Condorcet admissibility: Indeterminacy and path-dependence under majority voting on interconnected decisions

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Indeterminacy and path-dependence under majority voting on interconnected decisions

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Abstract

Judgement aggregation is a model of social choice where the space of social alternatives is the set of consistent evaluations ('views') on a family of logically interconnected propositions, or yes/no-issues. Unfortunately, simply complying with the majority opinion in each issue often yields a logically inconsistent collection of judgements. Thus, we consider the *Condorcet set*: the set of logically consistent views which agree with the majority in as many issues as possible. Any element of this set can be obtained through a process of *diachronic* judgement aggregation, where the evaluations of the individual issues are decided through a sequence of majority votes unfolding over time, with earlier decisions possibly imposing logical constraints on later decisions. Thus, for a fixed profile of votes, the ultimate social choice can depend on the order in which the issues are decided; this is called *path dependence*. We investigate the size and structure of the Condorcet set —and hence the scope and severity of path-dependence —for several important classes of judgement aggregation problems.

In the context of preference aggregation, pairwise majority voting often fails to produce unambiguous outcomes because a Condorcet winner might not exist. Similarly, in the general judgement aggregation problem where each social outcome is described as a pattern of yes/no answers to certain interrelated propositions, issue-wise majority voting frequently fails to produce consistent collective judgement sets; see Guilbaud (1966) and Kornhauser and Sager (1986). This observation has sparked off the recent literature on judgement aggregation, starting from the influential contribution List and Pettit (2002); see List and Puppe (2009) for a survey.

In this paper, we propose a natural solution to the problem of inconsistency of issue-wise majority voting that is based entirely on majoritarian principles: the *Condorcet set.*

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A collection of yes/no answers to all propositions is called a 'view.' A logically consistent view is Condorcet admissible if there is no other logically consistent view that endorses a strictly larger set of propositions which are supported by a majority of voters. The Condorcet set is defined as the set of all Condorcet admissible views; it is thus the largest set of views which are defensible on purely 'majoritarian' grounds —in particular it always coincides with the issue-wise majority view whenever the latter is consistent.

The normative content of Condorcet admissibility is non-trivial. We illustrate this in the context of preference aggregation. In particular, we show that the optimal choices associated with any Condorcet admissible preference ordering are contained in the top cycle of the majority tournament. In this manner, Condorcet admissibility yields a foundation of a weakening of the criterion of Smith consistency in voting theory. Notably, Condorcet admissibility, like Smith consistency, excludes otherwise natural-looking and putatively majoritarian voting rules such as the Simpson-Kramer (maxmin) rule.

The Condorcet set naturally arises from a 'diachronic' interpretation of the aggregation problem in which decisions unfold over time and previous decisions potentially restrict later decisions due to the logical interrelation of propositions; a corresponding model of judgement aggregation has been studied by List (2004). Here, we show that a consistent view (over all issues) is an element of the Condorcet set if and only if there exists an ordering of issues (a 'path') such that the given view arises from 'sequential majority voting,' i.e. from deciding each issue by majority voting in the specified ordering provided that the issue is not yet determined by previous decisions. Evidently, an aggregation problem will exhibit *path-dependence* whenever the Condorcet set has at least two elements, i.e. whenever (simultaneous) issue-by-issue majority voting is inconsistent.[1]

From an applied viewpoint, the diachronic perspective on the judgement aggregation problem is important since, in reality, many social decisions are made in a piecemeal fashion, with different aspects decided at different times. For example, jurisprudence is constrained by a body of 'legal precedents' (past court decisions) which accumulate over time. It is often viewed as important or even essential that later court decisions be consistent with earlier legal precedents.[2] If the extant body of legal precedents is regarded as an expression of some 'social decision', then this social decision is not made instantaneously, but is accumulated over many decades.

Unlike courts, legislatures have the power to override their own past decisions and reverse previous legislation. However, in many cases, the political or economic costs of doing this are so great that prior legislation is effectively irreversible; thus, legislatures are often *de facto* constrained by their past decisions.

The main goal of this paper is to investigate the properties of the Condorcet set, considered as a solution concept for judgement aggregation. A natural question concerns the size of the Condorcet set and thus the extent of path-dependence. In a nutshell, we show that, frequently, the Condorcet set can be large. In particular, we focus on the question on when the Condorcet set at a given profile contains views that either affirm or negate

[1]It follows from results in Nehring and Puppe (2007), an aggregation problem exhibits path-*in*dependence, i.e. the Condorcet set is a singleton for all profiles, only under very restrictive conditions: it must have the structure of a median space (see Section 1 below).

[2]Of course, a higher court can overrule the decision of a lower court, but not vice versa.

any particular proposition. We refer to such profiles as *issue-wise indeterminate*, and to aggregation problems as *issue-wise indeterminate* if they admit an issue-wise indeterminate profile. The central result of the paper, Theorem 4.3, characterizes issue-wise indeterminate profiles and aggregation problems in terms of the combinatorial structure of the logical interrelations characterizing the aggregation space. Using this general characterization, we show that many interesting aggregation problems are indeed issue-wise indeterminate, and that such indeterminacy can arise quite easily. In fact, outside the class of median spaces there do not seem to be many interesting aggregation problems that are *not* issue-wise indeterminate. Our results can be viewed as counterparts to the classical results of McKelvey (1979) for general judgment aggregation problems.

An even stronger notion of *total indeterminacy* arises if the Condorcet set equals the entire set of consistent views. While this can never happen in preference aggregation problems, it can happen in some interesting aggregation problems, but apparently only in very special cases. On the other hand, based on a quantitative measure of 'Condorcet entropy,' we also show that quite often aggregation problems admit profiles that are 'almost' totally indeterminate.

Another way to assess the severity of path-dependence is in terms of violations of unanimous judgments. We characterize the aggregation problems for which sequential majority voting can lead to the situation that a previous majority decision force the rejection of a proposition that is *unanimously* accepted. We show that this can occur (for some distribution of individual judgments) unless all logical interrelation between propositions are confined to implications in which the antecedent consists of the conjunction of at most *two* propositions. Evidently, this is a very special case; the result thus shows that unanimity violations are indeed typical. In special cases, however, one can prevent unanimity violations to occur by an appropriate design of the decision path; in particular, we show how this can be done in the case of preference aggregation adapting the so-called 'multi-stage elimination' tree of Shepsle and Weingast (1984).

A subsidiary goal of the paper is to illustrate the unity and diversity of (majoritarian) judgment aggregation theory. To this purpose, we show how the same core concepts and abstract characterizations apply to range of paradigmatic judgment aggregation problems which include preference aggregation, classification via equivalence relations, committee selection and resource allocation, and problems of 'horizontal equity.' The latter reflects a fundamental requirement of the rule of law, dating back at least to Aristotle, that equal cases should be treated equally, and similar cases similarly.

This paper is organized as follows. In §1, we introduce terminology and notation. In §2, we explore path-dependence in some paradigmatic judgement aggregation problems: preference aggregation, committee selection, resource allocation, and models of horizontal equity. In §3, we characterize when Condorcet admissible elements can override even a unanimous consensus of voters on some issues. In §4 we consider *issue-wise indeterminacy*, where the Condorcet set manifests disagreement about the truth-value of every proposition. In §5, we study *total indeterminacy*, where the Condorcet set contains *every* admissible view. Finally, in §6, we introduce *Condorcet entropy*: a quantitative measure of how close to total indeterminacy a judgement aggregation problem is. For ease of reading, all but the simplest proofs are relegated to an Appendix at the end of the paper.

1 Condorcet admissibility

1.1 Definition and basic facts

Let $K \in \mathbb{N}$, and let $[1...K]$ index a set of propositions or issues. An element $\mathbf{x} = (x_1, x_2, \ldots, x_K) \in \{0,1\}^K$ is called a *view*, and interpreted as an assignment of a truth value of 'true' or 'false' to each proposition.[3] Not all views are feasible, because there will be logical constraints between the propositions (determined by the structure of the underlying decision problem faced by society). Let $X \subseteq \{0,1\}^K$ be the set of 'admissible' or *consistent* views; a given set $X \subseteq \{0,1\}^K$ is also referred to as an 'aggregation problem.' An *anonymous profile* is a probability measure on X —that is, a function $\mu : X \longrightarrow [0,1]$ such that $\sum_{\mathbf{x}\in X} \mu(\mathbf{x}) = 1$ with the interpretation that, for all $\mathbf{x} \in X$, $\mu(\mathbf{x})$ is the proportion of the voters who hold the view $\mathbf{x}$. For any $Y \subseteq X$, define $\mu(Y) := \sum_{\mathbf{y}\in Y} \mu(\mathbf{y})$. Let $\Delta(X)$ be the set of all anonymous profiles. For any $\mu \in \Delta(X)$, any $k \in [1...K]$, let

$$\mu_k(1) \quad := \quad \mu\{\mathbf{x} \in X \; ; \; x_k = 1\} \tag{1}$$

be the total 'popular support' for the position "$x_k = 1$,' and let $\mu_k(0) := 1 - \mu_k(1)$. Let $\Delta^*(X) := \{\mu \in \Delta(X);\ \mu_k(1) \neq \frac{1}{2},\ \forall k \in [1...K]\}$ be the set of anonymous profiles where there is a strict majority supporting either 0 or 1 in each coordinate.[4] An *anonymous judgement aggregation rule* is a correspondence (i.e. multivalued function) $F : \Delta^*(X) \rightrightarrows \{0,1\}^K$. Most of the rules we will consider are *single-valued*, taking the form of a function $F : \Delta^*(X) \longrightarrow \{0,1\}^K$. For example, the *propositionwise majoritarian* judgement aggregation rule $\mathrm{Maj} : \Delta^*(X) \longrightarrow \{0,1\}^K$ is defined as follows: for any $\mu \in \Delta^*(X)$, $\mathrm{Maj}_k(\mu) := 1$ if $\mu_k(1) > \frac{1}{2}$, and $\mathrm{Maj}_k(\mu) := 0$ if $\mu_k(1) < \frac{1}{2}$.

It is quite common to find that $\mathrm{Maj}(\mu) \notin X$ —the 'majority ideal' can be inconsistent with the underlying logical constraints faced by society.[5] However, a basic principle of majoritarianism is that we should try to satisfy the majority's will in as many coordinates as possible. Formally, for any $\mathbf{x} \in X$, denote by $M(\mathbf{x}, \mu) := \{k \in [1...K] \; ; \; x_k = \mathrm{Maj}_k(\mu)\}$ the set of coordinates in which $\mathbf{x}$ coincides with the 'majority will.' We want $M(\mathbf{x}, \mu)$ to be as large as possible. In the setting of preference aggregation (where the coordinates encode the orderings between pairs of alternatives), this principle was first advocated by Condorcet (1785). Thus, we will say that an element $\mathbf{x} \in X$ is *Condorcet admissible* if there does not exist any $\mathbf{y} \in X$ such that $M(\mathbf{x}, \mu) \subsetneq M(\mathbf{y}, \mu)$. Let $\mathrm{Cond}(X, \mu) \subseteq X$ be the set of Condorcet admissible elements. A profile μ is called *Condorcet determinate* if $\mathrm{Cond}(X, \mu)$ is single-valued.

[3]There are several different terms in the literature: e.g., Dokow and Holzman (2010) speak of 'binary evaluations,' List and Puppe (2009) use the term 'judgement set.'

[4]Usually, judgement aggregation is considered on *all* of $\Delta(X)$. However, our goal in this paper is to investigate the multiplicity of solutions in the Condorcet set of a single profile; for this goal it is convenient to eliminate the 'spurious' multiplicities which arise when $\mu_k(1) = \frac{1}{2}$ for some $k \in [1...K]$. Thus, we will confine our attention to profiles in $\Delta^*(X)$ for expositional simplicity. (If the set of voters is large (respectively odd), then a profile in $\Delta(X) \setminus \Delta^*(X)$ is highly unlikely (resp. impossible) anyways.)

[5]Condorcet's 'voting paradox' is a well-known instance; another example that has received considerable interest recently is the so-called 'discursive dilemma,' see Kornhauser and Sager (1986), List and Pettit (2002), and List and Puppe (2009) for an overview of the subsequent literature.

For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \{0,1\}^K$, say that $\mathbf{y}$ is *between* $\mathbf{x}$ and $\mathbf{z}$ if, for all $k \in [1...K]$, $(x_k = z_k = 0) \Longrightarrow (y_k = 0)$ and $(x_k = z_k = 1) \Longrightarrow (y_k = 1)$. Furthermore, $\mathbf{y}$ is said to be *properly between* $\mathbf{x}$ and $\mathbf{z}$ if, in addition, $\mathbf{x} \neq \mathbf{y} \neq \mathbf{z}$. For any $\mathbf{x} \in X$ and $\mathbf{z} \in \{0,1\}^K$, write $\mathbf{x} \asymp \mathbf{z}$ if there exists no $\mathbf{y} \in X$ which is properly between $\mathbf{x}$ and $\mathbf{z}$.

Lemma 1.1 **(a)** *If* $\mathrm{Maj}(\mu) \in X$, *then* $\mathrm{Cond}\,(X, \mu) = \{\mathrm{Maj}(\mu)\}$.

(b) *Otherwise,* $\mathrm{Cond}\,(X, \mu) \;=\; \{\mathbf{x} \in X \; ; \; \mathbf{x} \asymp \mathrm{Maj}(\mu)\}$.

In this case, $|\mathrm{Cond}\,(X, \mu)\,| \geq 3$.

1.2 Condorcet admissible aggregation rules

An aggregation rule $F : \Delta^*(X) \rightrightarrows \{0,1\}^K$ is *Condorcet admissible* if $F(\mu) \subseteq \mathrm{Cond}\,(X, \mu)$ for any X and μ. The concept of Condorcet admissibility can be viewed as a normative extension of Condorcet consistency, i.e. the requirement that the majority view is chosen whenever it is feasible. It can be viewed as summarizing the normative implications (for single profiles) of the majoritarian viewpoint *per se.*

Condorcet admissible rules arise naturally in the following way. Consider a *gain* function $\phi : \left[-\frac{1}{2}, \frac{1}{2}\right] \longrightarrow \mathbb{R}$, assumed to be non-decreasing and odd (i.e. $\phi(r) = -\phi(-r)$) with $\phi(r) < 0$ for all $r < 0$ and $\phi(r) > 0$ for all $r > 0$. For any such function, and strictly positive weights $\lambda_k > 0$ with $\sum_{k \in [1...K]} \lambda_k = 1$, define the *weighted ϕ-support rule* $F_{\phi,\lambda} : \Delta^*(X) \rightrightarrows X$ by

$$F_{\phi,\lambda}(\mu) \quad := \quad \operatorname*{argmax}_{\mathbf{x} \in X} \sum_{k \in [1...K]} \lambda_k \cdot \phi[\mu_k(x_k) - \tfrac{1}{2}]. \tag{2}$$

In particular, if $\phi(r) = \mathrm{sign}(r)$ for all $r \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $\lambda_k = 1/K$ for all k, then $F_{\phi,\lambda}$ is the *Slater rule*:

$$\mathrm{Slater}\,(X, \mu) \quad := \quad \operatorname*{argmax}_{\mathbf{x} \in X} \#\big\{k \in [1...K] \; ; \; \mu_k(x_k) > \tfrac{1}{2}\big\}. \tag{3}$$

The Slater rule selects the consistent views that maximize the *number* of issues in which there is agreement with the majority will. This rule was first suggested by Slater (1961) in the setting of Arrovian preference aggregation, in which it selects the transitive orderings that agree with the majority tournament in the largest number of binary comparisons.[6]

On the other hand, suppose $\phi(r) = r$ for all $r \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $\lambda_k = 1/K$ for all k; the corresponding $F_{\phi,\lambda}$ is called the *median rule:*

$$\mathrm{Median}\,(X, \mu) \quad = \quad \operatorname*{argmax}_{\mathbf{x} \in X} \sum_{k \in [1...K]} \mu_k(x_k).$$

The median rule selects the consistent view(s) that maximize(s) the sum of the popular support over each issue. In the setting of Arrovian preference aggregation, it corresponds

[6] An alternative description of the Slater rule is as follows. Let d_H be the *Hamming distance*, i.e. $d_H(\mathbf{x}, \mathbf{y}) = \#\{k \in [1...K] \; ; \; x_k \neq y_k\}$ for all $\mathbf{x}, \mathbf{y} \in \{0,1\}^K$. It is easily verified that $\mathrm{Slater}\,(X, \mu) = \mathrm{argmin}_{\mathbf{x} \in X} \; d_H[\mathbf{x}, \mathrm{Maj}(\mu)]$; that is, the Slater rule selects the consistent view(s) that minimize the Hamming distance to the propositionwise majority view.

to the Kemeny (1959) rule, which has been analyzed by Young and Levenglick (1978) and Young (1986, 1988, 1995, 1997). As a general-purpose judgement aggregation rule, the median rule has been studied among others by Barthélémy and Monjardet (1981, 1988), Barthélémy (1989) and Barthélémy and Janowitz (1991).[7]

Proposition 1.2 *Any weighted support rule $F_{\phi,\lambda}$ is Condorcet admissible. Conversely, for any $\mu \in \Delta^*(X)$, any $\mathbf{x} \in \mathrm{Cond}(X, \mu)$, and any gain function ϕ, there exist weights $\lambda_k > 0$ such that $F_{\phi,\lambda}(\mu) = \{\mathbf{x}\}$.*

Normatively, there will frequently be additional considerations (that are not properly majoritarian) that privilege some Condorcet admissible views over others. Of particular interest here are considerations of symmetry; these give prominence to unweighted support rules which are studied in detail in Nehring and Pivato (2011b).

1.3 Sequential majority voting

Condorcet admissibility is closely related to 'sequential' majority voting, as follows. For any $\mathbf{y} = (y_k)_{k=1}^K \in \{0,1\}^K$ and $J \subset [1...K]$, define $\mathbf{y}_J := (y_j)_{j \in J} \in \{0,1\}^J$. For any $i \in [1...K] \setminus J$, say that y_i is X-*consistent* with $\mathbf{y}_J$ if there exists some $\mathbf{x} \in X$ with $\mathbf{x}_J = \mathbf{y}_J$ and $x_i = y_i$ —otherwise y_i is X-*inconsistent* with $\mathbf{y}_J$. A *path* through $[1...K]$ is a bijection $\gamma : [1...K] \longrightarrow [1...K]$. We now define the γ-*sequential majority* rule $F^\gamma : \Delta^*(X) \longrightarrow X$. Let $\mu \in \Delta^*(X)$ and let $\mathbf{z} := \mathrm{Maj}(\mu)$. Define $\mathbf{y} := F^\gamma(\mu) \in X$ inductively as follows:

- Define $y_{\gamma(1)} := z_{\gamma(1)}$.
- Inductively, let $J := \{\gamma(1), \gamma(2), \ldots, \gamma(n)\}$, and suppose we have already decided $\mathbf{y}_J$. Let $i := \gamma(n+1)$. If z_i is X-consistent with $\mathbf{y}_J$, then set $y_i := z_i$. Otherwise, set $y_i = \neg z_i$.[8]

From a 'normative' perspective, the support rules (2) are interesting because they have many nice properties which may be desirable in certain sorts of judgement aggregation (Nehring and Pivato, 2011b). However, from a 'descriptive' perspective, sequential majority rules may be more relevant, because they describe the historical process through which decisions are often made in the real world. In this setting, a key issue is path-dependence; for a related approach to path-dependence in diachronic judgement aggregation, see List (2004).

The profile μ is *path-dependent* if there exist paths γ and ξ such that $F^\gamma(\mu) \neq F^\xi(\mu)$. Path dependence is pernicious in at least two ways:

- Suppose the path γ is random and exogenous (e.g. the decisions on individual coordinates in $[1...K]$ are made on an *ad hoc* basis, in response to political or legal exigencies of random origin). If μ is path-dependent, then the ultimate social choice $F^\gamma(\mu)$ will be, to some extent, random and arbitrary.

[7] It is easily verified that $\mathrm{Median}(X, \mu) = \mathrm{argmin}_{\mathbf{x} \in X} \sum_{\mathbf{y} \in X} \mu(\mathbf{y}) \cdot d_H(\mathbf{x}, \mathbf{y})$; that is, the median rule selects the consistent view(s) that minimize the *average* Hamming distance to the views of the voters.

[8] "$\neg$" represents logical negation. That is: $\neg 1 := 0$ and $\neg 0 := 1$.

- Suppose the path γ is chosen by an 'agenda setter' (e.g. the chairperson of a committee). If μ is path-dependent, then the agenda setter can choose γ strategically, so as to manipulate the outcome $F^\gamma(\mu)$.

Path dependence is closely related to the Condorcet set because of the next result:

Proposition 1.3 *Let $X \subseteq \{0,1\}^K$.*

(a) *For any path γ through $[1...K]$, the rule F^γ is Condorcet admissible.*

(b) *Conversely, for any $\mu \in \Delta^*(X)$ and $\mathbf{x} \in \mathrm{Cond}(X, \mu)$, there exists a path γ such that $F^\gamma(\mu) = \mathbf{x}$.*

A profile $\mu \in \Delta^*(X)$ is ***path-independent*** if $F^\gamma(\mu) = F^\xi(\mu)$ for any two paths γ and ξ through $[1...K]$. The space X itself is called ***path-independent*** if every $\mu \in \Delta^*(X)$ is path-independent. Furthermore, X is ***majoritarian-consistent*** if $\mathrm{Maj}(\mu) \in X$ for all $\mu \in \Delta^*(X)$, i.e. if all profiles are Condorcet determinate. Let $J \subseteq [1...K]$ and consider $\mathbf{w} \in \{0,1\}^J$ which corresponds to a subset of judgements on the issues in J. The set J is the ***support*** of $\mathbf{w}$, denoted $\mathrm{supp}(\mathbf{w})$. We define $|\mathbf{w}| := |J|$. If $I \subseteq J$ and $\mathbf{v} \in \{0,1\}^I$, then we say $\mathbf{v}$ is a ***fragment*** of $\mathbf{w}$ (and write $\mathbf{v} \sqsubseteq \mathbf{w}$) if $\mathbf{v} = \mathbf{w}_I$. Furthermore, $\mathbf{w}$ is a ***forbidden fragment*** for X if, for all $\mathbf{x} \in X$, we have $\mathbf{x}_J \neq \mathbf{w}$. Finally, $\mathbf{w}$ is a ***critical fragment*** if it is a minimal forbidden fragment —that is, $\mathbf{w}$ is forbidden, and there exists no proper subfragment $\mathbf{v} \sqsubset \mathbf{w}$ such that $\mathbf{v}$ is forbidden.[9] Let $W(X)$ be the set of critical fragments for X, and let $\kappa(X) := \max\{|\mathbf{w}|\ ;\ \mathbf{w} \in W(X)\}$.

