-u)^{\frac{2k\theta}{\nu^2}-\frac{n}{2}-1} \\ &\quad\times\left(\frac{e^{kt}}{\sigma_0 u}\right)^{\frac{n}{2}-1}\left(\frac{\sigma_0 e^{-kt}u}{L(t)}\right)^{\frac{n}{2}-1}\exp\left(-\frac{\sigma_0 e^{-kt}u}{2L(t)}\right)du.\end{aligned}$$

Then, using the fact that $y^{\frac{n}{2}-1}\exp(-y)\leq C(n)$ for some positive constant $C(n)$, we can write

$$\begin{aligned}\mathbb{E}\left(\frac{1}{\sigma_t^n}\right) &\leq \frac{C(n)}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)L(t)}\left(\frac{e^{kt}}{\sigma_0}\right)^{\frac{n}{2}-1}\int_0^1 (1-u)^{\frac{2k\theta}{\nu^2}-\frac{n}{2}-1}du \\ &\leq \frac{C(n)}{L(t)}\left(\frac{e^{kt}}{\sigma_0}\right)^{\frac{n}{2}-1},\end{aligned}$$

which completes the proof.

Proposition 4.2. *Assume that $4\kappa\theta > 3\nu^2$, then $\sigma_t \in \mathbb{D}^{2,1}$ with*

$$\begin{aligned}D_\tau D_r\sigma_t &= \frac{\nu^2}{2}\left(\frac{\kappa\theta}{2}-\frac{\nu^2}{8}\right)\exp\left\{\int_{\tau\vee r}^t\left[-\frac{k}{2}-\left(\frac{k\theta}{2}-\frac{\nu^2}{8}\right)\frac{1}{\sigma_s^2}\right]ds\right\} \\ &\quad\times\int_{\tau\vee r}^t\exp\left\{\int_{\tau\vee r}^s\left[-\frac{k}{2}-\left(\frac{k\theta}{2}-\frac{\nu^2}{8}\right)\frac{1}{\sigma_u^2}\right]du\right\}\frac{1}{\sigma_s^3}ds\end{aligned}$$

for $\tau < t$ and 0 else. Furthermore if $2\kappa\theta > 3\nu^2$ we have $\sigma_t \in \mathbb{L}^{2,2}$ and

$$\mathbb{E}\left|D_\tau D_r\sigma_t\right|^2 \leq C(n,\sigma_0,T)\,\nu^2(t-r)(\ln t-\ln r)$$

where $C(n,\sigma_0,T)$ is a constant depending on n,σ_0 and T but not on t, τ or ν.

Proof. Without loss of generality we assume that $\tau > r$ and obtain formally

$$
\begin{aligned}
D_\tau D_r\sigma_t &= D_\tau \frac{\nu}{2}\exp\left\{\int_r^t \left[-\frac{\kappa}{2}-\left(\frac{\kappa\theta}{2}-\frac{\nu^2}{8}\right)\frac{1}{\sigma_t^2}\right]ds\right\}\\
&= \frac{\nu}{2}\exp\left\{\int_r^t \left[-\frac{\kappa}{2}-\left(\frac{\kappa\theta}{2}-\frac{\nu^2}{8}\right)\frac{1}{\sigma_t^2}\right]ds\right\}\int_\tau^t -\left(\frac{\kappa\theta}{2}-\frac{\nu^2}{8}\right)\cdot(-2)\frac{1}{\sigma_s^3}D_\tau\sigma_s ds\\
&= \frac{\nu^2}{2}\left(\frac{\kappa\theta}{2}-\frac{\nu^2}{8}\right)\exp\left\{\int_r^t \left[-\frac{k}{2}-\left(\frac{k\theta}{2}-\frac{\nu^2}{8}\right)\frac{1}{\sigma_s^2}\right]ds\right\}\times\\
&\quad \int_\tau^t \exp\left\{\int_\tau^s \left[-\frac{k}{2}-\left(\frac{k\theta}{2}-\frac{\nu^2}{8}\right)\frac{1}{\sigma_u^2}\right]du\right\}\frac{1}{\sigma_s^3}ds.
\end{aligned}
$$

Here we used that $D_\tau\sigma_s = 0$ for $\tau > r$ and $s \in [r,\tau)$. We will show that if $4\kappa\theta > 3\nu^2$ this expression is contained in $L^1(\Omega)$. This guarantees the existence of the second Malliavin derivative and furthermore that the expression just derived is in fact the second Malliavin derivative. In order to do this, note that for $r < \tau$ and $s < t$

$$
\exp\left\{\int_r^t \left[-\frac{\kappa}{2}-\left(\frac{\kappa\theta}{2}-\frac{\nu^2}{8}\right)\frac{1}{\sigma_s^2}\right]ds\right\} \leq \exp\left\{\int_\tau^s \left[-\frac{\kappa}{2}-\left(\frac{\kappa\theta}{2}-\frac{\nu^2}{8}\right)\frac{1}{\sigma_u^2}\right]du\right\},
$$

already follows from $2\kappa\theta \geq \nu^2$. This implies

$$
\begin{aligned}
|D_\tau D_r\sigma_t| &\leq C\int_\tau^t \exp\left\{\int_\tau^s \left[-\kappa-2\left(\frac{\kappa\theta}{2}-\frac{\nu^2}{8}\right)\frac{1}{\sigma_u^2}\right]du\right\}\frac{1}{\sigma_s^3}ds\\
&\leq C\int_\tau^t \exp\left\{-2\left(\frac{\kappa\theta}{2}-\frac{\nu^2}{8}\right)\int_\tau^s \frac{1}{\sigma_u^2}du\right\}\frac{1}{\sigma_s^3}ds
\end{aligned}
$$

where $C = \frac{\nu^2}{2}\left(\frac{\kappa\theta}{2}-\frac{\nu^2}{8}\right) \leq \nu^2\cdot\frac{\kappa\theta}{4}$. Similar as in the proof of Proposition 4.1 and Corollary 4.1 it follows that $\left|\exp\left\{-2\left(\frac{\kappa\theta}{2}-\frac{\nu^2}{8}\right)\int_\tau^s\frac{1}{\sigma_u^2}du\right\}\right| \leq 1$ and therefore the first statement of Proposition 4.2 easily follows from Lemma 4.2 with $n = 3$. The second statement can now be derived as follows. Applying the Cauchy Schwarz inequality we obtain $|D_\tau D_r\sigma_t|^2 \leq \nu^2\cdot\frac{\kappa\theta}{4}(t-\tau)\int_\tau^t\frac{1}{\sigma_s^6}ds$ and therefore, using Lemma 4.2 with $n = 6$, taking into account that $L(t) \geq \kappa t e^{-\kappa t}$, we obtain

$$
\begin{aligned}
\mathbb{E}|D_\tau D_r\sigma_t|^2 &\leq (t-r)\int_\tau^t \mathbb{E}\left(\frac{1}{\sigma_s^6}\right)ds\\
&\leq C(n)\cdot\frac{\theta\nu^2}{4\sigma_0^2}(t-r)\int_\tau^t\frac{e^{ks}}{s}\left(e^{ks}\right)^2 ds\\
&\leq C(n,\sigma_0,T)\,\nu^2(t-r)(\ln t-\ln r),
\end{aligned}
$$

with $C(n, \sigma_0, T) = C(n) \cdot \frac{\theta}{4\sigma_0^2} e^{\kappa T}$.