A particular role is played by spaces $X \subseteq \{0,1\}^K$ for which $\kappa(X) = 2$; these are known as ***median spaces*** in combinatorial mathematics. Their importance in the theory of aggregation has been emphasized by (Barthélémy and Monjardet, 1981; Nehring and Puppe, 2007, 2010). Note that a set of feasible views X is a median space if and only if all logical interrelations are confined to *simple* implications: for some j, k and all $\mathbf{x} \in X$, $x_j = 0$ implies that $x_k = 0$, or $x_j = 0$ implies that $x_k = 1$.

Proposition 1.4 *Let $X \subseteq \{0,1\}^K$. The following are equivalent:* [i] *X is path-independent;* [ii] *X is Condorcet determinate;* [iii] *X is a median space, i.e. $\kappa(X) = 2$.*

Proof: "[i] $\iff$ [ii]" follows immediately from Lemma 1.1 and Proposition 1.3. "[ii] $\iff$ [iii]" follows from (Nehring and Puppe, 2007, Fact 3.4). □

Notation. We define the elements $\mathbf{0}^K := (0, 0, \ldots, 0)$ and $\mathbf{1}^K := (1, 1, \ldots, 1)$ in $\{0,1\}^K$ (we will simply write "$\mathbf{0}$" and "$\mathbf{1}$" when K is clear from context). For any subset $J \subseteq [1...K]$, let $\mathbf{1}_J$ denote the vector $\mathbf{x} \in \{0,1\}^K$ such that $x_j = 1$ for all $j \in J$ and $x_k = 0$ for all $k \in [1...K] \setminus J$.

[9]Critical fragments are called 'critical families' in Nehring and Puppe (2007, 2010) and 'minimal infeasible partial evaluations (MIPEs)' in Dokow and Holzman (2010) .

2 Examples

2.1 Preference aggregation

Let $N \in \mathbb{N}$, let $[1...N]$ be some set of N social alternatives, let $K := N(N-1)/2$, and bijectively identify $[1...K]$ with a subset of $[1...N] \times [1...N]$ which contains exactly *one* element of the set $\{(n, m), (m, n)\}$ for each distinct $n, m \in [1...N]$. Then $\{0,1\}^K$ represents the space of all ***tournaments*** (i.e. complete, irreflexive, antisymmetric binary relations, or equivalently, complete directed graphs) on $[1...N]$. Let $X_N^{\mathrm{pr}} \subset \{0,1\}^K$ be the set of all tournaments representing total orderings (i.e. permutations) of $[1...N]$ (sometimes called the ***permutahedron***). Classical Arrovian aggregation of strict preference orderings is simply judgement aggregation on X_N^{pr}. For any profile $\mu \in \Delta^*(X_N^{\mathrm{pr}})$, the set $\mathrm{Cond}\,(X_N^{\mathrm{pr}}, \mu)$ is the set of preference orderings on $[1...N]$ such that no other ordering agrees with the μ-majority on a larger set of pairwise comparisons.

Note that in the case of preference aggregation, Condorcet admissibility always restricts the set of admissible elements, i.e. $\mathrm{Cond}\,(X_N^{\mathrm{pr}}, \mu)$ is never all of X_N^{pr}. For example, if a majority of voters strictly prefer a to b in the profile μ, then no element of $\mathrm{Cond}\,(X_N^{\mathrm{pr}}, \mu)$ can place a and b as nearest neighbours with $b \succ a$, because switching the social ranking between a and b while retaining all other comparisons would agree with the majority view on a strictly larger set of issues. Note also that Condorcet determinacy in the sense defined above is stronger than the usual notion of existence of a Condorcet winner. Indeed, Condorcet determinacy in our sense requires the existence of an entire ordering that agrees with the majority judgement in each binary comparison; it is thus equivalent to the existence of a Condorcet winner *on each subset of* $[1...N]$.

For any $\mathbf{x} \in \{0,1\}^K$, let $\overset{\mathbf{x}}{\prec}$ be the binary relation on $[1...N]$ defined by $\mathbf{x}$. Moreover, for any $\mu \in \Delta^*(X_N^{\mathrm{pr}})$, let $\overset{\mu}{\prec}$ be the binary relation defined by $\mathrm{Maj}(\mu)$ —the so-called ***majority tournament***. An element $\mathbf{c} \in X_N^{\mathrm{pr}}$ is called a ***directed Hamiltonian chain*** of $\mathrm{Maj}(\mu)$ if all nearest-neighbour orderings in $\overset{\mathbf{c}}{\prec}$ agree with the orderings specified by $\overset{\mu}{\prec}$. In other words, if we represent $\overset{\mathbf{c}}{\prec}$ as a linear directed graph C and represent $\overset{\mu}{\prec}$ as a complete directed graph D in the obvious way, then C is a (directed) subgraph of D.

Let $\overset{\mathbf{x}}{\preceq}_*$ be the transitive closure of $\overset{\mathbf{x}}{\prec}$, augmented by all pairs (n, n) for $n \in [1...N]$; then $\overset{\mathbf{x}}{\preceq}_*$ is a weak order (i.e. it is complete, reflexive and transitive). If $\overset{\mathbf{x}}{\approx}_*$ is the symmetric part of $\overset{\mathbf{x}}{\preceq}_*$, then $\overset{\mathbf{x}}{\approx}_*$ is an equivalence relation (one has $n \overset{\mathbf{x}}{\approx}_* m$ iff 'n and m belong to the same cycle of $\overset{\mathbf{x}}{\prec}$'). The $\overset{\mathbf{x}}{\approx}_*$-equivalence classes of $[1...N]$ are linearly ordered by the asymmetric part $\overset{\mathbf{x}}{\prec}_*$ of $\overset{\mathbf{x}}{\preceq}_*$ (one has $n \overset{\mathbf{x}}{\prec}_* m$ iff 'n is on a lower $\overset{\mathbf{x}}{\prec}$-cycle than m').[10]

Proposition 2.1 *Let* $\mu \in \Delta(X_N^{\mathrm{pr}})$.

(a) $\mathrm{Cond}\,(X_N^{\mathrm{pr}}, \mu) = \{\mathbf{x} \in X_N^{\mathrm{pr}};\ \overset{\mathbf{x}}{\prec}$ *is a Hamiltonian chain in* $\overset{\mu}{\prec}\}$.

(b) *For all* $n, m \in [1...N]$, $n \overset{\mu}{\prec}_* m$ *if and only if, for all* $\mathbf{x} \in \mathrm{Cond}\,(X_N^{\mathrm{pr}}, \mu)$, $n \overset{\mathbf{x}}{\prec} m$.

[10]For a choice-theoretic analysis of the relation $\overset{\mathbf{x}}{\preceq}_*$ see Duggan (2007).

As an illustrative example, consider the 4-permutahedron with alternatives a, b, c, d. Suppose that one third of the population endorses each of the preference orderings $a \succ b \succ c \succ d$, $b \succ c \succ d \succ a$ and $c \succ d \succ a \succ b$. For the corresponding majority tournament we have $c \overset{\mu}{\succ} a$, $d \overset{\mu}{\succ} a$, $a \overset{\mu}{\succ} b$, $b \overset{\mu}{\succ} c$, $b \overset{\mu}{\succ} d$, and $c \overset{\mu}{\succ} d$ (see Figure 1). By Proposition 2.1 (a), the Condorect efficient set consists of the following five orderings: $a \succ b \succ c \succ d$, $b \succ c \succ d \succ a$, $c \succ d \succ a \succ b$, $d \succ a \succ b \succ c$, $c \succ a \succ b \succ d$.

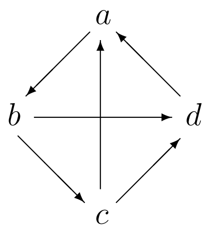

Figure 1: A majority tournament on four alternatives

Condorcet admissibility as applied in the present section concerns the aggregation of individual preferences into a social *preference*. What are the implications for social *choice* rules, i.e. for mappings that specify a chosen element for each non-empty subset? The following result gives an answer to this question. Given a profile μ, an element $n \in A \subseteq [1...N]$ is called ***Condorcet rationalizable in*** A if there exists $x \in \mathrm{Cond}\,(X, \mu)$ such that n is a maximal element of A with respect to the ordering x. For all non-empty $A \subseteq [1...N]$, denote by $C_*(A, \mu) \subseteq A$ the set of Condorcet rationalizable elements in A.

Proposition 2.2 *Let $\mu \in \Delta(X_N^{\mathrm{pr}})$ and $n \in A \subseteq [1...N]$. Then, $n \in C_*(A, \mu)$ if and only if n is $\overset{\mu}{\precsim}_*$-maximal in A.*

The result shows that the choice function C_* is *binary*, i.e. results from the maximization of a binary relation, since for all $A \subseteq [1...N]$,

$$C_*(A, \mu) = \{m \in A;\ n \overset{\mu}{\precsim}_* m \text{ for all } n \in A\}.$$

Proposition 2.2 implies in particular that, if the feasible set A agrees with the domain of preferences $[1...N]$, an alternative is Condorcet rationalizable if and only if it is an element of the maximal $\overset{\mu}{\approx}_*$-equivalence class, which is also known as the ***top cycle*** of the majority tournament $\overset{\mu}{\prec}$ (Moulin, 1988, p.253). However, if A is a proper subset of $[1...N]$, alternatives may be Condorcet rationalizable even if they are not in the top cycle. For example, if $A = \{a, b\}$ then b may be Condorcet rationalizable (together with a) even though a is preferred by a strict majority to b. While this may look counterintuitive at first, it makes good sense from the perspective of Condorcet admissibility as capturing the normative implications of majoritarianism *per se*. Consider, for instance, the triple a, b, c, and suppose that in the majority tournament $a \overset{\mu}{\succ} b$, $b \overset{\mu}{\succ} c$, $c \overset{\mu}{\succ} a$ the smallest supermajority margin is at $a \overset{\mu}{\succ} b$. Then, by an argument dating back to Condorcet, one can argue for the selection of the ordering $b \succ c \succ a$ as the unique 'best supported' social preference ordering, with an attendant choice of b over a in the feasible set $\{a, b\}$.[11]

As a corollary, the concept of 'Smith consistency,' i.e. the requirement that choice should always be from the top cycle (Smith, 1973; Moulin, 1988, p.241), turns out to be too strong a normative requirement for majoritarianism *as such*.[12]

While thus weaker than Smith consistency, Condorcet rationalizability has substantial and non-trivial implications beyond Condorcet consistency. An illustrative example is the Simpson-Kramer (maxmin) rule according to which the alternative with the highest minimal popular support in all binary comparisons is chosen. Consider the set of alternatives $\{a, b, c, d, e, m\}$ and a profile that gives equal weight to the following five preference orderings (see Duggan (2010)): $a \succ b \succ m \succ c \succ d \succ e$, $b \succ c \succ m \succ d \succ e \succ a$, $c \succ d \succ e \succ m \succ a \succ b$, $d \succ e \succ a \succ m \succ b \succ c$, $e \succ a \succ b \succ c \succ d \succ m$. As is easily verified, alternative m is the unique Simpson-Kramer winner since it looses with $2/5$ of the votes against every other alternative, while each other alternative looses against some alternative with only $1/5$ of the votes. However, the top cycle set is given by $\{a, b, c, d, e\}$; hence, by Proposition 2.1 (b), the unique Simpson-Kramer winner is never the top element of a Condorcet admissible preference ordering, and, by Proposition 2.2, it is not chosen by C_*.[13]

2.2 Committee selection and resource allocation

For any $\mathbf{x} \in \{0,1\}^K$, let $\|\mathbf{x}\| := \#\{k \in [1...K]\ ;\ x_k = 1\}$. Let $0 \leq I \leq J \leq K$, and define

$$X^{\text{com}}_{I,J;K} \quad := \quad \big\{\mathbf{x} \in \{0,1\}^K\ ;\ I \leq \|\mathbf{x}\| \leq J\big\}. \tag{4}$$

Heuristically, $[1...K]$ is a set of K 'candidates', and $X^{\text{com}}_{I,J;K}$ is the set of all 'committees' comprised of at least I and at most J of these candidates.[14] The next result follows immediately from Lemma 1.1:

Proposition 2.3 *Let $0 \leq I \leq J \leq K$ and let $\mu \in \Delta(X^{\text{com}}_{I,J;K})$. Let $M := \{k \in [1...K];$ $\mu[x_j] > \frac{1}{2}\}$ be the set of all 'candidates' receiving majority support.*

(a) *If $I \leq |M| \leq J$, then* $\mathrm{Cond}\,(X, \mu) = \{\mathbf{1}_M\}$.

(b) *If $|M| < I$, then* $\mathrm{Cond}\,(X, \mu) = \{\mathbf{1}_H\ ;\ M \subset H \subseteq [1...K],$ *and* $|H| = I\}$.

---

[11]This ordering is in fact chosen by the median rule, or more generally, by any unweighted suport rule.

[12]Smith consistency can be viewed as a version of Condorcet rationalizability where the domain of preferences used in the aggregation results from restricting the underlying preferences to the respective feasible sets, i.e. Smith consistency results from Condorcet rationalizability under a weak version of IIA.

[13]The example is by no means exceptional or 'non generic.' Indeed, Nehring and Pivato (2011a) provide a general method to construct profiles that generate any given set of supermajority ratios in the binary comparison between alternatives (possibly with a large set of voters).

[14]The committee selection problem appears to have a long history in social choice theory; according to McLean (1990) the aggregation problem corresponding to $X^{\text{com}}_{7,7;20}$ was studied by Ramon Lull already in the year 1274 in his 'Book of the Gentile and the Three Wise Men.' For a closely related model interpreted in terms of 'community standards,' see Miller (2009).

(c) *If* $|M| > J$*, then* $\mathrm{Cond}(X, \mu) = \{\mathbf{1}_H \ ; \ H \subseteq M \ \textit{and} \ |H| = J\}$.

Next, fix $M, D \in \mathbb{N}$, and consider the D-dimensional 'discrete cube' $[0...M]^D$. Then each element $\mathbf{x} \in [1...M]^D$ can be represented by a point $\Phi(\mathbf{x}) := \widetilde{\mathbf{x}} \in \{0,1\}^{D\times M}$ defined as follows:

$$\text{for all } (d,m) \in [1...D] \times [1...M], \qquad \widetilde{x}_{(d,m)} \quad := \quad \begin{cases} 1 & \text{if} \quad x_d \geq m; \\ 0 & \text{if} \quad x_d < m. \end{cases} \tag{5}$$

(For example, $\Phi(\mathbf{0}^D) = \mathbf{0}^{D\times M}$ and $\Phi(\mathbf{1}^D) = \mathbf{1}^{D\times M}$). This defines an injection $\Phi : [0...M]^D \longrightarrow \{0,1\}^{D\times M}$. Any subset of $P \subset [0...M]^D$ can thereby be represented as a subset $X := \Phi(P) \subset \{0,1\}^{D\times M}$. Judgement aggregation over X thus represents social choice over a D-dimensional 'policy space', where each voter's position represents her ideal point in P, the set of feasible policies. This framework is especially useful for resource allocation problems, as we now illustrate. Let

$$\begin{aligned} \triangle_M^D &:= \left\{ \mathbf{x} \in [0...M]^D \ ; \ \sum_{d=1}^{D} x_d \ = \ M \right\}, \\ \text{and} \quad X_{M,D}^{\triangle} &:= \Phi[\triangle_M^D] \subset \{0,1\}^{D\times M}. \end{aligned} \tag{6}$$

Geometrically, $\triangle_M^D$ is a 'discrete simplex'; points in $\triangle_M^D$ represent all ways of allocating M indivisible dollars amongst exactly D claimants. Thus, judgement aggregation over $X_{M,D}^{\triangle}$ describes a group which decides how to allocate a budget of M dollars towards D claimants by voting 'yea' or 'nay' to propositions of the form 'x_d should be at least m dollars' for each $d \in [1...D]$ and $m \in [1...M]$; see Lindner et al. (2010).

On the space $X_{M,D}^{\triangle}$, the Condorcet set allows for the following explicit characterization. For each $d \in [1...D]$ and μ, let $m_d^* := \mathrm{med}_d(\mu)$ denote the *median* in coordinate d (that is: m_d^* is the unique $m \in [0\ldots M]$ such that $\mu(x_{d,m}) > \frac{1}{2} > \mu(x_{d,m+1})\}$; this value exists because $\mu \in \Delta^*(X_{M,D}^{\triangle})$). It follows that $\mathrm{Maj}(\mu) = \Phi(m_1^*, \ldots, m_D^*)$ (see Lemma A.1(b)). Let $D(\mu) := \left(\sum_{d=1}^{D} m_d^*\right) - M$ be the 'majority deficit' corresponding to the profile μ. Note that the majority deficit can be positive or negative.

Proposition 2.4 *Let* $M, D \in \mathbb{N}$*, and let* $\mu \in \Delta^*(X_{M,D}^{\triangle})$.

- *If* $D(\mu) = 0$*, then* $\mathrm{Cond}\left(X_{M,D}^{\triangle}, \mu\right) = \mathrm{Maj}(\mu)$.
- *If* $D(\mu) > 0$*, then* $\mathrm{Cond}\left(X_{M,D}^{\triangle}, \mu\right) = \{\Phi(\mathbf{x});\ \ \mathbf{x} \in \triangle_M^D \textit{ and } x_d \in [m_d^* - D(\mu),\ m_d^*]$ *for all* $d \in [1\ldots D]\}$.
- *If* $D(\mu) < 0$*, then* $\mathrm{Cond}\left(X_{M,D}^{\triangle}, \mu\right) = \{\Phi(\mathbf{x});\ \ \mathbf{x} \in \triangle_M^D \textit{ and } x_d \in [m_d^*,\ m_d^* + |D(\mu)|]$ *for all* $d \in [1\ldots D]\}$.

Thus, a profile μ is Condorcet determinate if and only if $D(\mu) = 0$. Moreover, if there is positive (negative) deficit, the Condorcet set arises by allocating at most (at least) the median amount to each claimant while distributing the slack in any feasible way. In the space $X_{M,D}^{\triangle}$, the size of the Condorcet admissible set is thus directly determined by the absolute value of the majority deficit $| D(\mu) |$, which can be viewed as the 'degree of Condorcet-indeterminacy.'

Note also that, if $D > 2$, it is quite possible that $\mathrm{med}_d(\mu) = 0$ for all $d \in [1...D]$; for instance, this will be the case if there are sufficiently many voters and every voter is only interested in positive quantities of less than half of the goods $d \in [1...D]$ (provided that the relevant sets of goods differ across voters). By Proposition 2.4, one obtains $\mathrm{Cond}\left(X^{\Delta}_{M,D}, \mu\right) = X^{\Delta}_{M,D}$ in this case, so that the entailed path-dependence is maximal; in Section 5 below, we will refer to this as 'total indeterminacy' (cf. Example 5.1 (b) below).

Both $X^{\mathrm{com}}_{I,J;K}$ and $X^{\Delta}_{M,D}$ are examples of *allocation spaces*. Let $M, D \in \mathbb{N}$, and fix constants $0 \leq \underline{S} \leq \overline{S}$ and $0 \leq \underline{A}_d \leq \overline{A}_d \leq \overline{S}$ for all $d \in [1...D]$. Let $\mathbf{S} := (\underline{S}, \overline{S})$ and $\mathbf{A} := (\underline{A}_d, \overline{A}_d)_{d=1}^{D}$; the data $(D, \mathbf{S}, \mathbf{A})$ defines an *allocation polytope*:

$$P^D_{\mathbf{S},\mathbf{A}} \quad := \quad \left\{ \mathbf{m} \in [0...M]^D \ ; \ \underline{S} \leq \sum_{d=1}^{D} m_d \leq \overline{S} \text{ and } \underline{A}_d \leq m_d \leq \overline{A}_d, \quad \forall\ d \in [1...D] \right\}. \quad (7)$$

Let $X^D_{\mathbf{S},\mathbf{A}} := \Phi[P^D_{\mathbf{S},\mathbf{A}}] \subset \{0,1\}^{D\times M}$; then $X^D_{\mathbf{S},\mathbf{A}}$ is called an *allocation space*. Heuristically, $X^D_{\mathbf{S},\mathbf{A}}$ represents a set of feasible allocations of M indivisible 'resource units' (e.g. dollars, committee positions) amongst some set of D claimants, with lower and upper bounds on the total allocation (given by $\underline{S}$ and $\overline{S}$), and possibly with minimal and maximal amounts for each claimant (given by $(\underline{A}_d, \overline{A}_d)_{d=1}^{D}$). For example:

- Let $0 \leq I \leq J \leq K$. Set $\underline{S} := I$, $\overline{S} := J$, $M := 1$ and $D := K$, and set $\overline{a}_d := 0$ and $\overline{a}_d := 1$ for all $d \in [1...D]$. Then $P^D_{\mathbf{S},\mathbf{A}} = \left\{ \mathbf{a} \in \{0,1\}^K \ ; \ I \leq \sum_{k=1}^{K} a_k \leq J \right\}$. Thus, $X^D_{\mathbf{S},\mathbf{A}} = X^{\mathrm{com}}_{I,J;K}$.
- Let $M \in \mathbb{N}$. Set $\underline{S} = \overline{S} = M$, and set $\overline{A}_d = M$ and $\underline{a}_d = 0$ for all $d \in [1...D]$. Then $P^D_{\mathbf{S},\mathbf{A}} = \triangle^D_M$ is the 'discrete simplex' in eqn.(6). Thus, $X^D_{\mathbf{S},\mathbf{A}} = X^{\Delta}_{M,D}$.

Theorem 2.5 *Let X be any allocation space and let $\mu \in \Delta^*(X)$. Then* $\mathrm{Cond}\,(X, \mu) = \mathrm{Slater}\,(X, \mu)$.

Propositions 2.3 and 2.4 are corollaries to Theorem 2.5 applied to $X = X^{\mathrm{com}}_{I,J;K}$ and $X^{\Delta}_{M,D}$, respectively. It is worth emphasizing that, unlike in allocation spaces, the Slater rule does not, in general, exhaust the Condorcet set. A simple example is the 4-permutahedron. As in the example given in the previous subsection, assume that one third of the population endorses each of the preference orderings $a \succ b \succ c \succ d$, $b \succ c \succ d \succ a$ and $c \succ d \succ a \succ b$. As noted above, the Condorcet set consists of the following five orderings: $a \succ_1 b \succ_1 c \succ_1 d$, $b \succ_2 c \succ_2 d \succ_2 a$, $c \succ_3 d \succ_3 a \succ_3 b$, $d \succ_4 a \succ_4 b \succ_4 c$, $c \succ_5 a \succ_5 b \succ_5 d$. As is easily verified, $\succ_2$ agrees with the majority tournament in five binary comparisons $((a,c),(a,d),(b,c),(b,d),(c,d))$, whereas $\succ_4$ agrees with it only in three issues $((a,b),(a,d),(b,c))$; all other orderings agree with the majority tournament in exactly four issues. Thus, $\succ_2$ is uniquely chosen by the Slater rule.

2.3 Horizontal Equity

Consider a situation in which an institution, e.g. a parliament, a court, or an academic department, faces a collection of yes/no decisions $s \in [1...S]$ on a set of cases, say on the acceptability of public utterances, on the negligence of defendants, or on the employability of job candidates. We will refer to the elements of $[1...S]$ as ***cases***. Every voter and society must form an opinion on how these cases are to be decided. Let us call a set $C \subseteq [1...S]$ of *positive* decisions a ***standard***; a standard thus lists the public utterances that are deemed acceptable, the defendants that are considered to be guilty of negligence, the candidates that are considered to be employable, etc. The aggregation problem consists in determining for each profile of individual standards a social standard. In this context, a notion of horizontal equity may require that similar cases are to be treated similarly, i.e. that similar utterances, similar legal cases, similar candidates are to be assessed similarly. Formally, a ***norm (of horizontal equity)*** is a collection $\mathcal{C}$ of 'legitimate' standards (thus, a norm is a subset of $2^{[1...S]}$). The intended interpretation is that a norm describes a concept of similarity between cases with the elements of $\mathcal{C}$ as the collection of 'similarity clusters' of cases. This superordinate notion of similarity is shared by all individuals in the society in the sense that there is universal agreement that all legitimate standards, individual as well as social, have to be similarity clusters, i.e. elements of $\mathcal{C}$. The aggregation problem arises from the fact that different individuals may hold different standards, and that naive aggregation procedures may not produce a legitimate standard at the social level. The following example provides an illustration and will help to clarify the concepts.

Example 2.6 ('Unfree public speech') Suppose that a group such as a corporation or political party decides that it needs to form an opinion about which public utterances are acceptable. While ready to restrict freedom of speech in this manner, the group acknowledges, however, that it should proceed in a principled manner at least to the extent that it should apply a consistent standard to opinions uttered by different persons at different situations.[15]

Formally, this can be captured by letting each case correspond to a public utterance representing a particular position $s \in [1...S]$ in the political spectrum, and assume that these positions can be ordered from left to right such that, say, $1 < 2 < ... < S$. Every individual and society have to form an opinion about which public utterances are acceptable, i.e. have to determine a legitimate standard. In this example, a natural restriction is to require that a position that is in between two acceptable position with respect to the ordering $<$ must be acceptable as well, in other words, to require that legitimate standards must form *intervals* on the spectrum of positions. The underlying 'horizontal' equity norm is thus given by the collection $\mathcal{C}^{\text{line}}$ of all intervals with respect to the ordering $<$.

How, in this context, can one aggregate the individual standards into a social standard in order to decide on collective acceptability? A natural way to individuate the decisions is to identify each case $s \in [1...S]$ with an issue $k \in [1...K]$ on which individuals cast a vote. An element $\mathbf{x} \in \{0,1\}^K$ thus corresponds to a collection of cases with $x_k = 1$ ($x_k = 0$) denoting

[15] We are not advocating restrictions on freedom of speech, of course, but situations as sketched here happen empirically, with more or less respect for the 'rule of law.'

a positive (negative) decision in case k.[16] A typical element $C \in \mathcal{C}^{\text{line}}$ can thus be identified with a sequence $\mathbf{1}_C = (0...011...110...0)$ in which all 1's are adjacent to each other (form an interval) in the given left-to-right ordering. Let $X_K^{\text{line}} := \{\mathbf{1}_C \; ; \; C \in \mathcal{C}^{\text{line}}\} \subseteq \{0,1\}^K$. Given the proposed identification of cases and issues, majoritarianism demands to determine collective acceptability by letting individuals vote on each single case and to derive the social standard via Condorcet admissibility in the space X_K^{line}; note that the issue-wise majority view is in general not an admissible standard (i.e. not an element of X_K^{line}).

The following result characterizes Condorcet admissibility in the case of the horizontal equity norm $\mathcal{C}^{\text{line}}$. To state the result, observe that any non-empty subset $J \subseteq [1...S]$ can be uniquely written as the disjoint union of non-adjacent intervals $J = \bigsqcup_{n=1}^{N} I_n$; accordingly, any element $\mathbf{x} = \mathbf{1}_J \in \{0,1\}^K$ can be written as $\mathbf{1}_{\bigsqcup I_n}$ with disjoint and non-adjacent intervals $I_1, ..., I_N$.

Proposition 2.7 *Consider* $\mu \in \Delta^*(X_K^{\text{line}})$ *such that* $\text{Maj}(\mu) \neq \mathbf{0}$, *and write* $\text{Maj}(\mu) = \mathbf{1}_{\bigsqcup I_n}$. *Then* $\mathbf{x} = \mathbf{1}_J \in \text{Cond}\,(X_K^{\text{line}}, \mu)$ *if and only if there exist* $n, m \in [1 \ldots N]$ *such that* J *is the smallest interval containing* I_n *and* I_m.

To illustrate, let $\ell, r \in [1...S]$ with $\ell + 1 < r$, and suppose that one third of the population are strict leftists endorsing the standard $[1...\ell]$, one third are strict rightists endorsing the standard $[r...S]$, and the final third are strict liberalists endorsing the standard $[1...S]$. Then, $\text{Cond}\,(\mu) = \{\mathbf{1}_{[1...\ell]}, \mathbf{1}_{[r...S]}, \mathbf{1}_{[1...S]}\}$. This is easily verified by looking at the different sequences in the sequential majority voting over single positions. Call $[1...\ell]$ the 'left' positions, $[\ell + 1...r - 1]$ the 'middle' positions, and $[r...S]$ the 'right' positions. If a left and a middle position are already decided before the first right position is to be decided, then the social standard $[1...\ell]$ results; if a right and a middle position are already decided before the first left position is to be decided, then the social standard $[r...S]$ results; finally, if a left and a right position are already decided before the first middle position is to be decided, then the social standard $[1...S]$ results. Thus, depending on the particular sequence of majority decisions, either the left dogma, the right dogma, or complete permissiveness prevails. The acceptability of *every* position is therefore at the mercy of history (below, we will call this phenomenon 'issue-wise indeterminacy'). Note also that a profile μ is Condorcet determinate if and only if $\text{Maj}(\mu) \in X_K^{\text{line}}$. $\diamondsuit$

The diachronic perspective of judgement aggregation lends itself very naturally to the present interpretation in terms of horizontal equity standards. Anglo-Saxon law, to name an important example, develops over time with previous court decisions restricting present jurisdiction. Past court decisions thus represent an accumulated 'standard' that has to be taken into account in present legal practice. In a similar way, also other institutions may wish to submit to intertemporal consistency by referring to past decisions in recurring

[16] In view of the identification of the sets $[1...S]$ and $[1...K]$ one might wonder why we have distinguished them in the first place. The reason is that, while natural, the simple identification of each case with an issue in the judgement aggregation model yields only *one* possibility to model the underlying aggregation problem; for instance, issues could also be identified with appropriate *subsets* of cases.

cases; examples are the hiring policy of an academic department, or the admission process for office at public institutions.

Frequently, but perhaps not always, the intersection of two admissible standards will be admissible as a standard as well. Indeed, the eligibility under each of two different standards can be seen as a standard by itself. Moreover, the set $[1...S]$ itself can always be considered to be a standard, namely the standard that represents maximal permissibility. Formally, a ***convex structure***, or a ***convexity*** on a finite set S is a collection $\mathcal{C}$ of subsets of S such that $\mathcal{C}$ is closed under non-empty intersections and such that $S \in \mathcal{C}$.[17] The general idea behind our proposal to model horizontal social equity as an aggregation problem is as follows. As above, the set S consists of the possible cases, and the elements of $\mathcal{C}$ (the 'convex' sets) describe 'similarity clusters' of cases. The smallest similarity cluster common to all elements of a given set J is called its *convex hull*, $\mathrm{conv}(J) := \bigcap\{C \in \mathcal{C}\ ;\ J \subseteq C\}$. The convex structure describes a 'meta-standard' by requiring that all legitimate standards, individual as well as social, form similarity clusters. For instance, if in a legal context, the cases $s_1, s_2, ..., s_k$ are counted, say, as negligence, then all cases belonging to the same similarity cluster, i.e. all cases in $\mathrm{conv}(\{s_1, s_2, ..., s_k\})$ must be counted as negligence as well. The convex structure thus represents the 'rule of law,' i.e. the embodied equity norm corresponding to a fundamental principle of non-arbitrariness.

In case of the left-to-right spectrum in Example 2.6 above, the similarity clusters are the intervals, and legitimate yes-no classifications must respect the similarities derived from the underlying linear neighborhood structure. Another convex structure that will frequently be applicable is that of a 'taxonomic hierarchy.' A family of sets $\mathcal{C}^{\mathrm{tax}}$ is called a *taxonomic hierarchy* if, for all $C, D \in \mathcal{C}^{\mathrm{tax}}$, we have: either (1) $C \subset D$, or (2) $D \subset C$, or (3) C and D are disjoint. The elements of $\mathcal{C}^{\mathrm{tax}}$ are called *taxa*. A taxon $C \in \mathcal{C}^{\mathrm{tax}}$ is *minimal* if C does not contain any proper sub-taxa. Note that minimal taxa are not necessarily singletons. Taxonomic hierarchies arise in many contexts. Consider, for instance, the problem of determining which group of animals should be granted animal rights. A plausible horizontal equity norm would be to require that this group, whatever it otherwise may be, must form a biological taxon derived from the evolutionary tree of species. The meta-standard thus allows for individual disagreement about the appropriate specific taxon, say whether it be only the mammals or all vertebrates, but no disagreement about the fact that it must be a taxon.

As another example, consider the problem of regulating different industries. Horizontal equity requirements may be imposed on legislature here as anti-favoritism design: any regulation must be imposed on a natural class of sectors, say based on an industry classification system. Again, individual judges may disagree about the specific sectors which ought to be regulated, but any proposed regulation must apply equally to all members of a 'similarity cluster.' Suppose, for instance, that one third of the representatives of the regulation authority endorse regulation of all companies in the subsector #5221 ('Depository Credit Intermediation'), one third would like to see all companies in the disjoint subsector

[17] Convex structures have been studied in discrete mathematics (see, e.g. van de Vel (1993)) with the additional assumption that the empty set is a convex set (an element of $\mathcal{C}$). Note that any family of sets that is closed under non-empty intersections can be made a proper convex structure in this sense by including the empty set and the universal set.

#5222 ('Nondepository Credit Intermediation') regulated, and another third think that the entire industry #52 ('Finance and Insurance') should be regulated.[18] Then, if a separate vote is taken on the regulation of each company, the Condorcet admissible policies are to regulate the subsector #5221, the subsector #5222, or their smallest common supersector #522 ('Credit Intermediation and Related Activities').

An interesting and frequently applicable generalization of the taxonomic hierarchy model arises by assuming that the admissible similarity clusters are the unions of at most m taxa. For instance, in the context of regulation policy one might wish to treat companies that share a common legal status similarly *across* different sectors. Avoiding favoritism would still be guaranteed in this model but in a more flexible way. Note that the resulting family of subsets forms a convexity if and only if $m \leq 2$.

3 Unanimity preempted

Sequential majority voting, i.e. choosing from the Condorcet admissible set, can lead to violations of unanimous consent in some issues. A simple (and well-known) example is the following situation in the 4-permutahedron.

Example 3.1 Let three preference orderings $\succ_1, \succ_2, \succ_3$ on $\{a, b, c, d\}$ be given as follows: $d \succ_1 a \succ_1 b \succ_1 c$, $b \succ_2 c \succ_2 d \succ_2 a$, $c \succ_3 d \succ_3 a \succ_3 b$, and consider the anonymous profile μ such that $\mu(\succ_1) = \mu(\succ_2) = \mu(\succ_3) = \frac{1}{3}$. Let γ be any path that first decides the three binary comparisons $(a, b), (b, c), (c, d)$. Then, we obtain $a \overset{\mu}{\succ} b$, $b \overset{\mu}{\succ} c$ and $c \overset{\mu}{\succ} d$, each by a (2/3) majority; thus, by transitivity, the ordering $a \succ b \succ c \succ d$ is selected. In particular, $a \succ d$ although there is unanimous agreement on the opposite ranking of a and d. Our following analysis will show that such unanimity violations can occur as soon as there are at least 4 alternatives. $\diamondsuit$

For any $\mu \in \Delta^*(X)$ and $k \in [1...K]$, we say that μ is ***unanimous*** in coordinate k if $\mu_k(0) = 1$ or $\mu_k(1) = 1$. Let $\mathbf{x} \in X$; we say that $\mathbf{x}$ ***violates*** ***μ-unanimity*** in coordinate k if $\mu_k(x_k) = 0$. We say that ***Condorcet guarantees unanimity on*** X if, for all $\mu \in \Delta^*(X)$, there is no $\mathbf{x} \in \mathrm{Cond}(X, \mu)$ which violates μ-unanimity in any coordinate (equivalently: diachronic judgement aggregation never violates unanimity on X). Recall that $\kappa(X)$ is the size of the largest critical fragment in X. This section's main result is reminiscent of Proposition 1.4.

Theorem 3.2 *Let $X \subseteq \{0, 1\}^K$. Then Condorcet guarantees unanimity on X if and only if $\kappa(X) \leq 3$.*

As the proof of Theorem 3.2 shows, the possible violations of respect for unanimity when $\kappa(X) > 3$ require only three agents and can occur thus quite easily.

Example 3.3 (a) (Preference Aggregation) Let $N \in \mathbb{N}$, and let X_N^{pr} be as in §2.1. In the Appendix we show that

$$\kappa(X_N^{\mathrm{pr}}) = N.$$

[18] Numbers are taken from the North American Industry Classification System (NAICS).

Thus, Condorcet guarantees unanimity on $X^{\rm pr}_N$ if and only if $N \leq 3$.

(b) (Restricted preference aggregation) Let $\lhd$ be a partial order on $[1...N]$, and let $X^{\rm pr}_\lhd$ be the set of all linear orders on $[1...N]$ which extend $\lhd$. Judgement aggregation over $X^{\rm pr}_\lhd$ describes a variant of Arrovian preference aggregation where the voters unanimously agree with the preferences encoded in $\lhd$, and where the social preference order is also required (e.g. by the constitution) to agree with the preferences encoded in $\lhd$. For instance, suppose that all preferences are defined over the Cartesian product $M \times B$ where $M \subseteq \mathbb{R}$ represents a set of feasible amounts of public expenditure and B is a (finite) set of social states. Moreover, assume that the partial order $\lhd$ expresses monotonicity with respect to the first component in each social state ('more money is better') while refraining from any judgement across states, i.e. $(m, b) \lhd (m', b') :\Leftrightarrow [b = b'$ and $m < m']$.

A $\lhd$-*antichain* is a subset $A \subseteq [1...N]$ such that every pair of elements in A are $\lhd$-incomparable. Let width$(\lhd)$ be the largest cardinality of any $\lhd$-antichain. For instance, in the above example width$(\lhd) = |B|$. In the Appendix we prove that

$$\kappa(X^{\rm pr}_\lhd) = \text{width}(\lhd). \tag{8}$$

Thus, Condorcet guarantees unanimity on $X^{\rm pr}_\lhd$ if and only if width$(\lhd) \leq 3$. Equation (8) also shows that in the example, one obtains a median space if and only if $|B| = 2$, i.e. if there are only two different social states. In particular, in that case the majority view is always feasible, i.e. is a linear order (that evidently agrees with the preferences encoded in $\lhd$). Condorcet guarantees unanimity, however, also if $|B| = 3$, i.e. if there are three different social states.

(c) (Equivalence Relations) Let $N \in \mathbb{N}$, let $K := N(N-1)/2$, and identify $[1...K]$ with a subset of $[1...N] \times [1...N]$ containing exactly one of the pairs (n, m) or (m, n) for each $n \neq m \in [1...N]$. Thus, an element of $\{0, 1\}^K$ represents a symmetric, reflexive binary relation (i.e. undirected graph) on $[1...N]$. Let $X^{\rm eq}_N \subset \{0, 1\}^K$ be the set of all equivalence relations on $[1...N]$. In the Appendix we show that

$$\kappa(X^{\rm eq}_N) = N.$$

Thus, Condorcet guarantees unanimity on $X^{\rm eq}_N$ if and only if $N \leq 3$.

(d) (Committee Selection) Let $0 \leq I \leq J \leq K$, and let $X^{\rm com}_{I,J,K}$ be as defined in eqn.(4) of §2.2. A fragment $\mathbf{w}$ is $X^{\rm com}_{I,J,K}$-critical if and only if either (a) $|\mathbf{w}| = J + 1$ and $\mathbf{w} = (1, 1, \ldots, 1)$ or (b) $|\mathbf{w}| = K - I + 1$ and $\mathbf{w} = (0, 0, \ldots, 0)$. Thus,

$$\kappa(X^{\rm com}_{I,J;K}) = 1 + \max\{J,\ K - I\}.$$

For example, suppose $K = 4$ and $J = I = 2$; then $\kappa(X^{\rm com}_{2,2;4}) = 3$, so Condorcet guarantees unanimity on $X^{\rm com}_{2,2;4}$. However, if $K \geq 5$, then either $J \geq 3$ or $K - I \geq 3$, so that $\kappa(X^{\rm com}_{I,J;K}) \geq 4$; then Condorcet does not guarantee unanimity on $X^{\rm com}_{I,J;K}$.

(e) (Resource Allocation) Let $D, M \in \mathbb{N}$, and let $X^{\vartriangle}_{M,D}$ be as in eqn.(6) of §2.2. Suppose $M \geq D^2$ (which is always true if we denominate money in small enough units). In the Appendix we show that

$$\kappa(X^{\vartriangle}_{M,D}) = D.$$

Thus Condorcet guarantees unanimity on $X^{\triangle}_{M,D}$ if and only if $D \leq 3$. $\diamondsuit$

The application of Theorem 3.2 to the horizontal equity model in §2.3 above requires the computation of $\kappa(X_{\mathcal{C}})$ for any convex structure $\mathcal{C}$. The following result provides an explicit formula. Let $[1...K]$ have the convex structure $\mathcal{C}$; for simplicity, we assume in the following that $\emptyset \in \mathcal{C}$. As above, let $\mathrm{conv}(J) = \bigcap\{C \in \mathcal{C}\ ;\ J \subseteq C\}$, and note that $I \subseteq J$ implies $\mathrm{conv}(I) \subseteq \mathrm{conv}(J)$. Thus,

$$\bigcup_{I \subsetneq J} \mathrm{conv}(I) \quad \subseteq \quad \mathrm{conv}(J). \tag{9}$$

Say that J is *Carathéodory-independent* if the set inclusion (9) is strict. The *Carathéodory number* of $\mathcal{C}$ is defined as

$$\lambda(\mathcal{C}) := \max\{|J|;\ J \subseteq [1...K] \text{ is Carathéodory-independent}\},$$

(van de Vel, 1993, §II.1.5, p.166).

Proposition 3.4 *Let $\mathcal{C}$ be a convex structure on $[1...K]$. Then $\kappa(X_{\mathcal{C}}) = \lambda(\mathcal{C}) + 1$.*

For the linear convexity $\mathcal{C}^{\mathrm{line}}$ consisting of all intervals on the (ordered) set $[1...K]$, one obtains $\lambda(\mathcal{C}^{\mathrm{line}}) = 2$. Thus, by Proposition 3.4 and Theorem 3.2 Condorcet guarantees unanimity on the corresponding space $X_{\mathcal{C}^{\mathrm{line}}}$. Moreover, it is easily verified that, for any taxonomic hierarchy $\mathcal{C}^{\mathrm{tax}}$ on $[1...K]$, there exists an ordering of the elements of $[1...K]$ such that all taxa are intervals with respect to that ordering. It thus follows that Condorcet guarantees unanimity on the corresponding spaces X^{tax}.

3.1 On the possibility to design respect for unanimity

Theorem 3.2 above characterizes the aggregation problems for which respect for a unanimous vote is guaranteed *no matter* which sequence of majority decisions one chooses. As it turns out, most aggregation problems cannot guarantee respect for unanimity in this strong sense. In this subsection, we ask whether it is possible to overcome the problem *by design*. In other words, for a given aggregation problem, do there *exist* paths of majority decisions that guarantee respect for unanimity for all profiles? One might expect a negative answer since *prima facie* it is not clear how, by mere design of the sequence, one could 'detect' unanimities in contrast to mere majorities (in a profile-independent way). We consider two illustrative examples. The first pertains to the spaces $X^{\triangle}_{M,D}$ and confirms this negative intuition. The second example, on the other hand, shows that the problem *can* in fact be overcome in the context of preference aggregation.