5. An approximate option pricing formula for the Heston model

Let us consider the Heston stochastic volatility model with correlation ρ, which consists of a stock, a money market account with deterministic interest rate r and the volatility process v_t satisfying equations (1) and (2), where we assume that $dB_t \cdot dW_t = \rho dt$, with $\rho \in (-1, 1)$. It is well known that there exists a 2-dimensional Brownian motion $(Z_t, W_t)^\top$ on a filtered probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions, s.t. $B_t = \rho W_t + \sqrt{1-\rho^2} Z_t$. It is helpful in the following to think of the dynamic described by (1) and (2) as driven by$(Z_t, W_t)^\top$ rather than $(B_t, W_t)^\top$. We also assume that the dynamics is satisfied under the risk neutral measure chosen by the market and that this risk neutral measure is given by $\mathbb{P}$. This implies that $b = r$. In the following we work with the logarithmic price $X_t = \ln(S_t)$ rather then the actual price. The price of a contingent claim $h(X_T)$ at time t can then be computed via the formula $V_t = e^{-r(T-t)} \mathbb{E}\left(h(X_T)|\mathcal{F}_t\right)$. In the following let us fix a payoff function h and denote with $BS(t, x, \sigma)$ the price at time t of the corresponding contingent claim in the standard Black-Scholes model with constant volatility σ, given that the log price at time t is x. We assume that this payoff function $h : \mathbb{R} \to \mathbb{R}$ is continuous and piecewise continuously differentiable. Furthermore we denote with $\vartheta_t := \sqrt{\frac{1}{T-t} \int_t^T \sigma_s^2 ds}$ the average Heston future volatility starting from time t and with D the Malliavin derivative operator with respect to the Brownian motion W. The following proposition is in line with Theorem 3 in [3] and Theorem 3 in [4].

Proposition 5.1. *Consider the Heston model and assume that $2\kappa\theta \geq \nu^2$. Then*

$$V_t = \mathbb{E}\left(BS(t, X_t, \vartheta_t)|\, \mathcal{F}_t\right) + \frac{\rho}{2} \mathbb{E}\left(\int_t^T e^{-r(s-t)} H(s, X_s, \vartheta_s)\, \Lambda_s ds \middle|\, \mathcal{F}_t\right) \tag{5}$$

where $H(s,x,\sigma) := \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right) BS(s,x,\sigma)$ *and* $\Lambda_s := \left(\int_s^T D_s\sigma_r^2 dr\right)\sigma_s$.

Proof. Follows from Proposition 4.1 in connection with Theorem 3 in [4]

It follows from the classical Hull and White formula, see [9] that $\mathbb{E}\left(BS\left(t, X_t, \vartheta_t\right) \middle| \mathcal{F}_t\right)$ is the price of the contingent claim in the Heston model without correlation. Proposition 5.1 above therefore extends the classical Hull and White formula to the Heston model with correlation and gives interesting insight into how the correlation effects option prices. It says that this correlation effect is explicitly given by the second summand in equation (10). This fact is very useful in order to study price sensitivities with respect to ρ in the Heston stochastic volatility model or for the purpose of calibration of the model. In the following we propose various approximations for the correlation effect, which are computationally more accessible, and derive bounds for the error of these approximations. For this we consider maturities $T-t<1$ and assume that $\sigma^2<1$. From a financial point of view both assumptions are reasonable, as market parameters are all denoted on a yearly scale and maturity times of options are mostly less than one year, while annual volatility is usually in the range of less than 10%.

Lemma 5.1. *Assume* $2\kappa\theta \geq \nu^2$ *then* $\mathbb{E}\left[\left(\int_t^T \sigma_s^2 ds\right)^{-\frac{1}{2}} \middle| \mathcal{F}_t\right] \leq \frac{C(\sigma_t)}{(T-t)}$ *where* $C(\sigma_t)$ *is a constant depending on the current level of volatility but not on* t *explicitly.*

Proof. Since we are in a Markovian framework we can assume w.l.o.g. that $t=0$ and replace all conditional expectations by their unconditional counterparts. Using the identity $\frac{1}{x^\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty u^{\alpha-1}\exp(-ux)du$ while choosing $x=\int_0^T \sigma_s^2 ds$ and $\alpha = \frac{1}{2}$ we conclude that

$$\mathbb{E}\left[\left(\int_0^T \sigma_s^2 ds\right)^{-\frac{1}{2}}\right] = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^\infty u^{-\frac{1}{2}}\mathbb{E}\left[\exp\left(-u\left(\int_0^T \sigma_s^2 ds\right)\right)\right] du. \tag{6}$$

Using that the Heston volatility is in fact a time-transformed and scaled squared

Bessel process, we can write

$$\begin{aligned}\mathbb{E}\left[\exp\left(-u\left(\int_0^T \sigma_s^2 ds\right)\right)\right] &\leq \mathbb{E}\left[\exp\left(-u\left(\int_0^T e^{-\kappa s}\eta\left(\frac{\nu^2}{4\kappa}\left(e^{\kappa s}-1\right)\right)ds\right)\right)\right]\\ &\leq \mathbb{E}\left[\exp\left(-u\left(\frac{4}{\nu^2}\int_0^{\frac{\nu^2}{4\kappa}\left(e^{\kappa T}-1\right)}\eta\left(m\right)dm\right)\right)\right]\end{aligned}$$

with η a squared Bessel process of dimension $\delta = \frac{4\kappa\theta}{\nu^2}$. The following formula is well known, see for example [1]:

$$\begin{aligned}\mathbb{E}\left[\exp\left(-u\left(\frac{4}{\nu^2}\int_0^{\frac{\nu^2}{4\kappa}\left(e^{\kappa T}-1\right)}\eta\left(\alpha\right)d\alpha\right)\right)\right] &\leq \left(\cosh\left(\frac{\nu}{2\kappa}\left(e^{\kappa T}-1\right)\sqrt{2u}\right)\right)^{-\frac{\delta}{2}}\\ &\quad\times\exp\left(-\frac{\sqrt{2u}}{\nu}\cdot\sigma_0^2\tanh\left(\frac{\nu}{2\kappa}\left(e^{\kappa T}-1\right)\sqrt{2u}\right)\right).\end{aligned}$$

Substituting this into (6) it follows that

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^T \sigma_s^2 ds\right)^{-\frac{1}{2}}\right] &\leq \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^\infty u^{-\frac{1}{2}}\left(\cosh\left(\frac{\nu}{2\kappa}\left(e^{\kappa T}-1\right)\sqrt{2u}\right)\right)^{-\frac{\delta}{2}}\\ &\quad\times\exp\left(-\frac{\sqrt{2u}}{\nu}\cdot\sigma_0^2\tanh\left(\frac{\nu}{2\kappa}\left(e^{\kappa T}-1\right)\sqrt{2u}\right)\right)du\end{aligned}$$