Proposition 3.5 *Consider the spaces $X^{\triangle}_{M,D}$ as defined in §2.2. If $D \geq 4$, there does not exist a path γ such that $F^{\gamma}(\mu)$ respects unanimity for all profiles $\mu \in \Delta^*(X^{\triangle}_{M,D})$.*

Given this negative result for the spaces $X^{\triangle}_{M,D}$, the following positive result for preference aggregation is all the more remarkable. Recall that the uncovered relation $\overset{\mu}{\prec}_{uc}$ of the tournament $\overset{\mu}{\prec}$ is defined as follows (Moulin, 1988, p.254)

$$b \overset{\mu}{\prec}_{uc} a :\Leftrightarrow \left[b \overset{\mu}{\prec} a \text{ and for all } c \in [1...N] \ : \ c \overset{\mu}{\prec} b \Rightarrow c \overset{\mu}{\prec} a \right].$$

Proposition 3.6 *Consider the space X^{pr}_N of all linear orderings on $[1...N]$, and define a path ζ on the set of all pairs $(n,m) \in [1...N] \times [1...N]$ lexicographically such that (n,m) occurs before (n',m') in ζ if and only if [$n < n'$ or ($n = n'$ and $m < m'$)]. Then, for any μ, $F^{\zeta}(\mu)$ extends the uncovered relation corresponding to $\mathrm{Maj}(\mu)$. In particular, $F^{\zeta}(\mu)$ respects unanimity for all μ.*

The choice rule among alternatives induced by the path ζ defined in Proposition 3.6 admits the following simple recursive description. First step: let $A_1 := [1...N]$ be the initial pool of alternatives and set element $n_1 := 1$ as the initial top alternative. Step ℓ: Compare the current top alternative $n_{\ell-1}$ to all alternatives $m > n_{\ell-1}$ in the current pool $A_{\ell-1}$. If $n_{\ell-1}$ wins against all such elements, then $n_{\ell-1}$ is the final choice; otherwise, let the new top alternative n_ℓ be the smallest majority winner against $n_{\ell-1}$ and take A_ℓ as the old pool minus the majority losers against $n_{\ell-1}$. Evidently, this procedure yields a final choice after a finite number of steps, which by Proposition 3.6 cannot be unanimously defeated by another alternative.

It turns out that the choice rule induced by $F^{\zeta}(\mu)$ is the same as the one induced by the multi-stage elimination tree procedure of Shepsle and Weingast (1984) (cf. Section 9.4 in Moulin (1988)). In contrast to the multi-stage elimination tree which only produces a winning alternative, however, F^{ζ} yields an entire ordering over all alternatives. Moreover, to accomplish this, F^{ζ} only needs $\frac{N(N-1)}{2}$ binary comparisons, whereas the multi-stage elimination tree involves $(2^{N-1}) - 1$ binary comparisons. In particular, in the multi-stage elimination procedure, an alternative may face the same opponent several times; this does not happen along the path ζ.

4 Issue-wise indeterminacy

If the Condorcet set is not a singleton, how large can it be? The size of the Condorcet set matters under each of the motivations: How determinate is majoritarianism by itself as a normative criterion? How much leeway is there for path-dependence to matter? How much room is there for a strategic agenda-setter to manipulate the outcome by the choice of a suitable path?

We will address the question of the size of the Condorcet set in the following three sections based on different definitions of ‘size.’ In the present section, we take the Condorcet set at a profile to be ‘large’ if it admits elements containing both truth-values in every coordinate. Such profiles will be called ‘issue-wise indeterminate,’ and aggregation problems for which such profiles exist will likewise be called ‘issue-wise indeterminate.’

While exploring the mere existence of issue-wise indeterminate profiles may look like a worst case analysis, we also show that, frequently, such profiles are rather easy to come by. The characterization of issue-wise indeterminacy in this section can be viewed as a counterpart in judgement aggregation to the classical results by McKelvey (1979) on the indeterminacy of preference-based majority voting.

4.1 General characterization

Let $X \subset \{0,1\}^K$. For any $\mu \in \Delta^*(X)$, we define the set of *indeterminate issues* for μ:

$$\mathrm{Indet}(\mu) \quad := \quad \{k \in [1...K]\,;\, \exists\, \mathbf{x}, \mathbf{y} \in \mathrm{Cond}(X, \mu) \text{ such that } x_k \neq y_k\}.$$

Thus, $\mathrm{Indet}(\mu)$ is the set of issues in which either answer is compatible with the principle of majoritarianism as embodied in the notion of Condorcet admissibility. Diachronically speaking, $\mathrm{Indet}(\mu) = \{k \in [1...K];$ there exist paths γ and ζ such that $F_k^\gamma(\mu) \neq F_k^\zeta(\mu)\}$ (by Proposition 1.3). We say that μ is *issue-wise indeterminate* if $\mathrm{Indet}(\mu) = [1...K]$.

Example 4.1 (a) (Preference aggregation) Let X_N^{pr} be as defined in §2.1, and consider $\mu \in \Delta(X_N^{\mathrm{pr}})$. By Proposition 2.1 (b) we know that the 'determinate issues' (n, m) are exactly those for which either $n \overset{\mu}{\prec}_* m$ or $m \overset{\mu}{\prec}_* n$, in other words,

$$\mathrm{Indet}(\mu) = \{(n,m) \in [1...N] \times [1...N] \quad ; \quad n \overset{\mu}{\approx}_* m\}.$$

Note that it is easy to construct profiles such that all alternatives belong to the same cycle. For instance, the profile μ that assigns weight $1/N$ to the preference order $(n \prec n+1 \prec \cdots \prec N \prec 1 \prec \cdots \prec n-1)$ for each $n \in [1...N]$ is issue-wise indeterminate.

(b) (Linear Convexity) Consider the linear convexity $\mathcal{C}^{\mathrm{line}}$ on $[1...K]$ defined in §2.3, and suppose that, for some profile μ, we have $\mathrm{Maj}(\mu) = \mathbf{1}_{\bigsqcup I_n}$ with disjoint and non-adjacent intervals $I_1, ..., I_N$. For each $n \in [1 \ldots N]$, let $I_n = [k_n^- ... k_n^+]$ with $k_1^- \leq k_1^+ < k_2^- \leq k_2^+ < ... < k_N^- \leq k_N^+$. If $N = 1$, all issues are determinate, otherwise only the issues $k < k_1^-$ and $k > k_l^+$ (if existent) are determinate. Thus, it is again easy to construct issue-wise indeterminate profiles. $\diamondsuit$

The key to the analysis of this section is the following result. Let $\mu \in \Delta^*(X)$, and let $\mathbf{w}$ be a critical fragment for X, with $\mathrm{supp}(\mathbf{w}) = J$. We say μ *activates* $\mathbf{w}$ if $\mathrm{Maj}_J(\mu) = \mathbf{w}$. Let $W(X, \mu)$ be the set of X-critical fragments activated by μ.

Theorem 4.2 *For all $X \subseteq \{0,1\}^K$, and $\mu \in \Delta^*(X)$,* $\mathrm{Indet}(\mu) \;=\; \bigcup_{\mathbf{w} \in W(X,\mu)} \mathrm{supp}(\mathbf{w})$.

Theorem 4.2 states that issue k is indeterminate given μ whenever there exists a critical fragment activated by μ that fixes the answer in issue k; in particular, μ is issue-wise indeterminate if for each issue $k \in [1...K]$ such a critical fragment exists. In the preference aggregation example, any cycle of $\mathrm{Maj}(\mu)$ corresponds to a critical fragment activated by μ.

Evidently, Theorem 4.2 entails a characterization of when no issue is indeterminate, i.e. of when issue-wise majority voting is consistent for a given profile, and thus of Condorcet determinacy. This question has been addressed before by Dietrich and List (2010) and Pivato (2009). By Theorem 4.2, $\mathrm{Indet}(\mu) = \emptyset$ if and only if $W(X, \mu) = \emptyset$, i.e. if and only if μ does not activate any critical fragment in X. This readily translates into Dietrich and List's (2010) condition of 'majority consistency.' Dietrich and List (2010) also provide a number of other, simpler but only sufficient conditions for Condorcet determinacy of a given profile, among them the counterpart in the general judgement aggregation model of Sen's classical condition of 'value restriction' (see Sen (1970)). Pivato (2009) provides a class of sufficient geometric conditions for determinacy of a profile.

We say that X is *issue-wise indeterminate* if there exists some issue-wise indeterminate $\mu \in \Delta^*(X)$. Let $W_3(X)$ be the set of critical fragments for X of order 3 or more. For any $\mu \in \Delta^*(X)$, we have $W(X, \mu) \subseteq W_3(X)$, because a critical fragment of order two cannot be activated by μ. Let $\mathbf{x} \in \{0,1\}^K$; we say that $\mathbf{x}$ is *critical for* X if there exists a collection $\{\mathbf{w}_1, \ldots, \mathbf{w}_N\} \subseteq W_3(X)$ such that $\mathbf{w}_n \sqsubseteq \mathbf{x}$ for all $n \in [1...N]$ and $[1...K] = \bigcup_{n=1}^N \mathrm{supp}(\mathbf{w}_n)$. Let $\mathrm{Crit}(X) := \{\mathbf{x} \in \{0,1\}^K;\ \mathbf{x} \text{ is critical for } X\}$. Let $\mathrm{Maj}(X) := \{\mathrm{Maj}(\mu)\ ;\ \mu \in \Delta^*(X)\}$.

The following result is the central result of the paper, as it provides a simple combinatorial characterization of issue-wise indeterminacy. A profile is issue-wise indeterminate if and only if the majority ideal point at this profile is critical for X. Thus, an aggregation space is issue-wise indeterminate if and only if it gives rise to a majority ideal point that is critical for X.

Theorem 4.3 *Let* $X \subseteq \{0,1\}^K$.

- **(a)** *For any* $\mu \in \Delta^*(X)$, $\Big(\mu \textit{ is issue-wise indeterminate}\Big) \iff \Big(\mathrm{Maj}(\mu) \in \mathrm{Crit}(X)\Big)$.
- **(b)** $\Big(X \textit{ is issue-wise indeterminate}\Big) \iff \Big(\mathrm{Maj}(X) \cap \mathrm{Crit}(X) \neq \emptyset\Big)$.

Proof: (a) follows immediately from Theorem 4.2, and (b) follows from (a). □

To apply Theorem 4.3(a) to a profile $\mu \in \Delta^*(X)$, we need to determine whether $\mathrm{Maj}(\mu) \in \mathrm{Crit}(X)$. The following result provides an answer for the aggregation of preference orderings and equivalence relations, respectively; in either application, issue-wise indeterminacy arises very easily. In preference aggregation, this is a well-studied problem, see McKelvey (1979).

Proposition 4.4 *Let* $N \in \mathbb{N}$, *let* $K := N(N-1)$, *and bijectively identify* $[1...K]$ *with a subset of* $[1...N] \times [1...N]$ *containing exactly one of the pairs* (n, m) *or* (m, n) *for each* $n \neq m \in [1...N]$.

- **(a)** (Preference aggregation) *Let* $\mathbf{x} \in \{0,1\}^K$, *and interpret* $\mathbf{x}$ *as tournament on* $[1...N]$, *as in* §2.1. *Then* $\mathbf{x} \in \mathrm{Crit}(X_N^{\mathrm{pr}})$ *if and only if* topcycle($\mathbf{x}$) *contains every element of* $[1...N]$.
- **(b)** (Equivalence relations) *Let* $\mathbf{x} \in \{0,1\}^K$, *and interpret* $\mathbf{x}$ *as an undirected graph on* $[1...N]$, *as in Example* 3.3(c). *Then* $\mathbf{x} \in \mathrm{Crit}(X_N^{\mathrm{eq}})$ *if and only if the* $\mathbf{x}$*-graph is connected, but not complete.*[19]

[19]Recall that a graph is *complete* if every vertex is adjacent to every other vertex.

4.2 Issue-wise indeterminacy in McGarvey spaces

The characterization of issue-wise indeterminacy becomes particularly simple in spaces in which the set of majoritarian ideal points is completely unrestricted, i.e. in which $\mathrm{Maj}(X) = \{0,1\}^K$. In this case, we say that X is *McGarvey*. The McGarvey property is analyzed in detail in Nehring and Pivato (2011a), where numerous examples are given. Theorem 4.3 has the following simple corollary.

Corollary 4.5 *Let $X \subseteq \{0,1\}^K$ be McGarvey. Then* $\Big(X$ *is issue-wise indeterminate*$\Big) \iff \Big(\mathrm{Crit}(X) \neq \emptyset\Big)$.

Example 4.6 (a) (Preference Aggregation) X_N^{pr} is McGarvey (McGarvey, 1953); thus, if $N \geq 3$, then Proposition 4.4(a) implies that X_N^{pr} is issue-wise indeterminate (as we already saw in Example 4.1(a)).

(b) (Equivalence Relations) X_N^{eq} is McGarvey (Nehring and Pivato, 2011a, Example 3.9(a)). Thus, if $N \geq 3$, then Proposition 4.4(b) implies that X_N^{eq} is issue-wise indeterminate. $\diamondsuit$

The following result provides a simple and frequently applicable sufficient condition for issue-wise indeterminacy in the horizontal equity model of §2.3.[20]

Proposition 4.7 *Let $\mathcal{C}$ be a convex structure on $[1...K]$, and let $X_{\mathcal{C}}$ be defined as in §2.3. Then, $X_{\mathcal{C}}$ is McGarvey if and only if $\mathcal{C}$ contains all singletons. In this case, $X_{\mathcal{C}}$ is issue-wise indeterminate.*

The above results establish a strong connection between the McGarvey property and issue-wise indeterminacy. Indeed, we do not know of a single, 'naturally occurring' aggregation problem that is McGarvey but fails to be issue-wise indeterminate. One might thus conjecture that any non-degenerate McGarvey problem X possesses an element that is critical for X, and is hence issue-wise indeterminate. Yet the following example proves this to be wrong.

Example 4.8 Let X be the subset of $\{0,1\}^5$ defined by the two critical fragments $\mathbf{w}_1 = (1,1,0,0,*)$ and $\mathbf{w}_2 = (*,0,0,1,1)$.[21] By construction, we have $\mathrm{Crit}(X) = \emptyset$ (since $\mathbf{w}_1$ and $\mathbf{w}_2$ disagree in the second and fourth coordinate). Moreover, X has 28 elements which is more than 3/4 of $32 = |\{0,1\}^5|$; hence, by Proposition 2.4(a) of Nehring and Pivato (2011a) X is McGarvey. By Corollary 4.5, X is not issue-wise indeterminate. Note also that X contains $\mathbf{1}_k$ for all k; this shows that the property of closedness under non-empty intersections cannot be dropped in Proposition 4.7 above. $\diamondsuit$

[20] We are again assuming $\emptyset \in \mathcal{C}$; note that if $\emptyset \notin \mathcal{C}$ then $X_{\mathcal{C}}$ is indeterminate in a trivial way.

[21] Here, the "$*$" notation means that $\mathrm{supp}(\mathbf{w}_1) = \{1,2,3,4\}$, and $\mathrm{supp}(\mathbf{w}_2) = \{2,3,4,5\}$; see Notation A.2.

4.3 Aggregation problems that are not issue-wise indeterminate

Beyond the class of median spaces, are there any natural aggregation spaces that are *not* issue-wise indeterminate? These seems to be rare: at this point, we have found one such class of spaces in which issue-wise indeterminacy fails robustly. These are the 'comprehensive' spaces that are characterized by the requirement that any view affirming a larger set of propositions than some feasible view be feasible as well. For instance, in the context of the committee selection problem, a comprehensive aggregation problem is obtained if any super-committee of a feasible committee is feasible. Formally, for any $\mathbf{x}, \mathbf{y} \in \{0,1\}^K$, write $\mathbf{x} \leq \mathbf{y}$ if $x_k \leq y_k$ for all $k \in [1...K]$. A subset $X \subset \{0,1\}^K$ is ***comprehensive*** if, for any $\mathbf{x}, \mathbf{y} \in \{0,1\}^K$, if $\mathbf{x} \in X$ and $\mathbf{x} \leq \mathbf{y}$, then $\mathbf{y} \in \{0,1\}^K$ also. For example, $X^{\text{com}}_{I,K;K}$ is comprehensive, for any $I \leq K$.

We say X is ***nondegenerate*** if, for every $k \in [1...K]$, there is some $\mathbf{w} \in W_3(X)$ with $k \in \operatorname{supp}(\mathbf{w})$. (For example, $X^{\text{com}}_{K-1,K;K}$ is degenerate: all critical fragments have order 2). If X is degenerate, then $\text{Crit}(X)$ is obviously empty. Thus, nondegeneracy is necessary for issue-wise indeterminacy.

Proposition 4.9 *If X is comprehensive and nondegenerate, then* $\text{Crit}(X) = \{\mathbf{0}\}$. *Thus, X is issue-wise indeterminate if and only if* $\mathbf{0} \in \text{Maj}(X)$.

To illustrate, consider the class of comprehensive committee selection problems $X^{\text{com}}_{I,K;K}$. Here, Proposition 4.9 yields the following corollary.

Corollary 4.10 *Let $I > 0$. The space $X^{\text{com}}_{I,K;K}$ is issue-wise indeterminate if and only if $I < K/2$.*

To verify this, simply observe that $\mathbf{0} \in \text{Maj}(X^{\text{com}}_{I,K;K})$ only if $I < K/2$; indeed, if ever voter approves of more than half of the candidates, then at least one candidate must receive a majority of votes.

Comprehensiveness of the committees selection problem is essential for the conclusion of Corollary 4.10. Otherwise, issue-wise indeterminacy obtains almost without exception, as shown by the following result; note that $X^{\text{com}}_{I,J;K}$ is comprehensive if $J = K$, and isomorphic to a comprehensive problem if $I = 0$.

Proposition 4.11 *Consider $X^{\text{com}}_{I,J;K}$, and suppose that $I > 0$ and $J < K$. Then $X^{\text{com}}_{I,J;K}$ is issue-wise indeterminate unless $I = J = K/2$.*

4.4 How 'likely' is issue-wise indeterminacy?

Issue-wise indeterminacy characterizes full path-dependence in terms of the *existence* of a profile such that, in each issue, both answers are possible via a suitably chosen decision path. One may doubt the relevance of this concept and the corresponding analysis, since existence results might only describe 'worst cases' that are very special and unlikely to happen. One may thus wish to complement this by an analysis of how 'likely' it is to run into such a profile. Note that in the case of the aggregation of preference orderings and equivalence relations, respectively, Proposition 4.4 already suggests that issue-wise indeterminacy is far from being special and unlikely in the sense that it is obtained for a large set of profiles. In the following attempt at a more general analysis, we take the complexity of a profile as a proxy for its 'likelihood,' and assess the complexity of a profile in terms of a very simple but instructive measure, the number of agents needed to construct it.

Formally, for any $N \in \mathbb{N}$, let

$$\Delta^*_N(X) \quad := \quad \left\{ \mu \in \Delta^*(X) \ ; \ \forall\, \mathbf{x} \in X, \ \mu(\mathbf{x}) = \frac{n}{N} \text{ for some } n \in [0 \ldots N] \right\}.$$

Politically, $\Delta^*_N(X)$ is the set of profiles which can be generated by a population of exactly N voters. Geometrically, $\Delta^*_N(X)$ can be visualized as a discrete 'mesh' of density $1/N$ embedded in $\Delta^*(X)$. Let $\Delta^*_{\text{ind}}(X) := \{\mu \in \Delta^*(X) ; \ \mu \text{ is issue-wise indeterminate}\}$. We define

$$\eta(X) \quad := \quad \min\{N \in \mathbb{N} \ ; \ \Delta^*_N(X) \cap \Delta^*_{\text{ind}}(X) \neq \emptyset\}.$$

(with $\eta(X) = \infty$ if X is not issue-wise indeterminate). Politically, $\eta(X)$ is the minimum number of voters needed to construct an issue-wise indeterminate profile. Geometrically, $\eta(X)$ measures the thickness of $\Delta^*_{\text{ind}}(X)$: if $\eta(X) > N$, then $\Delta^*_{\text{ind}}(X)$ cannot contain a sphere of radius greater than $\frac{\epsilon}{N}$, where $\epsilon := \sqrt{1 - \frac{1}{K}}$. Thus, $\eta(X)$ measures the susceptibility of X to issue-wise indeterminacy: if $\eta(X)$ is small, then X is very susceptible.

Evidently, for any $X \subseteq \{0,1\}^K$, we have $\eta(X) \geq 3$. The following result shows that several common aggregation problems are very susceptible to issue-wise indeterminacy, since one obtains the minimal value $\eta(X) = 3$ for them.

Proposition 4.12 **(a)** *If X has a critical fragment of order K, then $\eta(X) = 3$.*

(b) *Let X^{pr}_N be as in* §2.1*. If $N \geq 3$, then $\eta(X^{\text{pr}}_N) = 3$.*

(c) *Let X^{eq}_N be as in Example* 3.3(c)*. If $N \geq 3$, then $\eta(X^{\text{eq}}_N) = 3$.*

(d) *Let $X^{\vartriangle}_{M,D}$ be as in eqn.*(6) *of* §2.2*. If $D \geq 3$, then $\eta(X^{\vartriangle}_{M,D}) = 3$.*

While for many aggregation problems, issue-wise indeterminate profiles are thus 'easy' to construct, the spaces $X^{\text{com}}_{I,J;K}$ (defined in §2.2 above) exhibit a more complex pattern. Consider, for instance, the space $X^{\text{com}}_{4,6;10}$, i.e. the space of all committees that contain at least 4 and at most 6 members of a set of 10 candidates. In the appendix, we show that the profiles μ that induce issue-wise indeterminacy must have either $\text{Maj}(\mu) = \mathbf{0}$ or $\text{Maj}(\mu) = \mathbf{1}$. Without loss of generality, by symmetry, suppose the former, i.e. $\mu_k(1) < \frac{1}{2}$ for all $k \in [1 \ldots K]$. Since each feasible view endorses at least 4 candidates, we have $\sum_k \mu_k(1) \geq 4$. Denoting by k^* the candidate with maximal popular support, we thus obtain $\frac{4}{10} \leq \mu_{k^*}(1) < \frac{1}{2}$. Satisfaction of this inequality requires at least five agents; together with Proposition 4.13(b) below we thus obtain $\eta(X^{\text{com}}_{4,6;10}) = 5$.

The argument just given can be generalized to give the lower bound on $\eta(X^{\text{com}}_{I,J;K})$ in part (a) of the following result.