Now, by substitution of $\tilde{m} := \frac{\nu}{2\kappa}\left(e^{\kappa T}-1\right)\sqrt{2u}$ we obtain

$$\mathbb{E}\left[\left(\int_0^T \sigma_s^2 ds\right)^{-\frac{1}{2}}\right] \leq \frac{C_1}{\left(e^{\kappa T}-1\right)}\int_0^\infty \left(\cosh\left(\tilde{m}\right)\right)^{-\frac{\delta}{2}}\exp\left(-\frac{\tilde{m}\sqrt{2}\kappa}{\nu^2\left(e^{\kappa T}-1\right)}\sigma_0^2\tanh\left(\tilde{m}\right)\right)d\tilde{m}$$

with $C_1 := \frac{2\sqrt{2}\kappa}{\nu}$ a constant. It is not difficult to see that since $\delta \geq 2$ the integral on the right hand side is finite and the last inequality can be written as $\mathbb{E}\left[\left(\int_0^T \sigma_s^2 ds\right)^{-\frac{1}{2}}\right] \leq \frac{C_1 \cdot I(\sigma_0)}{(e^{\kappa T}-1)}$ where $I(\sigma_0)$ denotes the value of the integral. Now we can use the fact that for positive κ we have $\left(e^{\kappa T}-1\right) \geq \kappa T$ and obtain $\mathbb{E}\left[\left(\int_0^T \sigma_s^2 ds\right)^{-\frac{1}{2}}\right] \leq \frac{C_1 \cdot I(\sigma_0)}{(e^{\kappa T}-1)} \leq \frac{C_1 \cdot I(\sigma_0)}{\kappa T} = \frac{C(\sigma_0)}{T}$ with $C(\sigma_0) = \frac{C_1 \cdot I(\sigma_0)}{\kappa}$.

Proposition 5.2. *Consider the Heston model and assume that* $2\kappa\theta \geq 3\nu^2$. *For* $t \in [0,T]$ *there exists a constant* $C(\sigma_t)$ *which does not depend on* t,ν *and* ρ *explicitly, such that* $\left|V_t - \mathbb{E}\left(BS\left(t, X_t; \vartheta_t\right) + \frac{\rho}{2} H\left(t, X_t, \vartheta_t\right)\left(\int_t^T \Lambda_s ds\right)\middle| \mathcal{F}_t\right)\right| \leq C(\sigma_t)\nu^2\rho^2(T - t)$.

Proof. It follows from Proposition 5.1 that

$$\begin{aligned}&\left|V_t - \mathbb{E}\left(BS\left(t, X_t, \vartheta_t\right) + \frac{\rho}{2} H\left(t, X_t, \vartheta_t\right)\left(\int_t^T \Lambda_s ds\right)\middle| \mathcal{F}_t\right)\right| \\ =\ &\left|\mathbb{E}\left(\frac{\rho}{2}\int_t^T e^{-r(s-t)} H\left(s, X_s, \vartheta_s\right)\Lambda_s ds - \frac{\rho}{2} H\left(t, X_t, \vartheta_t\right)\left(\int_t^T \Lambda_s ds\right)\middle| \mathcal{F}_t\right)\right| .\end{aligned}$$

Let us now consider the process $\frac{\rho}{2} e^{-rt} H\left(t, X_t, \vartheta_t\right)\left(\int_t^T \Lambda_u du\right)$. Obviously this process vanishes at $t = T$ and it follows from Proposition 3.1. as in the proof of Proposition 7 in [3] that

$$\begin{aligned}&\mathbb{E}\left\{\frac{\rho}{2}\int_t^T e^{-r(s-t)} H\left(s, X_s, \vartheta_s\right)\Lambda_s ds - \frac{\rho}{2} H\left(t, X_t, v_t\right)\left(\int_t^T \Lambda_u du\right)\middle| \mathcal{F}_t\right\} \\ =\ &\mathbb{E}\left\{\frac{\rho^2}{8}\int_t^T e^{-r(s-t)} G\left(s, X_s, \vartheta_s\right)\left(\int_s^T \Lambda_r dr\right)\Lambda_s ds\right. \\ &\left. + \frac{\rho^2}{4}\int_t^T e^{-r(s-t)} \frac{\partial H}{\partial x}\left(s, X_s, \vartheta_s\right)\left(\int_s^T D_s\Lambda_r dr\right)\sigma_s ds \middle| \mathcal{F}_t\right\} =: A_1 + A_2\end{aligned}$$

with $G(s, X_s, \vartheta_s) = \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right) H(s, X_s, \vartheta_s)$ and A_1 resp. A_2 the corresponding summands above. Let $\mathcal{G}_t$ be the $\sigma-$algebra generated by the Brownian motion (W_t) which drives the Heston volatility. Now the proof will be decomposed into two steps.

Step 1. Let us study the term A_1. From Lemma 2 in [4] we conclude that

$$\left|\mathbb{E}\left(\frac{\partial^n}{\partial x^n}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) BS\left(s, X_s, \vartheta_s\right)\middle| \mathcal{G}_t\right)\right| \leq C(\sigma_t)\rho\left(\int_t^T \sigma_s^2 ds\right)^{-\frac{1}{2}(n+1)} . \qquad (7)$$

Here $C(\sigma_t)$ is a constant whose value depends on the current σ_t. The fact that $D_r\sigma_\theta^2 = 2\sigma_\theta D_r\sigma_\theta$ and Hölder's inequality allow us to write

$$\int_t^T\left(\int_s^T \Lambda_r dr\right)\Lambda_s ds \leq \left(\int_t^T \sigma_s^2 ds\right)^2\left(\int_t^T\left(\int_r^T \left(D_r\sigma_\theta\right)^2 d\theta\right) dr\right) \qquad (8)$$

Then, (7) and (8) yield

$$\begin{aligned} A_1 &\leq C\frac{\rho^2}{8}\mathbb{E}\left[\left(\left(\int_t^T \sigma_s^2 ds\right)^{-5/2} + \left(\int_t^T \sigma_s^2 ds\right)^{-2} + \left(\int_t^T \sigma_s^2 ds\right)^{-3/2}\right)\int_t^T \left(\int_s^T \Lambda_r dr\right)\Lambda_s ds \middle| \mathcal{F}_t\right] \\ &\leq C\frac{\rho^2}{8}\mathbb{E}\left[\left(1 + \left(\int_t^T \sigma_s^2 ds\right)^{-1/2} + \left(\int_t^T \sigma_s^2 ds\right)^{1/2}\right)\left(\int_t^T \left(\int_r^T (D_r\sigma_\theta)^2 d\theta\right) dr\right) \middle| \mathcal{F}_t\right] \end{aligned}$$

and now, using the fact that $(D_r\sigma_\theta)^2$ is bounded by ν^2 it follows that

$$A_1 \leq C\nu^2\rho^2 (T-t)^2 \mathbb{E}\left[\left(1 + \left(\int_t^T \sigma_s^2 ds\right)^{-1/2} + \left(\int_t^T \sigma_s^2 ds\right)^{1/2}\right) \middle| \mathcal{F}_t\right].$$

The fact that $\mathbb{E}\left(\int_t^T \sigma_s^2 ds\right)^{1/2}$ is finite and Lemma 5.1, as well as $T - t < 1$ now imply that $A_1 \leq C(\sigma_t)\nu^2\rho^2 (T-t)$.