Proposition 4.13 *Consider the spaces $X^{\text{com}}_{I,J;K}$ as in eqn.*(4) *of §2.2.*

(a) *Suppose that* $0 < I < J < K$*; then*

$$\eta(X^{\rm com}_{I,J;K}) \quad \geq \quad \min\left\{\frac{K}{K-2I}, \ \frac{K}{2J-K}\right\}. \tag{10}$$

(b) *For any* $0 \leq I \leq J \leq K$*, we have* $\eta(X^{\rm com}_{I,J;K}) \leq K$*. Moreover, one obtains the following upper bounds for* $\eta(X^{\rm com}_{I,J;K})$*.*

[i] *If* $0 < I < K/2$*, then let* $N := \left\lceil \frac{I}{K-2I} \right\rceil$*. Then* $\eta(X^{\rm com}_{I,J;K}) \leq 2N+1$*.*

[ii] *If* $K/2 < J$*, then let* $N := \left\lceil \frac{K-J}{2J-K} \right\rceil$*. Then* $\eta(X^{\rm com}_{I,J;K}) \leq 2N+1$*.*

Combining parts (a) and (b) of Proposition 4.13, we obtain $\eta(X^{\rm com}_{4,6;10}) = 5$, as noted above; similarly, one obtains, for example, $\eta(X^{\rm com}_{6,8;14}) = 7$ and $\eta(X^{\rm com}_{5,6;11}) = 11$.

5 Total indeterminacy

Issue-wise indeterminacy means that, for some profile, any answer can be obtained in each issue by choosing a suitable decision path. An even stronger form of indeterminacy, which we shall henceforth refer to as *total* indeterminacy, occurs if, for some profile, any logically possible *combination* of answers across issues can be obtained via an appropriate decision path, i.e. if the corresponding Condorcet set contains *all* possible views.

Formally, a profile $\mu \in \Delta^*(X)$ is ***totally indeterminate*** if $\mathrm{Cond}(X,\mu) = X$. We say that X is ***totally indeterminate*** if there exists some $\mu \in \Delta^*(X)$ which is totally indeterminate over X.

Example 5.1 (a) Fix $J \in (\frac{K}{2}...K]$, and let $X^{\rm com}_{J,J;K} := \left\{\mathbf{x} \in \{0,1\}^K \ ; \ \|\mathbf{x}\| = J\right\}$. (Thus, $X^{\rm com}_{J,J;K}$ is the set of all 'committees' comprised of exactly J out of K candidates.) Let μ be the uniform distribution on $X^{\rm com}_{J,J;K}$. Then $\mathrm{Cond}\left(X^{\rm com}_{J,J;K}, \mu\right) = X^{\rm com}_{J,J;K}$, hence μ is totally indeterminate.

(Formally, this follows from Proposition 2.3(c). Intuitively, it is straightforward: if there are exactly J open slots and K viable candidates, and the slots are allocated on a 'first come, first serve' basis, then the slots will simply be allocated to the first J candidates.)

(b) Let $D \geq 3$, let $M \in \mathbb{N}$, and let $X^{\Delta}_{M,D}$ be as in eqn.(6) of §2.2. For all $d \in [1...D]$, let $\mathbf{x}^d$ be the element of $X^{\Delta}_{M,D}$ which allocates all M dollars towards claimant d. (Thus, for all $m \in [1...M]$, we have $x^d_{d,m} = 1$, while $x^d_{c,m} = 0$ for all $c \in [1...D] \setminus \{d\}$.) Let $\mu \in \Delta^*\left(X^{\Delta}_{M,D}\right)$ be the profile which allocates weight $1/D$ to each of $\mathbf{x}^1, \ldots, \mathbf{x}^D$. Then $\mathrm{Maj}(\mu) = \mathbf{0}$, so Proposition 2.4 implies that $\mathrm{Cond}\left(X^{\Delta}_{M,D}, \mu\right) = X^{\Delta}_{M,D}$, hence μ is totally indeterminate. In fact, it follows from Proposition 2.4 that μ is totally indeterminate *only* if $\mathrm{Maj}(\mu) = \mathbf{0}$, i.e. if $\mathrm{med}_d(\mu) = 0$ for all $d \in [1...D]$. Moreover, it also easily verified that μ is issue-wise indeterminate only if $\mathrm{Maj}(\mu) = \mathbf{0}$, thus a profile is issue-wise indeterminate if and only if it is totally indeterminate.

(The profile that allocates weight $1/D$ to each of $\mathbf{x}^1, \ldots, \mathbf{x}^D$ is not as contrived as it may seem. When the government engages in wealth redistribution, the elements of $[1...G]$ represent the potential recipients of government largesse (e.g. state governments seeking federal assistance; economic sectors seeking subsidies, etc.). If the redistribution is decided by a committee (e.g. the Senate), and each potential recipient controls roughly the same number of committee members (e.g. each state has two senators), then the resulting profile closely resembles this totally indeterminate profile). $\diamondsuit$

For any $\mathbf{x} \in X$ and $\mathbf{z} \in \{0,1\}^K$, recall that $\mathbf{x} \asymp \mathbf{z}$ if there is no $\mathbf{y} \in X \setminus \{\mathbf{x}\}$ which is ***between*** $\mathbf{x}$ and $\mathbf{z}$ (i.e. such that, for all $k \in [1...K]$ and $c \in \{0,1\}$, $(x_k = c = z_k) \Longrightarrow (y_k = c)$). We say that $\mathbf{z} \in \{0,1\}^K$ is a ***panopticon*** for X if $\mathbf{x} \asymp \mathbf{z}$ for all $\mathbf{x} \in X$ (this implies that $\mathbf{z} \notin X$, because any element of X would be between itself and every other element of X). Heuristically, from a panopticon, one can 'see' each element of X without the view being blocked by any other elements. For example, $\mathbf{1}$ is a panopticon for $X^{\mathrm{com}}_{J,J;K}$. Let $\mathrm{Pan}(X)$ be the set of all panoptica for X. The following result is this section's key observation:

Proposition 5.2 *Let* $X \subseteq \{0,1\}^K$.

(a) *For any* $\mu \in \Delta^*(X)$, $\Big(\mu$ *is totally indeterminate*$\Big) \iff \Big(\mathrm{Maj}(\mu) \in \mathrm{Pan}(X)\Big)$.

(b) $\Big(X$ *is totally indeterminate*$\Big) \iff \Big(\mathrm{Maj}(X) \cap \mathrm{Pan}(X) \neq \emptyset\Big)$.

Proof: (a) follows immediately from Lemma 1.1(b), and (b) follows from (a). $\Box$

Note that $\mathrm{Pan}(X) \neq \emptyset$ requires that all $\mathbf{x}, \mathbf{y} \in X$ have a Hamming distance ≥ 2. This is in itself a strong restriction; for instance, it precludes the permutahedron from having a panopticon, and hence from being totally indeterminate. Above, we have seen that the 'allocation' problems on $X^{\mathrm{com}}_{J,J;K}$ and $X^{\Delta}_{M,D}$ are totally indeterminate. Are there other natural classes of aggregation problems that are totally indeterminate? A necessary condition is that $\mathrm{Pan}(X) \neq \emptyset$. As in §4, it is natural to look at McGarvey problems, because for these $\mathrm{Pan}(X) \neq \emptyset$ is also sufficient for total indeterminacy. However, it is not clear that such problems exist. Heuristically, the problem is that, to have a panopticon, X must be a relatively 'small' subset of $\{0,1\}^K$, whereas to be McGarvey, X must be relatively 'large'. The next proposition illustrates this conflict.

Proposition 5.3 *Suppose* X *contains* $\mathbf{1}$ *and* $\mathbf{1}_k$*, for all* $k \in [1...K]$*. Then,* X *is McGarvey but* $\mathrm{Pan}(X) = \emptyset$.

Propositions 4.7, 5.2(b) and 5.3 together show that many naturally occurring aggregation problems are issue-wise indeterminate but not totally indeterminate. On the other hand, we will see in the next section that aggregation problems will frequently be 'almost' totally indeterminate.

6 Pushing the limits: Condorcet entropy and almost total indeterminacy

In this section, we propose a way to measure the relative size of the Condorcet set. As in other combinatorial contexts, it will be convenient to measure 'size' logarithmically. Specifically, for $\mu \in \Delta^*(X)$ let

$$h(\mu) \quad := \quad \frac{\log_2 |\mathrm{Cond}\,(X,\mu)\,|}{\log_2 |X|},$$

which we shall refer to as the *Condorcet entropy* of the profile μ.[22] Note that if $h(\mu) \approx 1$, then the profile μ is 'almost' totally indeterminate. For any $\mathbf{y} \in \{0,1\}^K$, let $X(\mathbf{y}) := \{\mathbf{x} \in X\ ;\ \mathbf{x} \asymp \mathbf{y}\}$. If $\mathbf{y} = \mathrm{Maj}(\mu)$, then Lemma 1.1(b) says $X(\mathbf{y}) = \mathrm{Cond}\,(X,\mu)$. Thus, we can define the *Condorcet entropy* of X:

$$h(X) \quad := \quad \sup_{\mu \in \Delta^*(X)} h(\mu) \quad = \quad \max_{\mathbf{y} \in \mathrm{Maj}(X)} \frac{\log_2 |X(\mathbf{y})|}{\log_2 |X|}.$$

The Condorcet entropy thus measures how close X is to being totally indeterminate. In particular, $h(X) = 1$ if and only if X is totally indeterminate. An important advantage of using the logarithmic entropy measure is that the Condorcet entropy of a Cartesian product is an average of the Condorcet entropies of its factors, and is thus independent of the number of factors.[23]

To illustrate, consider the linear convexity $\mathcal{C}_K^{\mathrm{line}}$ consisting of all intervals on the ordered set on $[1...K]$, and the corresponding space X_K^{line} (cf. §2.3). If $\mathbf{y}^* = (1,0,1,0,1,...) \in \{0,1\}^K$, then $X(\mathbf{y}^*)$ contains all elements of X_K^{line} that have exactly one array of an odd number of consecutive 1's starting at an odd numbered position in the set $[1...K]$. Evidently, 1/4 of the elements of X_K^{line} have this form if K is even, and slightly more if K odd. Moreover, $|X_K^{\mathrm{line}}| = K(K-1)/2$, so it takes roughly 2 bits less to encode the Condorcet set than the entire set X_K^{line}, out of $\log_2[K(K-1)/2]$. We thus obtain

$$h(X_K^{\mathrm{line}}) \quad \approx \quad 1 - \frac{2}{\log_2[K(K-1)/2]} \quad \approx \quad 1 - \frac{2}{2\log_2(K) - 1} \quad \xrightarrow[K\to\infty]{} \quad 1.$$

This shows that $h(X_K^{\mathrm{line}})$ converges to 1 as K grows, i.e. X_K^{line} is 'asymptotically' totally indeterminate. Formally, we say that a sequence $\{X_n\}_{n=1}^{\infty}$ of aggregation problems is *asymptotically totally indeterminate* if $\lim_{n\to\infty} h(X_n) = 1$.

Another natural example of a convexity is the *hypercube convexity* $\mathcal{C}_D^{\square}$ on $\{0,1\}^D$ which consists of all *subcubes* of $\{0,1\}^D$. Formally, a set $C \subseteq \{0,1\}^D$ belongs to $\mathcal{C}_D^{\square}$ if and only if

[22] For any $Y \subset \{0,1\}^K$, the ratio $(\log_2 |Y|)/K$ can be interpreted as the *information content* of Y relative to the *potential* information content available in $\{0,1\}^K$; this is often referred to as the *entropy* of Y (Cover and Thomas, 1991). Our notion of Condorcet entropy can thus be viewed as measuring the information content of $\mathrm{Cond}\,(X,\mu)$ as a fraction of the information content of X.

[23] For instance, without the logarithmic transformation the Condorcet entropy of the n-fold Cartesian product of any space that is not totally indeterminate would tend to 0 as n grows.

$C = \{0,1\}^{D_1} \times \{0\}^{D_2} \times \{1\}^{D_3}$ for some partition $\{D_1, D_2, D_3\}$ of D. Let $X_D^\square := \{\mathbf{1}_C \ ; \ C \in \mathcal{C}_D^\square\}$ (so that $X_D^\square \subseteq \{0,1\}^K$ with $K = 2^D$).

The following result states that besides the spaces X_K^{line} also the spaces $\mathcal{C}_D^\square$, the aggregation of preference orderings and the aggregation of equivalence relations are asymptotically totally indeterminate as the number of alternatives grows without bound.

Proposition 6.1 **(a)** *The sequence* $\{X_N^{\text{pr}}\}_{N=1}^\infty$ *is asymptotically totally indeterminate.*

(b) *The sequence* $\{X_N^{\text{eq}}\}_{N=1}^\infty$ *is asymptotically totally indeterminate.*

(c) *The sequence* $\{X_K^{\text{line}}\}_{K=1}^\infty$ *is asymptotically totally indeterminate.*

(d) *The sequence* $\{X_D^\square\}_{D=1}^\infty$ *is asymptotically totally indeterminate.*

Remark. In Proposition 6.1(a,b,c), we only establish $1 - h(X_N) \leq O\left(\frac{1}{\log_2(N)}\right)$, so the convergence may be slow. By contrast, the convergence in 6.1(d) is in fact very rapid with $1 - h(X_D^\square) \leq O\left(\frac{1}{D}\right)$.

Conclusion

The Condorcet set includes all minimally acceptable compromises between majoritarianism and logical consistency; arguably, it thus contains the output of a large class of judgement aggregation rules. It also provides a compact description of the possible outcomes of diachronic judgement aggregation: the way in which real social decisions often emerge from an uncoordinated sequence of *ad hoc* judgements unfolding over time. Unfortunately, the Condorcet set is quite large for almost any nontrivial judgement aggregation problem. In many plausible scenarios, path-dependence can override unanimous consensus on some propositions; in others, it can manipulate the truth value of *every* proposition. In some cases, *any* logically consistent outcome can arise from a suitably chosen path. In short: history matters.

Several problems remain open. For example, let P be the set of all paths on $[1...K]$. For any $\mu \in \Delta^*(X)$, diachronic judgement aggregation via simple majority defines a function $F : P \longrightarrow X$. Let ν be the uniform probability distribution on P; what is the distribution of $F \circ \nu$? If $F \circ \nu$ is almost-uniformly distributed on X, this represents an especially acute form of total indeterminacy. On the other hand, if $F \circ \nu$ is mostly concentrated on one or a few views, then this perhaps recommends these views as superior social choices. Also: what proportion of P violates unanimity?

We have assumed a diachronic process based on simple majority vote, because majority vote is the unique binary voting rule which is decisive, neutral, and anonymous (May, 1952). However, we could obtain greater path-independence by sacrificing decisiveness (e.g. using supermajoritarian voting in some coordinates), anonymity (e.g. using weighted voting rules) or both (e.g. using a system with vetoes or oligarchies). In particular, if we use a system of voting rules satisfying the *intersection property*, then the outcome is guaranteed to be logically consistent, and hence, path-independent (Nehring and Puppe, 2007, Proposition 3.4). For example, for all $k \in [1...K]$, let N_k be the order of the largest critical fragment containing k, and let $q_k := \max\{\frac{1}{2}, 1 - \frac{1}{N_k}\}$. Suppose we decide the truth value of k via q_k-supermajoritarian voting for each $k \in [1...K]$; then the outcome will be path-independent (Nehring and Puppe, 2007, Fact 3.4).

Is there an optimal tradeoff between decisiveness, neutrality, anonymity, and path-independence? One possibility: simple majorities could make 'provisional' rulings on the truth of certain propositions, but these rulings would only be treated as 'precedents' (i.e. binding on later decisions) if they exceeded some supermajority threshold —otherwise they could be overturned by a later, larger supermajority.

Finally, from the normative point of view, the indeterminacy of the Condorcet set poses the questions what further considerations can be invoked to pin down the view of the majority, and what rules these give rise to. In Section 1.2, we have briefly introduced the class of weighted support rules. Are there are other, well-motivated classes of rules that always select from the Condorcet set?

Appendix A: Proofs

Proof of Lemma 1.1 (a) is immediate from the definition of Condorcet admissibility, as is the first part of (b). Now suppose $\mathrm{Maj}(\mu) \notin X$; it remains only to show that $|\mathrm{Cond}\,(X,\mu)\,| \geq 3$. To see this, note that $\mathrm{Maj}(\mu) \notin X$ only if there is some X-critical fragment $\mathbf{w}$ which is activated by μ. This fragment $\mathbf{w}$ must have order 3 or more (if $|\mathbf{w}| = 2$, then $\mathbf{w}$ could not be activated by μ: if each coordinate received majority support, then a nonzero proportion of voters would endorse *both* coordinates of $\mathbf{w}$, which is impossible because $\mathbf{w}$ is forbidden).

Now, fix some coordinate $j \in \mathrm{supp}\,(\mathbf{w})$, and let $\mathbf{w}^j$ be the fragment obtained by deleting coordinate j from $\mathbf{w}$. Let $X_j := \{\mathbf{x} \in X\ ;\ \mathbf{w}^j \sqsubset \mathbf{x}\}$; then X_j is nonempty, because $\mathbf{w}^j$ is not forbidden, because $\mathbf{x}$ is critical. Let $\mathbf{x}$ be an element of X_j such that there exists no $\mathbf{y} \in X_j$ with $M(\mathbf{x},\mu) \subsetneq M(\mathbf{y},\mu)$.

Claim 1: $\mathbf{x} \in \mathrm{Cond}\,(X,\mu)$.

Proof: Suppose there was some $\mathbf{y} \in X$ with $M(\mathbf{x},\mu) \subsetneq M(\mathbf{y},\mu)$. Then we must have $\mathbf{y} \notin X_j$, which means $y_j = w_j$. But $\mathrm{supp}\,(\mathbf{w}) \setminus \{j\} \subseteq M(\mathbf{x},\mu) \subset M(\mathbf{y},\mu)$, so $\mathbf{w}^j \sqsubset \mathbf{y}$. Thus, $\mathbf{w} \sqsubset \mathbf{y}$. But $\mathbf{w}$ is forbidden, so this is impossible for any $\mathbf{y} \in X$. ◇ `Claim 1`

Thus, we can obtain an element of $\mathrm{Cond}\,(X,\mu)$ by contradicting the majority will in any single coordinate of $\mathbf{w}$; thus, there are at least as many different elements of $\mathrm{Cond}\,(X,\mu)$ as there are coordinates in $\mathbf{w}$ —hence $|\mathrm{Cond}\,(X,\mu)\,| \geq |\mathbf{w}| \geq 3$. □

Proof of Proposition 1.2 To show that any weighted support rule $F_{\phi,\lambda}$ is Condorcet admissible, take two views $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x}$ agrees with the majority view on a strictly larger set of issues than $\mathbf{y}$; say $\mathbf{x}$ agrees with $\mathrm{Maj}(\mu)$ on $K_\mathbf{x} \subseteq [1...K]$, and $\mathbf{y}$ agrees with

$\mathrm{Maj}(\mu)$ on $K_{\mathbf{y}} \subseteq [1...K]$, with $K_{\mathbf{x}} \supsetneq K_{\mathbf{y}}$. If $k \in K_{\mathbf{y}}$, or if $k \notin K_{\mathbf{x}}$, then $\mu_k(x_k) = \mu_k(y_k)$; on the other hand, if $k \in K_{\mathbf{x}} \setminus K_{\mathbf{y}}$, then

$$\mu_k(x_k) - \tfrac{1}{2} > 0 > \mu_k(y_k) - \tfrac{1}{2},$$

hence $\phi[\mu_k(x_k) - \frac{1}{2}] > \phi[\mu_k(y_k) - \frac{1}{2}]$ by our assumptions on ϕ. This shows that $\mathbf{y}$ does not maximize the sum on the right hand side of (2), and hence that $\mathbf{y}$ is not chosen by $F_{\phi,\lambda}$.

Conversely, let a gain function ϕ and a view $\mathbf{x} \in \mathrm{Cond}\,(X, \mu)$ be given. Suppose that $\mathbf{x}$ agrees with the majority view in the set $K_{\mathbf{x}} \subseteq [1...K]$ of issues, and let $m := |K_{\mathbf{x}}|$ (note that $m > 0$). If $m = K$, then $\mathbf{x}$ agrees with the majority view in all issues and is the unique Condorcet admissible view by Lemma 1.1. Thus, assume that $m < K$. For $\varepsilon > 0$, define $\lambda_k(\varepsilon) = \frac{1}{m} - \varepsilon$ if $k \in K_{\mathbf{x}}$ and $\lambda_k(\varepsilon) = \frac{m\varepsilon}{K-m}$ if $k \notin K_{\mathbf{x}}$. If ε is sufficiently small, we obtain $0 < \lambda_k(\varepsilon) < 1$ for all k. Moreover, for sufficiently small ε the view $\mathbf{x}$ is the unique maximizer of the right hand side of (2). Indeed, any other view $\mathbf{y} \in \mathrm{Cond}\,(X, \mu)$ disagrees with the majority view in least one issue $k \in K_{\mathbf{x}}$. Since, for all $k \notin K_{\mathbf{x}}$, $\lambda_k(\varepsilon)$ tends to zero, and $\phi[\mu_k(x_k) - \frac{1}{2}] > 0$ for all $k \in K_{\mathbf{x}}$, this shows that $\mathbf{y}$ is not chosen by $F_{\phi,\lambda(\varepsilon)}$ for sufficiently small ε. □

Proof of Proposition 1.3. Let $\mathbf{z} := \mathrm{Maj}(\mu)$.

(a) Let γ be a path through $[1...K]$, and let $\mathbf{x} := F^{\gamma}(\mu)$. We must show that $\mathbf{x} \in \mathrm{Cond}\,(X, \mu)$. Let $\mathbf{y} \in X$, and suppose $\mathbf{y}$ is between $\mathbf{x}$ and $\mathbf{z}$.

Claim 1: *For all $t \in [1...K]$, we have $x_{\gamma(t)} = y_{\gamma(t)}$.*

Proof: (by induction on t) First, $x_{\gamma(1)} = z_{\gamma(1)}$ (by definition of F^{γ}). Thus, $y_{\gamma(1)} = x_{\gamma(1)}$ also (because $\mathbf{y}$ is between $\mathbf{x}$ and $\mathbf{z}$).