Step 2. Let us study the term A_2. Using again Hölder's inequality we can write

$$\begin{aligned} \int_t^T \left(\int_s^T D_s\Lambda_r dr\right)\sigma_s ds \leq \left(\int_t^T \sigma_s^2 ds\right)\int_t^T \int_r^T (D_r\sigma_\alpha)^2 d\alpha dr \\ + \left(\int_t^T \sigma_s^2 ds\right)^{\frac{3}{2}} \left(\int_t^T \left(\int_s^T \left(\int_r^T (D_sD_r\sigma_\alpha)^2 d\alpha\right) dr\right) ds\right)^{\frac{1}{2}} \end{aligned} \tag{9}$$

Then, using (7) and (9) in a similar way as in *Step 1* we obtain

$$\begin{aligned} A_2 &\leq \frac{\rho^2}{4}\mathbb{E}\left[\left(1 + \left(\int_t^T \sigma_s^2 ds\right)^{-\frac{1}{2}}\right)\left(\int_t^T \int_r^T (D_r\sigma_\alpha)^2 d\alpha dr\right) \middle| \mathcal{F}_t\right] \\ &\quad + C(\sigma_t)\frac{\rho^2}{4}\mathbb{E}\left[\int_t^T \left(\int_s^T \left(\int_r^T (D_sD_r\sigma_\alpha)^2 d\alpha\right) dr\right) ds \middle| \mathcal{F}_t\right] \end{aligned}$$

Now Proposition 4.1, Proposition 4.2. and our assumption $T - t < 1$ enable us to deduce that $A_2 \leq C(\sigma_t)\nu^2\rho^2(T-t)$.

Remark 5.1. Let us briefly illustrate how the result in Proposition 5.2. should be interpreted in a dynamic framework. As one can obviously see, the approximation is getting better with a quadratic rate, as the factor ν decreases. The situation is similar for ρ. As the constant $C(\sigma_t)$ however depends implicitly on t through σ_t we can not say, that as time to maturity decreases, our approximation is getting better in general. In fact a large change in the volatility during a trading day may lead to the result that our approximation tomorrow is in fact worse then today. This effect however is entirely caused by the random volatility. Putting aside this effect and fixing the volatility artificially in time, then the accuracy of the approximation increases at least linearly with decreasing time to maturity.

Let us now consider the following approximation for the correlation effect :

$$\frac{\rho}{2} H\left(t, X_t, \vartheta_t^*\right) \mathbb{E}\left(\left.\int_t^T \Lambda_s ds\right| \mathcal{F}_t\right) \tag{10}$$

with $\vartheta_t^* = \sqrt{\frac{1}{T-t} \int_t^T \mathbb{E}\left(\sigma_s^2 \middle| \mathcal{F}_t\right) ds}$ and as an approximation of the option price

$$BS\left(t, X_t; \vartheta_t^*\right) + \frac{\rho}{2} H\left(t, X_t, \vartheta_t^*\right) \mathbb{E}\left(\left.\left(\int_t^T \Lambda_s ds\right)\right| \mathcal{F}_t\right) \tag{11}$$

We will later need the following lemma, which is related to equation (7), but for the specific case considered here gives a slightly better approximation.

Lemma 5.2. *Let $BS(t, x, \sigma)$ denote the Black-Scholes price in the log-stock price x. Then there exists a constant such that for all times to maturity $T - t < 1$ we have*

$$\left|\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)^2 BS\left(t, x, \sigma\right)\right| \leq C\sigma^{-2}(T-t)^{-\frac{3}{2}}$$

Proof. Applying the chain rule of differential calculus with $S = e^x$, the well known formulas for the greeks delta and vega can be used to obtain

$$\begin{aligned}
\frac{\partial}{\partial x} BS(t, x, \sigma) &= N(d_1) e^x \\
\frac{\partial^2}{\partial x^2} BS(t, x, \sigma) &= \left(\frac{\partial}{\partial x}\left(N(d_1)\right)\right) + N(d_1)\right) e^x = \left(\frac{N'(d_1)}{\sigma\sqrt{T-t}} + N(d_1)\right) e^x.
\end{aligned}$$

where d_1 denotes the classical Black-Scholes parameter. Therefore

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) BS\left(t, x, \sigma\right) = \frac{N'(d_1)}{\sigma\sqrt{T-t}} e^x$$

Further differentiation now shows that

$$\begin{aligned}\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\right)^2 BS\left(t,x,\sigma\right) &= \left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\right)\left(\frac{N'(d_1)}{\sigma\sqrt{T-t}}e^x\right)\\ &= \left(\frac{N'''(d_1)}{\sigma^2(T-t)^{3/2}}+\frac{N''(d_1)}{\sigma(T-t)}\right)e^x\end{aligned}$$

The result then follows, since all derivatives of the standard normal distribution function $N(x)$ are bounded and furthermore $(T-t)^{3/2}$ dominates $(T-t)$ for $T-t<1$.

The following proposition represents an analytical result on the quality of this approximation.

Proposition 5.3. *Assume that* $2\kappa\theta \geq 3\nu^2$ *and define* $\vartheta_t^* = \sqrt{\frac{1}{T-t}\int_t^T \mathbb{E}\left(\sigma_s^2 \middle| \mathcal{F}_t\right) ds}$ *for* $t\in[0,T]$*. Then there exists a constant* $C(\sigma_t)$ *which does not depend explicitly on* t *and* ν *s.t.* $\left|V_t - BS\left(t,X_t;\vartheta_t^*\right)-\frac{\rho}{2}H\left(t,X_t,\vartheta_t^*\right)\mathbb{E}\left(\left(\int_t^T \Lambda_s ds\right)\middle|\mathcal{F}_t\right)\right| \leq C(\sigma_t)\nu^2(T-t)$