Now, let $J := \{\gamma(1), \ldots, \gamma(t-1)\}$, and suppose inductively that $\mathbf{x}_J = \mathbf{y}_J$; we will show that $x_{\gamma(t)} = y_{\gamma(t)}$. If $x_{\gamma(t)} = z_{\gamma(t)}$, then $y_{\gamma(t)} = x_{\gamma(t)}$ (because $\mathbf{y}$ is between $\mathbf{x}$ and $\mathbf{z}$). If $x_{\gamma(t)} \neq z_{\gamma(t)}$, then this must be because $z_{\gamma(t)}$ is X-inconsistent with $\mathbf{x}_J$. But then $z_{\gamma(t)}$ is also X-inconsistent with $\mathbf{y}_J$ (by induction), so we must also have $y_{\gamma(t)} \neq z_{\gamma(t)}$ (because $\mathbf{y} \in X$). Thus, $y_{\gamma(t)} = x_{\gamma(t)}$. ◇ `Claim 1`

Thus, if $\mathbf{y}$ is between $\mathbf{z}$ and $\mathbf{x}$, then Claim 1 implies that $\mathbf{y}$ *is* $\mathbf{x}$. Thus, $\mathbf{x} \asymp \mathbf{z}$; hence $\mathbf{x} \in \mathrm{Cond}\,(X, \mu)$, as desired.

(b) Let $\mathbf{x} \in \mathrm{Cond}\,(X, \mu)$; we must find a path γ such that $F^{\gamma}(\mu) = \mathbf{x}$. Let $J := |M(\mathbf{x}, \mu)|$, and let $\gamma : [1...K] \longrightarrow [1...K]$ be a path such that $\gamma[1 \ldots J] = M(\mathbf{x}, \mu)$. Thus, $x_{\gamma(j)} = z_{\gamma(j)}$ for all $j \in [1 \ldots J]$, while $x_{\gamma(j)} \neq z_{\gamma(j)}$ for all $j \in [J+1 \ldots K]$. Let $\mathbf{y} := F^{\gamma}(\mu)$.

Claim 2: *For all $t \in [1...J]$, $y_{\gamma(t)} = x_{\gamma(t)}$.*

Proof: (by induction on t) First, $y_{\gamma(1)} = z_{\gamma(1)}$ by definition of F^{γ}; hence $y_{\gamma(1)} = x_{\gamma(1)}$.

Let $t \in [1...J]$, let $I := \{\gamma(1), \ldots, \gamma(t-1)\}$ and suppose inductively that $\mathbf{y}_I = \mathbf{x}_I$. Now, $x_{\gamma(t)}$ is X-consistent with $\mathbf{x}_I$ (because $\mathbf{x} \in X$); hence $z_{\gamma(t)}$ is X-consistent with $\mathbf{y}_I$ (because $\mathbf{y}_I = \mathbf{x}_I$ by induction hypothesis, while $z_{\gamma(t)} = x_{\gamma(t)}$ by definition of γ and J). Thus, $y_{\gamma(t)} = z_{\gamma(t)}$; hence $y_{\gamma(t)} = x_{\gamma(t)}$. ◇ `Claim 2`

Claims 2 implies $\mathbf{y}_J = \mathbf{x}_J$; hence $\mathbf{y}$ is between $\mathbf{x}$ and $\mathbf{z}$ (by definition of J). But $\mathbf{y} \in X$ and $\mathbf{x} \asymp \mathbf{z}$; thus we must have $\mathbf{y} = \mathbf{x}$ —in other words, $F^\gamma(\mu) = \mathbf{x}$, as desired. □

Proof of Proposition 2.1. (a) We will give two proofs —one based directly on Condorcet admissibility, and one based on a 'diachronic' construction, via Proposition 1.3.

Direct proof of (a) "$\subseteq$" (by contrapositive) Let $\mathbf{x} \in X_N^{\mathrm{pr}}$ represent the ordering $(a_1 \overset{\mathbf{x}}{\prec} a_2 \overset{\mathbf{x}}{\prec} a_3 \overset{\mathbf{x}}{\prec} \cdots \overset{\mathbf{x}}{\prec} a_N)$, and suppose $\overset{\mathbf{x}}{\prec}$ is *not* a Hamiltonian chain in $\overset{\mu}{\prec}$. Then there exists $n \in [1...N)$ such that $a_n \overset{\mu}{\succeq} a_{n+1}$. Define the ordering $\overset{\mathbf{y}}{\prec}$ by switching the positions of a_n and a_{n+1} in $\overset{\mathbf{x}}{\prec}$. That is:

$$a_1 \overset{\mathbf{y}}{\prec} a_2 \overset{\mathbf{y}}{\prec} \cdots \overset{\mathbf{y}}{\prec} a_{n-2} \overset{\mathbf{y}}{\prec} a_{n-1} \overset{\mathbf{y}}{\prec} a_{n+1} \overset{\mathbf{y}}{\prec} a_n \overset{\mathbf{y}}{\prec} a_{n+2} \overset{\mathbf{y}}{\prec} a_{n+3} \overset{\mathbf{y}}{\prec} \cdots \overset{\mathbf{y}}{\prec} a_N.$$

Observe that $\overset{\mathbf{y}}{\prec}$ agrees with $\overset{\mathbf{x}}{\prec}$ in every pairwise ordering except the ordering of $\{a_n, a_{n+1}\}$. Thus, $M(\mathbf{y}, \mu) = M(\mathbf{x}, \mu) \cup \{(a_n \prec a_{n+1})\}$. Thus, $M(\mathbf{x}, \mu)$ is not maximal, so $\mathbf{x} \notin \mathrm{Cond}\,(X_N^{\mathrm{pr}}, \mu)$.

"$\supseteq$" Let $(a_1 \overset{\mu}{\prec} a_2 \overset{\mu}{\prec} \cdots \overset{\mu}{\prec} a_N)$ be any Hamiltonian chain in $\overset{\mu}{\prec}$. Let $\mathbf{x} \in X_N^{\mathrm{pr}}$ be the transitive closure of this Hamiltonian chain (so that $a_1 \overset{\mathbf{x}}{\prec} a_2 \overset{\mathbf{x}}{\prec} \cdots \overset{\mathbf{x}}{\prec} a_N$). Then $M(\mathbf{x}, \mu) \supseteq \{(a_1 \prec a_2), (a_2 \prec a_3), \ldots, (a_{N-1} \prec a_N)\}$. Furthermore, for any $\mathbf{y} \in X_N^{\mathrm{pr}}$, if $M(\mathbf{y}, \mu) \supseteq \{(a_1 \prec a_2), (a_2 \prec a_3), \ldots, (a_{N-1} \prec a_N)\}$, then clearly $\mathbf{y} = \mathbf{x}$. Thus, $M(\mathbf{x}, \mu)$ is maximal, so $\mathbf{x} \in \mathrm{Cond}\,(X_N^{\mathrm{pr}}, \mu)$.

Diachronic proof of (a) "$\subseteq$" Let $\gamma : [1...K] \longrightarrow [1...K]$ be a path, and let $\mathbf{x} = F^\gamma(\mu)$; we must show that $\mathbf{x}$ is a Hamiltonian chain in $\mathrm{Maj}(\mu)$. To see this, let $n_1, n_2 \in [1...N]$ be any nearest neighbours in the ordering $\overset{\mathbf{x}}{\prec}$; then the ordering between n_1 and n_2 must have been decided directly through majority vote, not indirectly through transitivity constraints (because a transitivity constraint cannot force the ordering between nearest neighbours). Thus, every nearest-neighbour ordering specified by $\mathbf{x}$ agrees with the order specified by $\mathrm{Maj}(\mu)$. Thus, $\mathbf{x}$ is a Hamiltonian chain in $\mathrm{Maj}(\mu)$.

"$\supseteq$" Conversely, let $\mathbf{x}$ be a Hamiltonian chain in $\mathrm{Maj}(\mu)$, which represents the linear ordering $m_1 \overset{\mathbf{x}}{\prec} m_2 \overset{\mathbf{x}}{\prec} \cdots \overset{\mathbf{x}}{\prec} m_N$ of $[1...N]$. Let γ be any path such that $\gamma(1) = (m_1, m_2)$, $\gamma(2) = (m_2, m_3)$, $\ldots$, $\gamma(N-1) = (m_{N-1}, m_N)$ (the rest of γ is arbitrary). Thus, in the first $N-1$ steps, γ decides the pair-orderings $m_1 \prec m_2$, $m_2 \prec m_3$, $\ldots$, $m_{N-1} \prec m_N$, each through majority vote. At this point, the rest of $F^\gamma(\mu)$ is forced to equal $\mathbf{x}$ by transitivity constraints.

(b) "$\Rightarrow$" (by contrapositive) Let $\mathbf{x} \in \mathrm{Cond}\,(X_N^{\mathrm{pr}}, \mu)$ with $a_1 \overset{\mathbf{x}}{\prec} a_2 \overset{\mathbf{x}}{\prec} \cdots \overset{\mathbf{x}}{\prec} a_N$, and suppose that for some $k < l$, $m = a_k$ and $n = a_l$, so that $m \overset{\mathbf{x}}{\prec} n$ by transitivity of $\overset{\mathbf{x}}{\prec}$. By part (a), the nearest-neighbour orderings in $\overset{\mathbf{x}}{\prec}$ agree with the orderings specified by $\overset{\mu}{\prec}$, hence $m = a_k \overset{\mu}{\prec} a_{k+1}$, $a_{k+1} \overset{\mu}{\prec} a_{k+2}, \cdots, a_{l-1} \overset{\mu}{\prec} a_l = n$, i.e. $m \overset{\mu}{\preceq}_* n$.

"$\Leftarrow$" (by contrapositive) Conversely, suppose that $m \overset{\mu}{\preceq}_* n$, and assume without loss of generality that $m \neq n$. Let $A := \{a_k, a_{k+1}, \cdots, a_l\}$ be a minimal set such that $a_k = m$,

$a_l = n$ and $a_k \stackrel{\mu}{\prec} a_{k+1}$, $a_{k+1} \stackrel{\mu}{\prec} a_{k+2}, \cdots, a_{l-1} \stackrel{\mu}{\prec} a_l$. Consider any path γ that decides the issues $(a_k, a_{k+1}), (a_{k+1}, a_{k+2}), \ldots, (a_{l-1}, a_l)$ first, and let $\mathbf{x} = F^\gamma(\mu)$. By minimality of A, there are no transitivity constraints among these first $l-k$ decisions, thus they all agree with the majority view, i.e. $m = a_k \stackrel{\mathbf{x}}{\prec} a_{k+1} \stackrel{\mathbf{x}}{\prec} \cdots \stackrel{\mathbf{x}}{\prec} a_l = n$. Hence, $m \stackrel{\mathbf{x}}{\prec} n$ by transitivity. □

Proof of Proposition 2.2. "$\Longrightarrow$" Let $n \in C_*(A, \mu)$, and choose $x \in \mathrm{Cond}\,(X, \mu)$ such that n is $\stackrel{\mathbf{x}}{\prec}$-maximal in A. Let $\stackrel{\mathbf{x}}{\prec}$ be given by $a_1 \stackrel{\mathbf{x}}{\prec} a_2 \stackrel{\mathbf{x}}{\prec} \cdots \stackrel{\mathbf{x}}{\prec} a_N$, and let $n = a_k$, so that $A \subseteq \{a_1, \cdots a_k\}$. By Proposition 2.1 (a), we have $a_1 \stackrel{\mu}{\prec} a_2$, $a_2 \stackrel{\mu}{\prec} a_3, \cdots, a_{k-1} \stackrel{\mu}{\prec} a_k$, and hence $a_l \stackrel{\mu}{\preceq}_* a_k$ for all $l = 1 \ldots k$. This shows that $n = a_k$ is $\stackrel{\mu}{\preceq}_*$-maximal in A.

"$\Longleftarrow$" Let n be $\stackrel{\mu}{\preceq}_*$-maximal in A. Consider the set $\{a_1, \ldots, a_M\}$ of all $\stackrel{\mu}{\preceq}_*$-maximal elements in A, and suppose $n = a_M$ and $a_1 \stackrel{\mu}{\prec} a_2 \stackrel{\mu}{\prec} \cdots \stackrel{\mu}{\prec} a_M \stackrel{\mu}{\prec} a_1$. Choose $\mathbf{y} \in X_N^{\mathrm{pr}}$ representing an ordering $\stackrel{\mathbf{y}}{\prec}$ such that a_M is $\stackrel{\mathbf{y}}{\prec}$-maximal in A, and $a_1 \stackrel{\mathbf{y}}{\prec} a_2 \stackrel{\mathbf{y}}{\prec} a_3 \stackrel{\mathbf{y}}{\prec} \cdots \stackrel{\mathbf{y}}{\prec} a_M$. If $\mathbf{y} \in \mathrm{Cond}\,(X, \mu)$, then we're done. If $\mathbf{y} \notin \mathrm{Cond}\,(X, \mu)$, then there exists some $\mathbf{x} \in \mathrm{Cond}\,(X, \mu)$ such that $M(\mathbf{y}, \mu) \subseteq M(\mathbf{x}, \mu)$. By Proposition 2.1 (b), $\{a_1, \ldots, a_M\}$ form the top M elements of $\stackrel{\mathbf{x}}{\prec}$ in A, and a_M must be maximal among them, because

$$M(\mathbf{x}, \mu) \quad \supseteq \quad M(\mathbf{y}, \mu) \quad \supseteq \quad \{(a_1 \prec a_2),\ (a_2 \prec a_3),\ \ldots,\ (a_{M-1} \prec a_M)\}.$$

It follows that a_M is the $\stackrel{\mathbf{x}}{\prec}$-maximal element of A, as desired. □

Proposition 2.4 follows from Theorem 2.5. To prove Theorem 2.5, we need a lemma and some further notation. Recall (footnote 6) that the Slater rule minimizes the Hamming distance to $\mathrm{Maj}(\mu)$. The next lemma clarifies what this means in the discrete cube $[0...M]^D$. For any $\mathbf{x}, \mathbf{y} \in [0...M]^D$, we define

$$d_1(\mathbf{x}, \mathbf{y}) \quad := \quad \sum_{d=1}^{D} |x_d - y_d|.$$

(This is the metric induced by the ℓ^1 norm on $\mathbb{R}^D$). Let $\Delta([0...M]^D)$ be the set of all probability distributions on $[0...M]^D$ —that is, all functions $\nu : [0...M]^D \longrightarrow [0,1]$ such that $\sum_{\mathbf{m} \in [0...M]^D} \nu(\mathbf{m}) = 1$. For any $\nu \in \Delta([0...M]^D)$ and all $d \in [1...D]$, we define a nonincreasing function $\nu_d : [0 \ldots M{+}1] \longrightarrow [0,1]$ by

$$\nu_d(n) \quad := \quad \sum \Big\{ \nu(\mathbf{m})\ ;\ \mathbf{m} = (m_1, \ldots, m_D) \in [0...M]^D \text{ and } \ m_d \geq n \Big\}, \tag{A1}$$

for every $n \in [0...M]$, while $\nu_d(M+1) := 0$. Let

$$\Delta^*([0...M]^D) \quad := \quad \Big\{ \nu \in \Delta([0...M]^D)\ ;\ \nu_d(m) \neq \tfrac{1}{2},\ \forall\, n \in [0...M] \text{ and } \ \forall\, d \in [1...D] \Big\}.$$

If $\nu \in \Delta^*([0...M]^D)$, then the *median* of ν is the (unique) point $\mathbf{m}^* \in [0...M]^D$ such that, for all $d \in [1...D]$, we have $\nu_d(m_d^*) > \frac{1}{2} > \nu_d(m_d^* + 1)$.

Lemma A.1 *Define* $\Phi : [0...M]^D \longrightarrow \{0,1\}^{D\times M}$ *by equation* (5) *in §2.2.*

(a) Φ *is an isometry from the metric* d_1 *on* $[0...M]^D$ *to the Hamming metric on* $\{0,1\}^{D\times M}$*. That is: for all* $\mathbf{x}, \mathbf{y} \in [0...M]^D$*, we have* $d_H(\Phi(\mathbf{x}), \Phi(\mathbf{y})) = d_1(\mathbf{x}, \mathbf{y})$*.*

(b) *Let* $P \subseteq [0...M]^D$ *and let* $X := \Phi[P] \subset \{0,1\}^{D\times M}$*. Let* $\mu \in \Delta^*(X)$*. Define* $\nu \in \Delta([0...M]^D)$ *by* $\nu(\mathbf{m}) := \mu[\Phi(\mathbf{m})]$ *for all* $\mathbf{m} \in [0...M]^D$*. Then* $\nu \in \Delta^*([0...M]^D)$*. If* $\mathbf{m}^*$ *is the median of* ν*, then* $\Phi(\mathbf{m}^*) = \mathrm{Maj}(\mu)$*.*

(c) *Thus, if* S *is the set of all points in* P *of minimal* d_1*-distance from* $\mathbf{m}^*$*, then* $\mathrm{Slater}\,(X, \mu) = \Phi[S]$*.*

For any $r > 0$, let $\diamondsuit(\mathbf{m}^*, r) := \{\mathbf{n} \in \mathbb{Z}^D\ ;\ d_1(\mathbf{n}, \mathbf{m}^*) = r\}$ be the 'sphere' of radius r around the median $\mathbf{m}^*$ in the d_1 metric on $\mathbb{Z}^D$ —this will be a polytope with 2^D faces, each of which is a D-dimensional simplex. (For example, if $D = 3$, then $\diamondsuit(\mathbf{m}^*, r)$ is a regular octahedron). In Lemma A.1(c), we have $S = P \cap \diamondsuit(\mathbf{m}^*, r)$ for some $r \in \mathbb{N}$. If $P = \triangle^D_M$ is the simplex defined in eqn.(6), then typically this intersection will be between P and one of the two simplicial faces of $\diamondsuit(\mathbf{m}^*, r)$ which lies parallel to P —see Figure 2.

Proof of Lemma A.1. (a) Let $\mathbf{x} = (x_1, \ldots, x_D)$ and $\mathbf{y} = (y_1, \ldots, y_D)$ be elements of $[0...M]^D$. Let $\widetilde{\mathbf{x}} := \Phi(\mathbf{x})$ and $\widetilde{\mathbf{y}} := \Phi(\mathbf{y})$ (elements of $\{0,1\}^{M\times D}$). For all $d \in [1...D]$, if $x_d < y_d$, then we have $\widetilde{x}_{(d,m)} = 0 \neq 1 = \widetilde{y}_{(d,m)}$ for all $m \in [x_d + 1 \ldots y_d]$. If $y_d < x_d$, then we have $\widetilde{y}_{(d,m)} = 0 \neq 1 = \widetilde{x}_{(d,m)}$ for all $m \in [y_d + 1 \ldots x_d]$. Either way, $\widetilde{x}_{(d,m)} = \widetilde{y}_{(d,m)}$ for all other $m \in [1...M]$, so that $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{y}}$ differ in exactly $|x_d - y_d|$ of the coordinates $(d, 1), \ldots, (d, M)$. This holds for all $d \in [1..D]$; hence $d_H(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) = |x_1 - y_1| + \cdots + |x_D - y_D| = d_1(\mathbf{x}, \mathbf{y})$.

(b) Let $\mathbf{x} := \mathrm{Maj}(\mu)$; we must show that $\mathbf{x} = \Phi(\mathbf{m}^*)$. For all $d \in [1...D]$ and $n \in [1...M]$, combining the defining equations (1), (5), and (A1) yields $\mu_{(d,n)}(1) = \nu_d(n)$. Thus,

$$\begin{aligned}\left(x_{(d,n)} = 1\right) \quad &\Longleftrightarrow \quad \left(\mu_{(d,n)}(1) > \tfrac{1}{2}\right) \Longleftrightarrow \left(\nu_d(n) > \tfrac{1}{2}\right) \\ &\Longleftrightarrow \quad \left(m^*_d \geq n\right) \underset{(5)}{\Longleftrightarrow} \left(\Phi(\mathbf{m}^*)_{(d,n)} = 1\right).\end{aligned}$$

Thus, $x_{(d,n)} = \Phi(\mathbf{m}^*)_{(d,n)}$, for all $d \in [1...D]$ and $n \in [1...M]$, as desired.

(c) This follows by combining (a) and (b) $\square$

Proof of Theorem 2.5. Let $\mathbf{m}^* \in [0...M]^D$ be the median point from Lemma A.1(b); thus, $\Phi(\mathbf{m}^*) = \mathrm{Maj}(\mu)$. Note that $\mathbf{m}^*$ must automatically satisfy the constraints $\underline{A}_d \leq m^*_d \leq \overline{A}_d$, for all $d \in [1...D]$ (because every point in X satisfies these constraints). There are three cases: either $\underline{S} \leq \sum_{d=1}^D m^*_d \leq \overline{S}$, or $\sum_{d=1}^D m^*_d > \overline{S}$, or $\sum_{d=1}^D m^*_d < \underline{S}$.

If $\underline{S} \leq \sum_{d=1}^D m^*_d \leq \overline{S}$, then $\mathrm{Maj}(\mu) \in X$, in which case Lemma 1.1(a) implies that $\mathrm{Cond}\,(X, \mu) = \mathrm{Maj}(\mu)$. Clearly, $\mathrm{Slater}\,(X, \mu) = \mathrm{Maj}(\mu)$ also, so we're done.