Proof. We can write

$$\begin{aligned}&\left|V_t - BS\left(t,X_t;\vartheta_t^*\right)-\frac{\rho}{2}H\left(t,X_t,\vartheta_t^*\right)\mathbb{E}\left(\left(\int_t^T \Lambda_s ds\right)\middle|\mathcal{F}_t\right)\right|\\ \leq\;& \left|V_t - \mathbb{E}\left(BS\left(t,X_t;\vartheta_t\right)+\frac{\rho}{2}H\left(t,X_t,\vartheta_t\right)\left(\int_t^T \Lambda_s ds\right)\middle|\mathcal{F}_t\right)\right|\\ &+\left|\mathbb{E}\left(BS\left(t,X_t;\vartheta_t\right)\middle|\mathcal{F}_t\right)-BS\left(t,X_t;\vartheta_t^*\right)\right|\\ &+\frac{\rho}{2}\left|\mathbb{E}\left(\left(H\left(t,X_t,\vartheta_t\right)-H\left(t,X_t,\vartheta_t^*\right)\right)\left(\int_t^T \Lambda_s ds\right)\middle|\mathcal{F}_t\right)\right|\\ =\;& B_1+B_2+B_3.\end{aligned}$$

with B_1, B_2 and B_3 the corresponding summands from above. We conclude from the last proposition that $B_1 \leq C(\sigma_t)\nu^2\rho^2(T-t)$ and we are left with the expressions B_2 and B_3. Let us study the expression B_2 first. Notice that

$$\vartheta_t^* = \sqrt{\frac{1}{T-t}\left(M_t - \int_0^t \sigma_s^2 ds\right)}, \vartheta_t = \sqrt{\frac{1}{T-t}\left(M_T - \int_0^t \sigma_s^2 ds\right)}$$

where $M_t := \int_0^T \mathbb{E}\left(\sigma_s^2 \middle| \mathcal{F}_t\right) ds$. It is not difficult to verify the following:

$$
\begin{aligned}
M_t &= \int_t^T \left[\sigma_t^2 e^{-\kappa(s-t)} + \theta\left(1 - e^{-\kappa(s-t)}\right)\right] ds + \int_0^t \sigma_s^2 ds \qquad (12) \\
dM_t &= \int_t^T \left[\kappa\sigma_t^2 e^{-\kappa(s-t)} dt + e^{-\kappa(s-t)} d\sigma_t^2 - \kappa\theta e^{-\kappa(s-t)}\right] ds = \nu\sigma_t \left(\int_t^T e^{-\kappa(s-t)} ds\right) dW_t
\end{aligned}
$$

Using the classical Itô formula and the relationship between the Greeks

$$
\frac{\partial BS}{\partial \sigma}(s, x, \sigma) \frac{1}{\sigma(T-s)} = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) BS(s, x, \sigma) \qquad (13)
$$

we deduce that

$$
\begin{aligned}
B_2 &= \mathbb{E}\left(BS(t, X_t; \vartheta_t) \middle| \mathcal{F}_t\right) - BS(t, X_t; \vartheta_t^*) \\
&= \mathbb{E}\left(BS\left(t, X_t; \sqrt{\frac{1}{T-t}\left(M_T - \int_0^t \sigma_s^2 ds\right)}\right) \middle| \mathcal{F}_t\right) \\
&\quad - \mathbb{E}\left(BS\left(t, X_t; \sqrt{\frac{1}{T-t}\left(M_t - \int_0^t \sigma_s^2 ds\right)}\right) \middle| \mathcal{F}_t\right) \\
&= \nu^2 \mathbb{E}\left(\int_t^T \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)^2 BS\left(t, X_t; \sqrt{\frac{1}{T-t}\left(M_u - \int_0^t \sigma_s^2 ds\right)}\right)\right. \\
&\qquad \left. \times \left(\int_u^T e^{-k(s-u)} ds\right)^2 \sigma_u^2 du \middle| \mathcal{F}_t\right)
\end{aligned}
$$

We can now conclude from lemma 5.2. that

$$
\begin{aligned}
B_2 &\leq \nu^2 \mathbb{E}\left(\int_t^T C\left[\frac{1}{T-t}\left(M_u - \int_0^t \sigma_s^2 ds\right)\right]^{-1} \left(\int_u^T e^{-\kappa(s-u)} ds\right)^2 \sigma_u^2 du \middle| \mathcal{F}_t\right) \\
&= C\nu^2 (T-t) \mathbb{E}\left(\int_t^T \left[\int_t^T \mathbb{E}\left(\sigma_s^2 \middle| \mathcal{F}_u\right) ds\right]^{-1} \left(\int_u^T e^{-\kappa(s-u)} ds\right)^2 \sigma_u^2 du \middle| \mathcal{F}_t\right).
\end{aligned}
$$

Now, using that $t < u$, the definition of M_t and equation (12) we obtain

$$
\begin{aligned}
\left[\int_t^T \mathbb{E}\left(\sigma_s^2 \middle| \mathcal{F}_u\right) ds\right]^{-1} &\leq \left[\int_u^T \mathbb{E}\left(\sigma_s^2 \middle| \mathcal{F}_u\right) ds\right]^{-1} \\
&= \left[\int_u^T \left(\sigma_u^2 e^{-\kappa(s-u)} + \theta\left(1 - e^{-\kappa(s-u)}\right)\right)\right]^{-1} \\
&\leq \sigma_u^{-2} \left(\int_u^T e^{-\kappa(s-u)} ds\right)^{-1}
\end{aligned}
$$

as $\theta\left(1-e^{-\kappa(s-u)}\right) \geq 0$ for all $s \geq u$. Back-substitution gives

$$\begin{aligned} B_2 &\leq C\nu^2(T-t)\mathbb{E}\left(\int_t^T \sigma_u^{-2}\left(\int_u^T e^{-\kappa(s-u)}ds\right)^{-1}\left(\int_u^T e^{-\kappa(s-u)}ds\right)^2 \sigma_u^2 du \middle| \mathcal{F}_t\right) \\ &\leq C\nu^2(T-t)\int_t^T\int_u^T e^{-\kappa(s-u)}dsdu \leq \frac{C}{\kappa}\nu^2(T-t)^2 \end{aligned}$$

The latter is bounded from above by $\frac{C}{\kappa}\nu^2(T-t)$ for all $T-t<1$. Let us finally consider the term B_3. Proposition 3.1 and (13) imply that

$$\begin{aligned} B_3 &= \mathbb{E}\left(\left(H\left(t,X_t,\vartheta_t\right)-H\left(t,X_t,\vartheta_t^*\right)\right)\left(\int_t^T \Lambda_s ds\right)\middle| \mathcal{F}_t\right) \\ &= \nu\mathbb{E}\left[\int_t^T\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\right)H\left(t,X_t,\sqrt{\frac{1}{T-t}\left(M_u-\int_0^t \sigma_s^2 ds\right)}\right)\right. \\ &\qquad\qquad \left.\times\left(D_u\int_t^T \Lambda_s ds\right)\sigma_u\left(\int_u^T e^{-\kappa(s-u)}ds\right)du\middle| \mathcal{F}_t\right] \\ &+ \nu\mathbb{E}\left[\int_t^T\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\right)^2 H\left(t,X_t,\sqrt{\frac{1}{T-t}\left(M_u-\int_0^t \sigma_s^2 ds\right)}\right)\right. \\ &\qquad\qquad \left.\times\left(\int_t^T \Lambda_s ds\right)\sigma_u^2\left(\int_u^T e^{-\kappa(s-u)}ds\right)^2 du\middle| \mathcal{F}_t\right] \end{aligned}$$

Now, similar arguments as used for B_2 give us that $B_3 \leq C(\sigma_t)\nu^2\left(T-t\right)$.