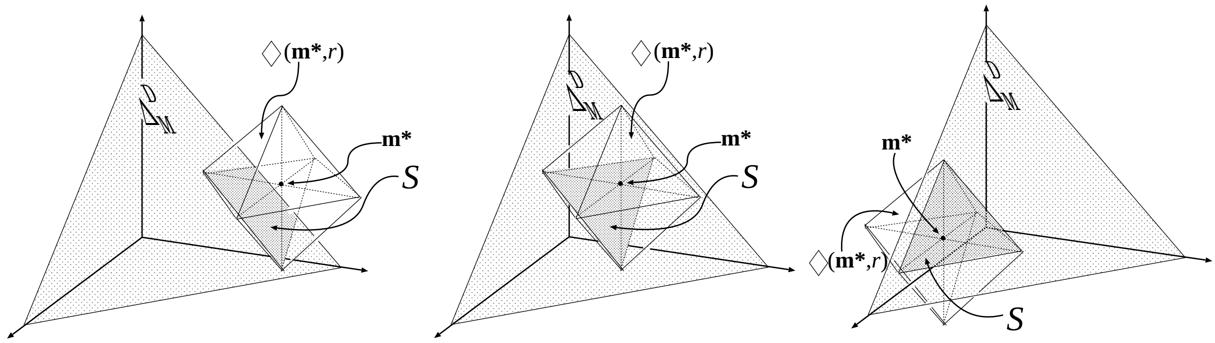

Figure 2: The Condorcet (=Slater) rule on the simplex. Here, $\mathrm{Maj}(X^{\triangle}_{M,D}, \mu) = \Phi[\mathbf{m}^*]$ and $\mathrm{Cond}\left(X^{\triangle}_{M,D}, \mu\right) = \mathrm{Slater}\left(X^{\triangle}_{M,D}, \mu\right) = \Phi[S]$, where S is the intersection of $\triangle^D_M$ with another, parallel simplex. Figures (a) and (b) illustrate the case when $\mathbf{m}^*$ is above the simplex. (c) illustrates the case when $\mathbf{m}^*$ is below the simplex.

If $\sum_{d=1}^{D} m^*_d > \overline{S}$, then $\mathbf{m}^*$ lies 'above' the polytope (as in Figures 2(a,b)). We define

$$\begin{aligned} O &:= \left\{\mathbf{n} \in \mathbb{Z}^D \ ; \ n_d \leq m^*_d, \ \forall \ d \in [1..D]\right\} \\ \text{and} \quad P^+ &:= \left\{\mathbf{m} \in [0...M]^D \ ; \ \sum_{d=1}^{D} m_d \ = \ \overline{S} \text{ and } \ \underline{A}_d \leq m_d \leq \overline{A}_d, \ \ \forall \ d \in [1...D]\right\}. \end{aligned}$$

Intuitively, O is the 'negative orthant' with its origin at $\mathbf{m}^*$, and P^+ is the 'top face' of the allocation polytope $P^D_{\mathbf{S},\mathbf{A}}$ defined in eqn.(7). Let $S := O \cap P^+$; then $\mathrm{Cond}\,(X, \mu) = \Phi[S]$. However, if $R := \sum_{d=1}^{D} m^*_d \ - \ \overline{S}$, then we have $S = \Diamond(\mathbf{m}^*, R) \cap P^D_{\mathbf{S},\mathbf{A}}$ —in other words, S is the set of points in $P^D_{\mathbf{S},\mathbf{A}}$ minimizing the d_1-distance to $\mathbf{m}^*$; thus, Lemma A.1(c) says $\mathrm{Slater}\,(X, \mu) = \Phi[S]$. Thus, $\mathrm{Slater}\,(X, \mu) = \mathrm{Cond}\,(X, \mu)$, as claimed.

If $\sum_{d=1}^{D} m^*_d < \underline{S}$, then the argument is similar, except that now $\mathbf{m}^*$ lies 'below' $P^D_{\mathbf{S},\mathbf{A}}$ (as in Figures 2(c)), so we define O to be the 'positive' orthant based at $\mathbf{m}^*$, and we intersect it with the 'bottom' face of $P^D_{\mathbf{S},\mathbf{A}}$. □

Proof of Proposition 2.7. "$\Longleftarrow$" Let $\mathrm{Maj}(\mu) = \mathbf{1}_{\bigsqcup I_n}$ with disjoint and non-adjacent intervals $I_1, \ldots I_N$, and let J be the smallest interval containing I_n and I_m. Let ℓ_n be the left-most element of I_n, and r_m the right-most element of I_m. Since $\mathbf{x} = \mathbf{1}_J$ respects the majority view in the issues $\ell_n - 1$, ℓ_n, r_m and $r_m + 1$, and since any element of X^{line}_K has to represent an interval, it is evident that $\mathbf{x} \in \mathrm{Cond}\,(X^{\text{line}}_K, \mu)$.

"$\Longrightarrow$" (by contrapositive) Let J be an interval such that its left-most element does *not* coincide with a left-most element of some I_n, or its right-most element does *not* coincide with a right-most element of some I_m. Then, obviously, the majority view can be respected in at least one additional issue. For instance, if the left-most element ℓ_J of J is in the interior of some I_n, then $\mathbf{1}_{J'} \in X_K^{\text{line}}$ with $J' = J \cup \{\ell_J - 1\}$ respects the majority view in a strictly larger set of issues. □

Notation A.2. Let $J \subset [1...K]$, and let $\mathbf{w} \in \{0,1\}^J$. We define $[\mathbf{w}] := \{\mathbf{x} \in \{0,1\}^K;\ \mathbf{w} \sqsubseteq \mathbf{x}\}$. Thus, $\mathbf{w}$ is X-forbidden if and only if $X \cap [\mathbf{w}] = \emptyset$. It is sometimes convenient to express $\mathbf{w}$ as an element of $\{0,1,*\}^K$, where we define $w_k = *$ for all $k \notin J$. For example, suppose $J = \{i, i+1, \ldots, j\}$ for some $i \leq j \leq K$; then we would write $\mathbf{w} = (\underbrace{*, \ldots, *}_{i-1}, w_i, w_{i+1}, \ldots, w_j, \underbrace{*, \ldots, *}_{K-j})$.

Proof of Theorem 3.2. Let $\mu \in \Delta^* (X)$, and let $\mathrm{Unan}(\mu) := \{\mathbf{x} \in \{0,1\}^K;\ \mathbf{x}$ does not violate μ-unanimity$\}$.

"$\Longrightarrow$" (by contrapositive) Let $\mathbf{w}$ be an X-critical fragment with $|\mathbf{w}| = N+1$ for some $N \geq 3$. By reordering $[1...K]$ if necessary, we can write

$$\mathbf{w} = (w_1, w_2, \ldots, w_N, w_{N+1}, *, *, \ldots, *),$$

where $w_1, \ldots, w_{N+1} \in \{0,1\}$. For all $n \in [1 \ldots N{+}1]$, let $\mathbf{w}^n := (w_1, \ldots, w_{n-1}, *, w_{n+1}, \ldots, w_{N+1}, *, *, \ldots, *)$. For all $n \in [1 \ldots N{+}1]$, there exists some $\mathbf{x}^n \in [\mathbf{w}^n] \cap X$ (because $\mathbf{w}^n$ is not X-forbidden because $\mathbf{w}$ is X-critical). Define $\mu \in \Delta^* (X)$ by $\mu(\mathbf{x}^n) = 1/N$ for all $n \in [1...N]$. Then $\mu_n[w_n] = (N-1)/N \geq 2/3$ (because $N \geq 3$) for all $n \in [1...N]$, while $\mu_{N+1}[w_{N+1}] = 1$, so $\mathrm{Maj}(\mu) \in [\mathbf{w}]$. Now, let γ be the path $(1, 2, 3, \ldots, K)$, and let $\mathbf{x} := F^\gamma(\mu)$; then $\mathbf{x} \in \mathrm{Cond}\,(X, \mu) \cap [\mathbf{w}^{N+1}]$ (because $\mathbf{w}^{N+1}$ is not X-forbidden). But $x_{N+1} \neq w_{N+1}$ (because $\mathbf{w}$ *is* X-forbidden). Thus, $\mathbf{x} \notin \mathrm{Unan}(\mu)$ (because $\mu_{N+1}[w_{N+1}] = 1$). Thus, Condorcet violates unanimity.

"$\Longleftarrow$" Suppose that all critical fragments have order 3 or less. If $\mathrm{Maj}(\mu) \in X$, then $\mathrm{Cond}\,(X, \mu) = \{\mathrm{Maj}(\mu)\}$ and we're done, because $\mathrm{Maj}(\mu) \in \mathrm{Unan}(\mu)$.

Suppose $\mathrm{Maj}(\mu) \notin X$, and let $\mathbf{y} \in \mathrm{Cond}\,(X, \mu)$.

Claim 1: *Let $k \in [1...K]$. If $y_k \neq \mathrm{Maj}_k(\mu)$, then $\mu_k(\mathrm{Maj}_k(\mu)) < 1$.*

Proof: If $y_k \neq \mathrm{Maj}_k(\mu)$, then there exist $i, j \in [1...K] \setminus \{k\}$ and some X-critical fragment $\mathbf{w} = (w_i, w_j, w_k) \sqsubset \mathrm{Maj}(\mu)$ such that $y_i = w_i$ and $y_j = w_j$, thereby forcing $y_k \neq w_k$. (If there was no such critical fragment, then we could change y_k to $\mathrm{Maj}_k(\mu)$ without leaving X —hence $\mathbf{y} \notin \mathrm{Cond}\,(X, \mu)$).

By contradiction, suppose $\mu_k[w_k] = 1$. Note that $\mu_i[w_i] > \frac{1}{2}$ and $\mu_j[w_j] > \frac{1}{2}$ (because $\mathbf{w} \sqsubset \mathrm{Maj}(\mu)$). Thus, $\mu_i[\neg w_i] < \frac{1}{2}$ and $\mu_j[\neg w_j] < \frac{1}{2}$, and of course, $\mu_k[\neg w_k] = 0$. But $X \setminus [\mathbf{w}] \subset [\neg w_i] \cup [\neg w_j] \cup [\neg w_k]$; thus, $\mu(X \setminus [\mathbf{w}]) \leq \mu[\neg w_i] + \mu[\neg w_j] + \mu[\neg w_k] < \frac{1}{2} + \frac{1}{2} + 0 = 1$. But if $\mu(X \setminus [\mathbf{w}]) < 1$, then $\mu(X \cap [\mathbf{w}]) > 0$, which means $[\mathbf{w}] \cap X \neq \emptyset$, contradicting the fact that $\mathbf{w}$ is an X-forbidden fragment.

By contradiction, we must have $\mu_k[w_k] < 1$. ◇ Claim 1

The contrapositive of Claim 1 says that $y_k = \mathrm{Maj}_k(\mu)$ whenever $\mu_k(\mathrm{Maj}_k(\mu)) = 1$; thus, $\mathbf{y} \in \mathrm{Unan}(\mu)$. This holds for all $\mathbf{y} \in \mathrm{Cond}\,(X, \mu)$; hence $\mathrm{Cond}\,(X, \mu) \subseteq \mathrm{Unan}(\mu)$, as desired. □

Proof of Example 3.3(a). (This is a special case of the proof of Example 3.3(b) below).

"$\kappa(X_N^{\rm pr}) \leq N$": Any $X_N^{\rm pr}$-forbidden fragment corresponds to a 'preference cycle' $a_1 \prec a_2 \prec \cdots \prec a_M \prec a_1$ (creating a transitivity violation), for some $a_1, \ldots, a_M \in [1...N]$; such a cycle can have at most N elements. Furthermore, this $X_N^{\rm pr}$-forbidden fragment is critical if it only specifies the ordering between nearest neighbours in the preference cycle (otherwise it has redundant coordinates which can be removed to obtain a forbidden subfragment). Thus a $X_N^{\rm pr}$-critical fragment requires at most N coordinates.

"$\kappa(X_N^{\rm pr}) \geq N$": Let $\mathbf{w}$ be the fragment of order N representing the set of N assertions: $\{1 \prec 2,\ 2 \prec 3,\ 3 \prec 4,\ \ldots,\ (N-1) \prec N,\ N \prec 1\}$. Then $\mathbf{w}$ is forbidden (it creates a cycle), but no subfragment of $\mathbf{w}$ is forbidden; hence $\mathbf{w}$ is $X_N^{\rm pr}$-critical. □

Notation: For any subset $I \subset [1...K]$, we define $\mathbf{1}_I^*$ to be the fragment $\mathbf{x}$ such that $x_i := 1$ for all $i \in I$ and $x_k := *$ for all $k \in [1...K] \setminus I$.

Proof of Example 3.3(b). Let $W := \text{width}(\lhd)$. We must show that $\kappa(X_\lhd^{\rm pr}) = W$.

Recall that $K = N(N-1)/2$ and $X_N^{\rm pr} \subset \{0,1\}^K$ is defined by identifying $[1...K]$ with some subset of $[1...N]\times[1...N]$ containing exactly one element of each pair $\{(n,m),\ (m,n)\}$ for each distinct $n, m \in \mathbb{N}$. By reordering $[1...K]$ if necessary, we can suppose that the partial ordering $\lhd$ is obtained by fixing the values of the coordinates $[J{+}1 \ldots K]$ for some $J \in [1...K]$. Thus, we can regard $X_\lhd^{\rm pr}$ as a subset of $\{0,1\}^J$.

"$\kappa(X_\lhd^{\rm pr}) \geq W$" Let $A := \{a_1, \ldots, a_W\} \subseteq [1...N]$ be an antichain of cardinality W. Let $I \subset [1...J]$ be the set of coordinates representing the W pairs $\{(a_1,a_2),(a_2,a_3),\ldots,(a_{W-1},a_W),$ $(a_W,a_1)\}$. Thus, the fragment $\mathbf{1}_I^*$ represents the (cyclical) binary relation "$a_1 \lhd a_2 \lhd \cdots \lhd a_N \lhd a_1$". Thus, $\mathbf{1}_I^*$ is a $(X_\lhd^{\rm pr})$-forbidden fragment.

Now, for any linear ordering of A, there is some element of $X_\lhd^{\rm pr}$ which encodes an extension of this ordering (because A is an antichain of $\lhd$). Thus, no subfragment of $\mathbf{1}_I^*$ is $(X_\lhd^{\rm pr})$-forbidden; hence $\mathbf{1}_I^*$ is a $(X_\lhd^{\rm pr})$-critical fragment. Clearly, $|\mathbf{1}_I^*| = W$. Thus, $\kappa(X_\lhd^{\rm pr}) \geq W$.

"$\kappa(X_\lhd^{\rm pr}) \leq W$" (by contradiction) Let $\mathbf{v}$ be a $(X_\lhd^{\rm pr})$-critical fragment, and suppose $|\mathbf{v}| > W$. Then $\mathbf{v}$ must encode a cyclical binary relation "$b_1 \check{\prec} b_2 \check{\prec} \cdots \check{\prec} b_V \check{\prec} b_1$", for some subset $B := \{b_1, \ldots, b_V\} \subset [1...N]$. Let $I \subset [1...J]$ be the set of V coordinates representing the V pairs $\{(b_1,b_2),(b_2,b_3),\ldots,(b_{V-1},b_V),\ (b_V,b_1)\}$;

Claim 1: $\mathbf{v} = \mathbf{1}_I^*$*; hence* $|\mathbf{v}| = V$.

Proof: (by contradiction) If $\text{supp}\,(\mathbf{v})$ contained any coordinates not linking two elements from the set $\{b_1, \ldots, b_V\}$, then we could remove this coordinate without destroying the cycle. Likewise, if $\text{supp}\,(\mathbf{v})$ contained (b_n, b_m) for any $n, m \in [1...V]$ with $m \not\equiv n \pm 1 \pmod V$, then we could remove this coordinate without breaking the cycle. Either way we would obtain a shorter $(X_\lhd^{\rm pr})$-forbidden fragment, contradicting the minimality of $\mathbf{v}$. ◇ Claim 1

Now, $|B| = V$, and Claim 1 implies that $V = |\mathbf{v}|$, while $|\mathbf{v}| > W$ by hypothesis. Thus, $|B| > W$, so B cannot be a $\lhd$-antichain. Thus, there exist some $n, m \in [1...V]$ such that $b_n \lhd b_m$. Clearly, $n \not\equiv m \pm 1 \pmod V$, because $(n, n+1)$ and $(n, n-1)$ are represented by elements of $[1...J]$ (because they form part of the support of $\mathbf{v}$). If $n < m$, then we can remove the elements $\{b_{n+1}, \ldots, b_{m-1}\}$ and get a shorter cycle:

$$b_1 \overset{\vee}{\prec} b_2 \overset{\vee}{\prec} \cdots \overset{\vee}{\prec} b_n \lhd b_m \overset{\vee}{\prec} b_{m+1} \overset{\vee}{\prec} \cdots \overset{\vee}{\prec} b_{V-1} \overset{\vee}{\prec} b_V \overset{\vee}{\prec} b_1.$$

If $n > m$, then we can remove the elements $\{b_1, \ldots, b_{m-1}\}$ and $\{b_{n+1}, \ldots, b_V\}$ and get a shorter cycle:

$$b_m \overset{\vee}{\prec} b_{m+1} \overset{\vee}{\prec} \cdots \overset{\vee}{\prec} b_{n-1} \overset{\vee}{\prec} b_n \lhd b_m.$$

Either way, we can construct a *smaller* $(X^{\mathrm{pr}}_{\lhd})$-forbidden fragment, which contradicts the minimality of $\mathbf{v}$. By contradiction, we must have $V \leq W$. This argument holds for any critical fragment $\mathbf{v}$. Thus, $\kappa(X^{\mathrm{pr}}_{\lhd}) \leq W$. □

Proof of Example 3.3(c). "$\kappa(X^{\mathrm{eq}}_N) \leq N$": Any X^{eq}_N-forbidden fragment corresponds to some 'broken cycle' of the form: $a_1 \sim a_2 \sim \cdots \sim a_M \not\sim a_1$ (creating a transitivity violation), for some $a_1, \ldots, a_M \in [1...N]$; such a broken cycle can have at most N elements. Furthermore, this X^{eq}_N-forbidden fragment is critical only if it specifies only the (non)equivalences between nearest neighbours in the broken cycle (otherwise it has redundant coordinates which can be removed to obtain a forbidden subfragment). Thus an X^{eq}_N-critical fragment requires at most N coordinates.

"$\kappa(X^{\mathrm{eq}}_N) \geq N$": let $\mathbf{w} \in X^{\mathrm{eq}}_N$ be the fragment of order N representing the set of N assertions $\{1 \equiv 2,\ 2 \equiv 3,\ 3 \equiv 4,\ \ldots,\ (N-1) \equiv N,\ N \not\equiv 1\}$. Then $\mathbf{w}$ is X^{eq}_N-forbidden (it violates transitivity), but no subfragment of $\mathbf{w}$ is forbidden; hence $\mathbf{w}$ is X^{eq}_N-critical. □

Proof of Example 3.3(e). "$\kappa(X^{\triangle}_{M,D}) \leq D$": Let $\mathbf{w}$ be a $X^{\triangle}_{M,D}$-critical fragment. Suppose there exist $d \in [1...D]$ and $n < m \in [1...M]$ such that $w_{(d,n)} = 0$ and $w_{(d,m)} = 1$. Then we could eliminate all other coordinates from $\mathbf{w}$ to obtain a $X^{\triangle}_{M,D}$-forbidden fragment containing only these two coordinates. Thus, if $\mathbf{w}$ was $X^{\triangle}_{M,D}$-critical, then we must have $\mathbf{w} = (w_{(d,n)}, w_{(d,m)})$; hence $|\mathbf{w}| = 2$.

So, for all $d \in [1...D]$ assume that there do *not* exist any $n < m \in [1...M]$ such that $w_{(d,n)} = 0$ and $w_{(d,m)} = 1$.

Claim 1: *For each* $d \in [1...D]$, $\mathrm{supp}\,(\mathbf{w})$ *contains at most* one *of the coordinates* $\{(d,1),\ (d,2), \ldots, (d,M)\}$.

Proof: (by contradiction) Let $1 \leq n < m \leq M$, and suppose $(d,n) \in \mathrm{supp}\,(\mathbf{w})$ and $(d,m) \in \mathrm{supp}\,(\mathbf{w})$. If $w_{(d,n)} = 1 = w_{(d,m)}$ then we can remove coordinate $w_{(d,n)}$ and still have a $X^{\triangle}_{M,D}$-forbidden fragment. If $w_{(d,n)} = 0 = w_{(d,m)}$, then we can remove coordinate $w_{(d,m)}$ and still have a $X^{\triangle}_{M,D}$-forbidden fragment. Now suppose $w_{(d,n)} = 1$ and $w_{(d,m)} = 0$. If $\|\mathbf{w}\| > M$ then we can remove $w_{(d,m)}$ and still have a $X^{\triangle}_{M,D}$-forbidden fragment. If $\|\mathbf{w}\| < M$, then we can remove $w_{(d,n)}$ and still have a $X^{\vartriangle}_{M,D}$-forbidden fragment.

In any case, we can remove a coordinate from $\mathbf{w}$ to obtain a smaller $X^{\vartriangle}_{M,D}$-forbidden fragment; hence $\mathbf{w}$ is not $X^{\vartriangle}_{M,D}$-critical. $\diamondsuit$ Claim 1

Claim 1 implies $|\mathbf{w}| \leq D$. This holds for any $X^{\vartriangle}_{M,D}$-critical fragment, so $\kappa(X^{\vartriangle}_{M,D}) \leq D$.