Let us now make things more transparent by evaluating the expression

$$\frac{1}{2}H\left(t,X_t,\vartheta_t^*\right)\mathbb{E}\left(\int_t^T \Lambda_s ds\middle| \mathcal{F}_t\right) \tag{14}$$

which determines the effect of correlation on option prices in the Heston model. We have to evaluate ϑ_t^* and $\mathbb{E}\left(\int_t^T \Lambda_s ds\middle| \mathcal{F}_t\right)$. Since the framework is a Markovian one, we can assume without loss of generality that $t=0$. In this case we have to evaluate the quantities $\vartheta_0^* = \sqrt{\frac{1}{T}\int_0^T \mathbb{E}\left(\sigma_s^2\right)ds}$ and $\mathbb{E}\left(\int_0^T \Lambda_s ds\right)$. Let us start with the computation of ϑ_t^*. It follows from $\sigma_s = \sqrt{v_t}$ and the dynamics of (v_t) by taking expectations and solving the corresponding ordinary ODE for the expectation, that

$\mathbb{E}(\sigma_s^2) = \mathbb{E}(v_s) = \theta + (v_0 - \theta)e^{-\kappa s}$. From this it follows that

$$(\vartheta_0^*)^2 = \frac{1}{T}\int_0^T \mathbb{E}\left(\sigma_s^2\right) ds = \theta + \frac{(v_0-\theta)}{T}\int_0^T e^{-\kappa s} ds = \theta + \frac{(v_0-\theta)\left(1-e^{-\kappa T}\right)}{\kappa T}.$$

Now consider the expression $\mathbb{E}\left(\int_0^T \Lambda_s ds\right)$. By definition of Λ_s we have that

$$\mathbb{E}\left(\int_0^T \Lambda_s ds\right) \quad = \quad \mathbb{E}\left(\int_0^T \left(\int_s^T \mathbb{E}\left(D_s\sigma_r^2\middle|\mathcal{F}_s\right) dr\right)\sigma_s ds\right)$$

Lemma 5.3. *Assume* $2\kappa\theta \geq \nu^2$ *then* $\mathbb{E}\left(D_s\sigma_r^2\middle|\mathcal{F}_s\right) = \nu\exp\left(-\kappa\left(r-s\right)\right)\sqrt{v_s}$.

Proof. Notice that it follows from Corollary 4.2 and the Clark-Ocone formula that

$$v_r = \sigma_r^2 = \mathbb{E}(\sigma_r^2) + \int_0^r \mathbb{E}\left(D_s\sigma_r^2\middle|\mathcal{F}_s\right) dW_s \tag{15}$$

On the other hand consider the process defined by the stochastic integral equation

$$\tilde{v}_r \quad = \quad \theta + (v_0-\theta)e^{-\kappa r} + \nu\int_0^r \exp\left(-\kappa\left(r-s\right)\right)\sqrt{\tilde{v}_s} dW_s.$$

Taking differentials of $\tilde{v}_r$ leads to

$$\begin{aligned} d\tilde{v}_r &= -\kappa\left[(v_0-\theta)e^{-\kappa r} + \nu\int_0^r \exp\left(-\kappa\left(r-s\right)\right)\sqrt{\tilde{v}_s}dW_s\right] dr + \nu\sqrt{\tilde{v}_r}dW_r \\ &= \kappa\left(\theta - \tilde{v}_r\right) + \nu\sqrt{\tilde{v}_r}dW_r \end{aligned}$$

We therefore see that $(\tilde{v}_r)$ has the same differential as (v_r) and since $\mathbb{E}(\tilde{v}_r) = \mathbb{E}(v_r)$ we have $\tilde{v}_r = v_r$. This leads to

$$\mathbb{E}(\sigma_r^2) + \int_0^r \mathbb{E}\left(D_s\sigma_r^2\middle|\mathcal{F}_s\right) = \mathbb{E}(v_r) + \nu\int_0^r \exp\left(-\kappa(r-s)\right)\sqrt{v_s}dW_s$$

and since $\mathbb{E}(\sigma_r^2) = \mathbb{E}(v_r)$ Lemma 5.2 follows from the uniqueness of this representation.

By application of Lemma 5.2. we now obtain

$$\mathbb{E}\left(\int_0^T \Lambda_s ds\right) = \mathbb{E}\left(\int_0^T \left(\int_s^T \mathbb{E}\left(D_s\sigma_r^2 \middle| \mathcal{F}_s\right) dr\right)\sigma_s ds\right)$$
$$= \nu\mathbb{E}\left(\int_0^T \left(\int_s^T \exp\left(-\kappa\left(r-s\right)\right) dr\right)\sigma_s^2 ds\right) = \nu\int_0^T \left(\int_s^T \exp\left(-\kappa\left(r-s\right)\right) dr\right)\mathbb{E}\left(\sigma_s^2\right) ds$$
$$= \nu\int_0^T \left(\int_s^T \exp\left(-\kappa\left(r-s\right)\right) dr\right)\left(\theta+(v_0-\theta)e^{-\kappa s}\right) ds$$

These integrals can easily be evaluated and we obtain

$$\mathbb{E}\left(\int_0^T \Lambda_s ds\right) = \frac{\nu}{\kappa^2}\cdot\left[\theta(\kappa-2)+v_0+e^{-\kappa T}\left(\kappa T(\theta-v_0)+2\theta-v_0\right)\right].$$

With these explicit expressions for ϑ_0^* and $\mathbb{E}\left(\int_0^T \Lambda_s ds\right)$ expression (14) which by the previous discussion approximates the effect of correlation on option prices becomes semi explicit, depending on the corresponding option valuation formula in the Black-Scholes model. If this value does not admit an explicit expression one can use Monte Carlo methods in order to compute it. For a standard European call option however we derive a fully explicit expression, where H is given by

$$\frac{1}{2}H\left(0,x,\sigma\right) = \frac{e^x}{2\sigma\sqrt{2\pi T}}\exp\left(-\frac{d_1^2}{2}\right)\left(1-\frac{d_1}{\sigma\sqrt{T}}\right),$$

with $d_1 = \frac{x-\ln K+rT}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$. The effect of correlation on option prices using our approximation can then be obtained in explicit form by substituting the corresponding expressions above in (18). The case of a European call is important, not as a particular application of our method in practice, but in order to test its quality, with respect to the benchmark [8]. The following figures illustrates the goodness of our approximation. Figure 1 represents the error of our approximation from Proposition 5.3 relative to the option price computed using a standard analytic Heston pricer for plain vanilla calls, such as it is available at *http://kluge.in-chemnitz.de/tools/pricer/*. The model parameters have been chosen as $\kappa = 8$,

$\theta = 0.04$, $\nu = 0.1$, $r = 0.0953$, $\sigma_0^2 = 0.0225$, $S_0 = 100$, $T = 0.1$ and $K = 100$. Figure 1 documents that our approximation is rather accurate. The figure in the lower left corner indicates that the larger part of the error is produced by replacing $\mathbb{E}\left(BS\left(t, X_t, \vartheta_t\right) \middle| \mathcal{F}_t\right)$ in Proposition 5.1 with $BS\left(t, X_t; \vartheta_t^*\right)$ in Proposition 5.3, while the error contributed by our approximation of the correlation effect decreases to zero as the correlation ρ decreases to zero. Figure 2 shows the dependence of the accuracy of our approximation on time to maturity $T - t$. The second graph in particular shows a linear relationship for small times to maturity, as predicted by Proposition 5.3. Figure 3 shows total error and percentage error as function of time to maturity, where parameters has changed to $\kappa = 2$, $\theta = 0.015$, $\nu = 0.2$, which violates the strong coefficients assumption $2\kappa\theta \geq 3\nu^2$ in Proposition 5.3. Comparing with figure 2 we see that absolute and percentage errors are significantly higher and do not appear to flatten out in the observed time interval.

6. Conclusions

We have proved that under the usual coefficient condition $2\kappa\theta \geq \nu^2$ the Heston stochastic volatility v_t as well as its square root σ_t are Malliavin differentiable and have given compact formulas for their derivatives. Under stricter conditions on the coefficients we have shown that the second Malliavin derivatives also exist. These two results are key results in so far as that they open the door for applications of Malliavin calculus in the framework of the Heston stochastic volatility model. We have discussed an explicit application by deriving and approximate option pricing formula for the Heston model, which is extremely accurate and easy to compute. Furthermore we derived analytical expressions which control the error of this approximation.

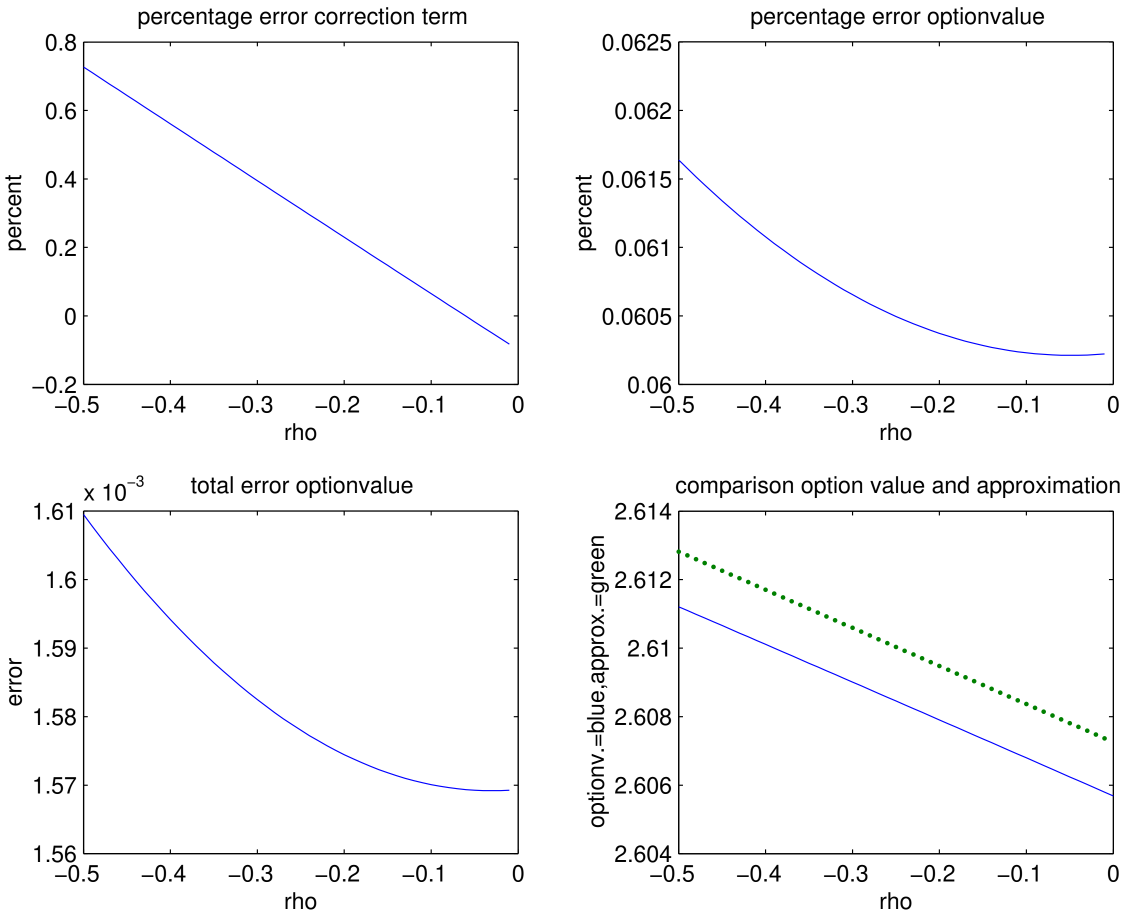

FIGURE 1: Error of approx. from Prop. 5.3 as function of ρ

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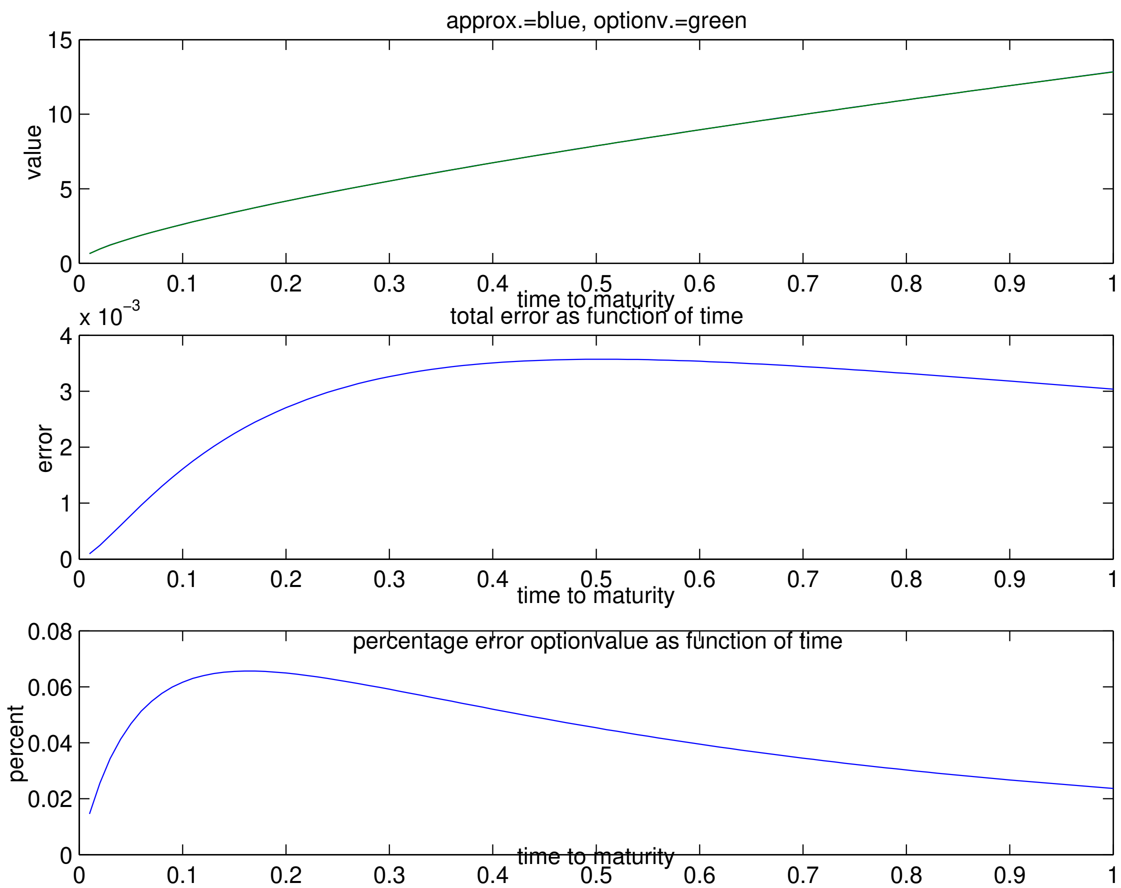

FIGURE 2: Error of approx. from Prop. 5.3 as function of $T - t$

and the "2'nd Workshop on Stochastic Equations" in Jena 2006. The second author wishes to thank the Organizers Prof. Schael, Prof. Riedel as well as Prof. Engelbert and gratefully acknowledges travel grants from the DFG and the Royal Society. Both authors wish to thank an anonymous referee for carefully reading an earlier version of this paper and giving very helpful comments and advice, which helped to improve this paper.

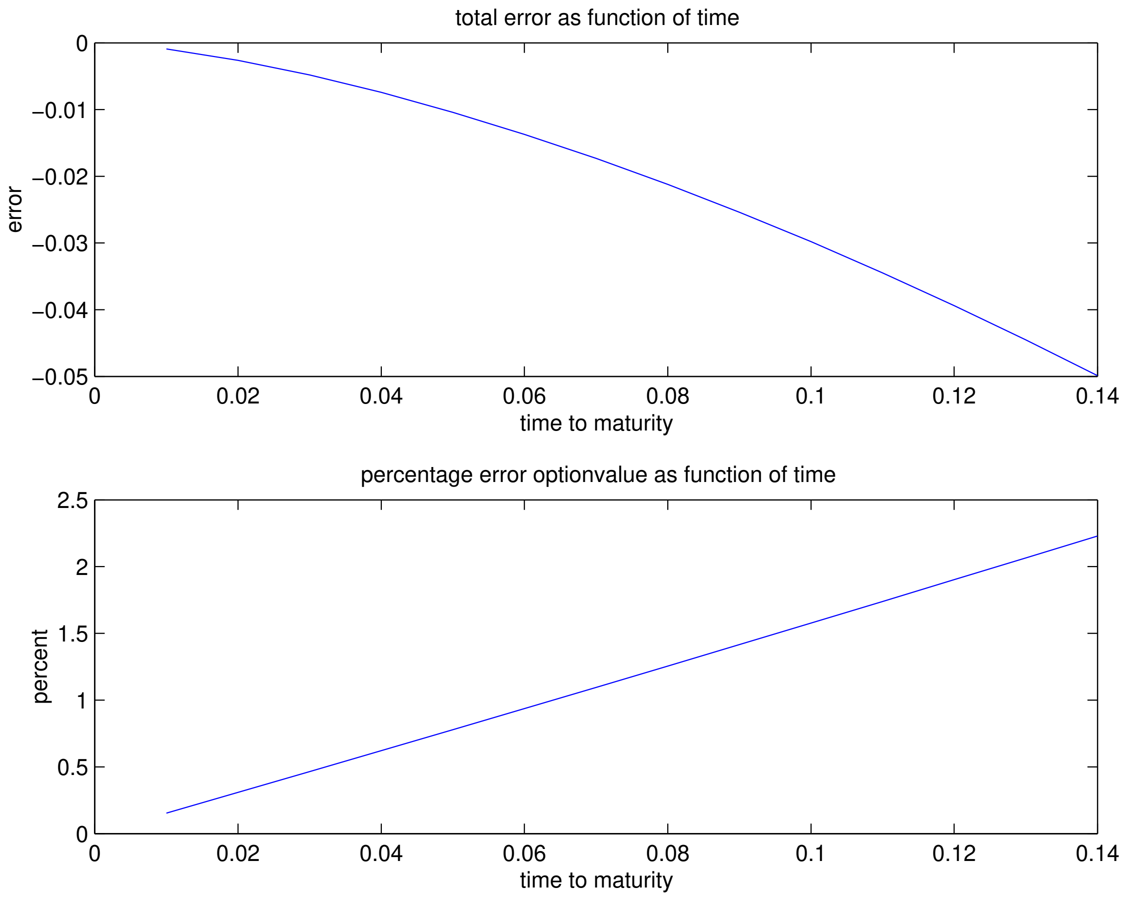

FIGURE 3: Error of approx. from Prop. 5.3 as function of $T - t$ with $2\kappa\theta < 3\nu^2$

References

[1] Borodin, A.N., Salminen, P. Handbook of Brownian motion-facts and formulae. Second edition. Probability and its Applications. Birkhäuser Verlag, Basel (2002).

[2] Bossy, M. Diop, A. An efficient discretization scheme for one dimensional SDE's with a diffusion coefficient function of the form $|x|^\alpha$,$\alpha \in [1/2, 1)$. Rapport de recherche, Institut Nationale de Recherche en Informatique et en Automatique (INRIA), No. 5396, Decembre 2004

[3] Alos, E. An extension of the Hull and White formula with applications to option pricing approximation. Finance and Stochastics, volume 10 (3), pg.353-365 (2006)

[4] Alos,E; Leon, J.A.; Vives, J. On the shorttime behavior of the implied volatility for jump-diffusion models with stochastic volatility. UPF working paper. (2006)

[5] Alos, E.; Nualart, D. An extension of Itô's formula for anticipating processes. Journal of Theoretical Probability, 11 (2) (1998)

[6] Detemple, J.; Garcia, R.; Rindisbacher, M. Representation formulas for Malliavin derivatives of diffusion processes. Finance and Stochastics Volume 9, Number 3 (2005)

[7] Geman, H. Bessel Processes, Asian Options and Perpetuities. Mathematical Finance, Vol. 3, No. 4 (1993)

[8] Heston, S.L.; A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options; Review of Financial Studies, Vol.6, Issue 2 (1993)

[9] Hull, J.; White, A. The Pricing of Options on Assets with Stochastic Volatilities. The Journal of Finance, Vol. XLII No. 2 (1987)

[10] Karatzas, I.; Shreve, S.-E. Brownian motion and stochastic calculus. Graduate Texts in Mathematics, 113. New York etc. Springer-Verlag (1988)

[11] Nualart, D. The Malliavin calculus and related topics. Probability and Its Applications. Springer-Verlag (1995).

[12] Yor, M. Some Aspects of Brownian Motion. Part 1 : Some Special Functionals. Lecture Notes in Math, ETH Zurich. Basel : Birkhaeuser (1992)