"$\kappa(X^{\vartriangle}_{M,D}) \geq D$": Let $m \in [1...M]$ be the smallest number such that $m\,D > M$, and consider the fragment $\mathbf{w}$ of order D defined by the D assertions $\{x_1 \geq m, x_2 \geq m, \ldots, x_D \geq m\}$. This fragment is $X^{\vartriangle}_{M,D}$-forbidden because it requires allocating a total of $m\,D > M$ 'dollars'. No subfragment of $\mathbf{w}$ is $X^{\vartriangle}_{M,D}$-forbidden, because $(D-1)\,m = mD - m \underset{(*)}{\leq} mD - D = (m-1)\,D \underset{(\dagger)}{\leq} M$. (Here $(*)$ is because $D \leq m$ because $D^2 \leq M$, and $(\dagger)$ is because m is the *smallest* number with $m\,D > M$). Thus, $\mathbf{w}$ is critical. Thus, $\kappa(X^{\vartriangle}_{M,D}) \geq |\mathbf{w}| = D$. $\Box$

The proof of Proposition 3.4 requires the following observation:

Lemma A.3 *Let $\mathcal{C}$ be a convex structure on $[1...K]$. Any $X_{\mathcal{C}}$-critical fragment has the form $\mathbf{w} = (\mathbf{1}^*_J, 0_k)$, where $J \subset [1...K]$ is a Carathéodory-independent set and $k \in [1...K] \setminus \mathrm{conv}(J)$. Thus, $|\mathbf{w}| = |J| + 1$.*

Proof: Let $\mathbf{w}$ be a fragment with support $I \subset [1...K]$. Suppose $I = I_0 \sqcup I_1$, where $w_i = 0$ for all $i \in I_0$ and $w_i = 1$ for all $i \in I_1$. Then $\mathbf{w}$ is $X_{\mathcal{C}}$-forbidden if and only if $I_0 \cap \mathrm{conv}(I_1) \neq \emptyset$. We can strip out all coordinates in $I_0 \setminus \mathrm{conv}(I_1)$, and all but one of the coordinates in $I_0 \cap \mathrm{conv}(I_1)$, and still have a $X_{\mathcal{C}}$-forbidden fragment; hence we can assume that $I_0 = \{k\}$ some $k \in \mathrm{conv}(I_1)$. Next, find the smallest subset $J \subseteq I_1$ such that $k \in \mathrm{conv}(J)$, and strip out all coordinates in $I_1 \setminus J$; the result is still $X_{\mathcal{C}}$-forbidden fragment. At this point, we have $k \in \mathrm{conv}(J)$ and $k \notin \mathrm{conv}(J)$ for all $J' \subsetneq J$. Thus, J is Carathéodory-independent. $\Box$

Proof of Proposition 3.4. To see that $\kappa(X_{\mathcal{C}}) \leq \lambda(\mathcal{C}) + 1$, let $\mathbf{w}$ be any $X_{\mathcal{C}}$-critical fragment, and let $J \subset [1...K]$ be the Carathéodory-independent set described in Lemma A.3. Then $|\mathbf{w}| = |J| + 1 \leq \lambda(C) + 1$.

To see that $\kappa(X_{\mathcal{C}}) \geq \lambda(\mathcal{C}) + 1$, Let $J \subset [1...K]$ be a maximal Carathéodory-independent set, so that $|J| = \lambda(C)$. Let $\mathbf{w}$ be as in Lemma A.3. Then $\mathbf{w}$ is critical, and $|\mathbf{w}| = |J| + 1 = \lambda(C) + 1$. $\Box$

Proof of Proposition 3.5. Recall from equation (5) in §2.2 the way in which $X^{\vartriangle}_{M,D}$ is embedded in $\{0,1\}^{D\times M}$. Let $\gamma : [1\ldots DM] \longrightarrow [1\ldots D] \times [1\ldots M]$ be any path. For all $d \in [1\ldots D]$, let $t(d) := \min\{t \in [1\ldots DM];\ \ \gamma(t) = (d,m) \text{ for some } m \in [1,\ M/3)\}$. Find $d_1, d_2, d_3 \in [1\ldots D]$ such that $t(d_1) < t(d_2) < t(d_3) < t(d)$ for all other $d \in [1\ldots D]$. For all $j \in \{1,2,3\}$, let $\mathbf{x}^j$ be the element of $X^{\vartriangle}_{M,D}$ which allocates all M dollars towards claimant d_j. (Thus, for all $m \in [1...M]$, we have $x^j_{d_j,m} = 1$, while $x^j_{c,m} = 0$ for all $c \in [1...D] \setminus \{d_j\}$.) Then let $\mu \in \Delta(X^{\Delta}_{M,D})$ be the profile with $\mu[\mathbf{x}^j] = \frac{1}{3}$ for $j \in \{1, 2, 3\}$. Observe that $\mathrm{Maj}(\mu) = \mathbf{0}$. Furthermore, observe that $\mu_{d,m}(0) = 1$ (unanimity) for all $m \in [1 \ldots M]$ and $d \in [1 \ldots D] \setminus \{d_1, d_2, d_3\}$ (this set is nonempty because $D \geq 4$).

By construction, we must have $F^{\gamma}(\mu)_{d_1,m_1} = F^{\gamma}(\mu)_{d_2,m_2} = F^{\gamma}(\mu)_{d_3,m_3} = 0$ for some $m_1, m_2, m_3 \in [1,\ M/3)$. Thus, $F^{\gamma}(\mu)$ allocates less than $M/3$ dollars towards each of claimants d_1, d_2, d_3. Thus, $F^{\gamma}(\mu)$ must allocate more than 0 dollars towards some other claimant. That is, $F^{\gamma}(\mu)_{d,m} = 1$ for some $m \in [1...M]$ and $d \in [1 \ldots D] \setminus \{d_1, d_2, d_3\}$. But this violates unanimity in coordinate (d, m). □

Proof of Proposition 3.6. Consider the path ζ specified in the statement of the proposition, and denote by $\overset{\mu}{\prec}_{\zeta}$ the ordering in X^{pr}_N generated by $F^{\zeta}(\mu)$.

Claim 1: *For any μ, $\overset{\mu}{\prec}_{\zeta}$ extends $\overset{\mu}{\prec}_{uc}$.*

Proof: The claim is obvious if $N \leq 2$. By induction, suppose that the claim holds for all sets of cardinality $l < N$. Take n, m such that $m \overset{\mu}{\prec} n$ and, for all k, $k \overset{\mu}{\prec} m \Rightarrow k \overset{\mu}{\prec} n$. If $m = 1$ or $n = 1$, we evidently obtain $m \overset{\mu}{\prec}_{\zeta} n$. Thus, let $m, n \neq 1$. Now observe that no binary comparison between alternative 1 and any other alternative $k = 2, \ldots, N$ can ever force $n \overset{\mu}{\prec}_{\zeta} m$, since by assumption either $[1 \overset{\mu}{\prec} n$ and $1 \overset{\mu}{\prec} m]$, $[n \overset{\mu}{\prec} 1$ and $m \overset{\mu}{\prec} 1]$, or $[m \overset{\mu}{\prec} 1$ and $1 \overset{\mu}{\prec} n]$. Hence the relative ranking between n and m in the ordering $\overset{\mu}{\prec}_{\zeta}$ coincides with their relative ranking induced by the application of F^{ζ} to the set $[2...N]$. Thus, by the induction hypothesis, $m \overset{\mu}{\prec}_{\zeta} n$. ◇ `Claim 1`

Let n, m be such that $n \overset{\mu}{\prec}_{\zeta} m$ but $m \overset{\mu}{\prec} n$. By Claim 1, there must exist $k \in [1...N]$ such that $n \overset{\mu}{\prec} k$ and $k \overset{\mu}{\prec} m$. The latter two relations imply that there is at least one voter who strictly prefers m to k *and* strictly prefers k to n, hence by transitivity, also m to n. Thus, $\overset{\mu}{\prec}_{\zeta}$ respects unanimous judgements. □

For the proof of Theorem 4.2 we need a series of further auxiliary results. The crux of path-dependence is that earlier precedents can override the majority will in a later decision. An explanation of how this happens requires a close analysis of the way that paths interact with critical fragments. If ζ is a path, and $k \in J \subset [1...K]$, then we say ζ "covers every other element of J before reaching k" if there is some $t \in [1 \ldots K]$ such that $\zeta(t) = k$ and $J \subseteq \zeta([1 \ldots t])$. Recall that $W(X, \mu)$ is the set of X-critical fragments activated by profile μ. Let $\mathbf{w} \in W(X, \mu)$ and let $J := \mathrm{supp}\,(\mathbf{w})$. If ζ is a path, and $k \in J$, then we say that ζ *focuses* $\mathbf{w}$ on k if:

(F1) ζ covers every other element of J before reaching k; and

(F2) for all $j \in J \setminus \{k\}$, we have $F^{\zeta}_j(\mu) = w_j = \mathrm{Maj}_j(\mu)$; hence

(F3) $F^{\zeta}_k(\mu) = \neg w_k \neq \mathrm{Maj}_k(\mu)$.

The following result characterizes exactly when diachronic aggregation violates a majority in a particular coordinate.

Proposition A.4 *Let $X \subseteq \{0,1\}^K$. Let $\mu \in \Delta^*(X)$, let $k \in [1...K]$, and let ζ be a path through $[1...K]$. Then:*

$$\left(F_k^\zeta(\mu) \neq \mathrm{Maj}_k(\mu)\right) \iff \left(\textit{There is some } \mathbf{w} \in W(X,\mu) \textit{ such that } \zeta \textit{ focuses } \mathbf{w} \textit{ on } k\right).$$

The "$\Longleftarrow$" direction of Proposition A.4 follows immediately from (F3). The proof of the "$\Longrightarrow$" direction of Proposition A.4 involves a certain combinatorial construction.

If $I, J \subseteq [1...K]$; then fragments $\mathbf{v} \in \{0,1\}^I$ and $\mathbf{w} \in \{0,1\}^J$ are *compatible* if $v_k = w_k$ for all $k \in I \cap J$ (hence, if $I \cap J = \emptyset$, then $\mathbf{v}$ and $\mathbf{w}$ are always compatible). In this case, we define $\mathbf{v} \uplus \mathbf{w} \in \{0,1\}^{I \cup J}$ by $(\mathbf{v} \uplus \mathbf{w})_i = v_i$ for all $i \in I$ and $(\mathbf{v} \uplus \mathbf{w})_j = w_j$ for all $j \in J$.

A *forbidden tree* of *height* 1 is a pair $T := (\mathbf{w}, j)$, where $\mathbf{w}$ is a critical fragment (called the *wood* of T) and $j \in \mathrm{supp}(\mathbf{w})$. We say that j is the *root* of T.

For any $h \geq 2$, we inductively define a *forbidden tree* of *height* h to be a system $T := (\mathbf{w}, j; T_1, \ldots, T_N)$, such that:

(T1) $\mathbf{w}$ is a critical fragment, and $j \in \mathrm{supp}(\mathbf{w})$. (Here, j is called the *root* of T, and $\mathbf{w}$ is the *trunk* of T.)

(T2) For all $n \in [1...N]$, T_n is a forbidden tree of height $h-1$ or less, whose root j_n is an element of $\mathrm{supp}(\mathbf{w}) \setminus \{j\}$.

(T3) For all $n \in [1...N]$, if subtree T_n has trunk $\mathbf{w}^n$, then $w^n_{j_n} = \neg w_{j_n}$.

(T4) For all $n \in [1...N]$, let $\widetilde{\mathbf{w}}^n$ be the wood of T_n, and let $J_n := \mathrm{supp}(\widetilde{\mathbf{w}}^n) \setminus \{j_n\}$. Finally let $J_0 := J \setminus \{j_1, \ldots, j_n\}$. Then the subfamilies $\mathbf{w}_{J_0}$, $\widetilde{\mathbf{w}}^1_{J_1}, \widetilde{\mathbf{w}}^2_{J_2}, \ldots, \widetilde{\mathbf{w}}^N_{J_N}$ are all compatible. The *wood* of T is the fragment $\widetilde{\mathbf{w}} := \mathbf{w}_{J_0} \uplus \widetilde{\mathbf{w}}^1_{J_1} \uplus \widetilde{\mathbf{w}}^2_{J_2} \uplus \cdots \uplus \widetilde{\mathbf{w}}^N_{J_N}$.

Note: The wood of T includes the root of T, but *not* the roots of its subtrees $T_1, \ldots, T_N$. In contrast, the *support* of T is defined inductively:

$$\mathrm{supp}(T) \quad := \quad \mathrm{supp}(\mathbf{w}) \cup \bigcup_{n=1}^{N} \mathrm{supp}(T_n).$$

Example A.5 Let $K = 36$, and identify $[1...K]$ with a 6×6 grid as shown in Figure 3(A). Let $J_1 := \{1, 2, 3\}$ and suppose $\mathbf{w}^1 := (0,0,0) \in \{0,1\}^{J_1}$ is critical [see Figure 3(B)]. Then $T_1 := (\mathbf{w}^1, 2)$ is a forbidden tree of height 1 [see Figure 3(C)]. Let $J_2 := \{2, 8, 14, 20, 26, 32\}$ and suppose $\mathbf{w}^2 := (1,1,1,1,1,1) \in \{0,1\}^{J_2}$ is critical [see Figure 3(D)]. Then $T_2 := (\mathbf{w}^2, 26; T_1)$ is a forbidden tree of height 2 [see Figure 3(E)]. Let $J_3 := \{25, 26, 27, 28, 29\}$ and suppose $\mathbf{w}^3 := (0,0,0,0,0) \in \{0,1\}^{J_3}$ is critical [see Figure 3(F)]. Then $T_3 := (\mathbf{w}^3, 28; T_2)$ is a forbidden tree of height 3 [see Figure 3(G)]. Let $J_4 := \{14, 15, 16, 17, 18\}$ and suppose $\mathbf{w}^4 := (1,0,0,0,0) \in \{0,1\}^{J_4}$ is critical [see Figure 3(H)]. Then $T_4 := (\mathbf{w}^4, 16)$ is a forbidden tree of height 1 [see Figure 3(I)]. Finally, let $J_5 := \{16, 22, 28, 34\}$ and suppose $\mathbf{w}^5 :=$

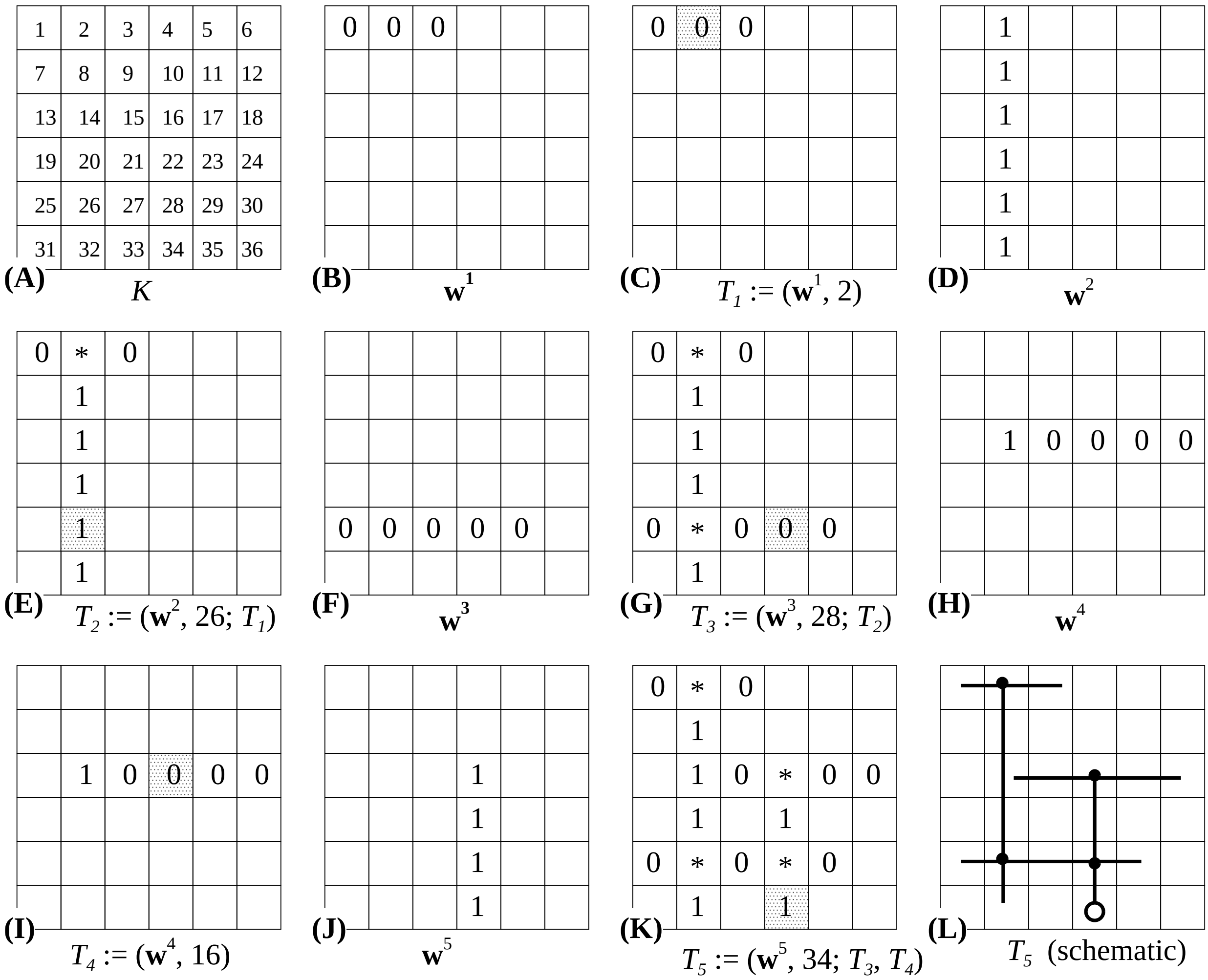

Figure 3: Construction of forbidden trees. Boxes labelled '0' or '1' are part of the wood of the tree. Boxes labelled '$*$' are in the support of the tree, but not its wood. The shaded boxes are the roots of the trees. See Example A.5 for explanation.

$(1,1,1,1) \in \{0,1\}^{J_5}$ is critical [see Figure 3(J)]. Then $T_5 := (\mathbf{w}^5, 34; T_3, T_4)$ is a forbidden tree of height 4, shown in Figure 3(K). A 'schematic' of T_5 is shown in Figure 3(L). Observe that two 'branches' of T_5 overlap in coordinate 14, but they are compatible because $w^2_{14} = 1 = w^4_{14}$; this is the significance of condition (T4). The *support* of T_5 is all entries in Figure 3(K) containing '0', '1', or '$*$'. The *wood* of T_5 is all entries in Figure 3(K) containing a '0' or a '1' (but *not* a '$*$'). $\diamondsuit$

For any $\mu \in \Delta^*(X)$, we say that μ ***activates*** the tree T if μ activates the wood of T.

Lemma A.6 *Let $X \subseteq \{0,1\}^K$.*

(a) *Let $\widetilde{\mathbf{w}}$ be the wood of a forbidden tree. Then $\widetilde{\mathbf{w}}$ is itself a forbidden fragment for X.*

(b) *Let $\mu \in \Delta^*(X)$, let $k \in [1...K]$, and let ζ be a path such that $F_k^\zeta(\mu) \neq \mathrm{Maj}_k(\mu)$. Then k is the root of a forbidden tree T activated by μ, such that:*

(b1) *ζ covers every other element of* $\mathrm{supp}(T)$ *before reaching k; and*

(b2) *if $\widetilde{\mathbf{w}}$ is the wood of T, then for all $j \in \mathrm{supp}(\widetilde{\mathbf{w}}) \setminus \{k\}$ we have $F_j^\zeta(\mu) = \mathrm{Maj}_j(\mu) = \widetilde{w}_j$.*

Proof: (a) (by induction on height) If T is a tree of height 1, then $T := (\mathbf{w}, j)$, and its wood is the forbidden fragment $\mathbf{w}$ by definition.

Now let $h \geq 2$ and inductively suppose the claim is true for all trees of height less than h. Let $T := (\mathbf{w}, j, T_1, \ldots, T_N)$ be a forbidden tree of height h, with wood $\widetilde{\mathbf{w}}$ and let $\widetilde{J} = \mathrm{supp}(\widetilde{\mathbf{w}})$. Let $\mathbf{x} \in X$, and suppose (by contradiction) that $\mathbf{x}_{\widetilde{J}} = \widetilde{\mathbf{w}}$. Let $J := \mathrm{supp}(\mathbf{w})$. For all $n \in [1...N]$, let forbidden tree T_n have root $j_n \in J$.

Claim 1: (a) *For all $n \in [1...N]$, we have $x_{j_n} = w_{j_n}$.*

(b) *Also,* $\mathbf{x}_{J \setminus \{j_1,\ldots,j_N\}} = \mathbf{w}_{J \setminus \{j_1,\ldots,j_N\}}$.

Proof: (a) Fix $n \in [1...N]$. Let T_n have wood $\mathbf{w}^n$ and let $J_n := \mathrm{supp}(\mathbf{w}^n)$; then $\widetilde{\mathbf{w}}_{J_n \cap \widetilde{J}} = \mathbf{w}^n_{J_n \cap \widetilde{J}}$ by definition of 'wood' in condition (T4). Thus, $\mathbf{x}_{J_n \cap \widetilde{J}} = \mathbf{w}^n_{J_n \cap \widetilde{J}}$, because $\mathbf{x}_{\widetilde{J}} = \widetilde{\mathbf{w}}$ by hypothesis. But $J_n \cap \widetilde{J} = J_n \setminus \{j_n\}$, so we have $\mathbf{x}_{J_n \setminus \{j_n\}} = \mathbf{w}^n_{J_n \setminus \{j_n\}}$. We must then have $x_{j_n} = \neg w^n_{j_n}$, because $\mathbf{w}^n$ is a forbidden fragment (by induction hypothesis). Thus, $x_{j_n} = w_{j_n}$, because $w_{j_n} = \neg w^n_{j_n}$ by (T3).

(b) $J \setminus \{j_1, \ldots, j_N\} = \widetilde{J} \cap J$ and $\widetilde{\mathbf{w}}_{\widetilde{J} \cap J} = \mathbf{w}_{\widetilde{J} \cap J}$ (by definition of 'wood' in (T4)); thus $\mathbf{x}_{J \setminus \{j_1,\ldots,j_N\}} = \mathbf{w}_{J \setminus \{j_1,\ldots,j_N\}}$ (because $\mathbf{x}_{\widetilde{J}} = \widetilde{\mathbf{w}}$ by hypothesis). ◇ `Claim 1`

Claim 1 implies that $\mathbf{x}_J = \mathbf{w}$. But $\mathbf{w}$ is a forbidden fragment (by (T1)). Contradiction.

Thus, for all $\mathbf{x} \in X$ we must have $\mathbf{x}_{\widetilde{J}} \neq \widetilde{\mathbf{w}}$; hence $\widetilde{\mathbf{w}}$ is a forbidden fragment, as desired.

(b) Suppose $k = \gamma(t)$ for some $t \in [1...K]$. If $F_k^\zeta(\mu) \neq \mathrm{Maj}_k(\mu)$, then we must have $t \geq 2$. We will prove the claim by induction on t.

Base case. If $t = 2$, then let $J := \{\zeta(1), \zeta(2)\}$ and $\mathbf{w} := \mathrm{Maj}_J(\mu)$; then $\mathbf{w}$ must be a critical fragment, so $(\mathbf{w}, k)$ is a forbidden tree of height 1.

Induction. If $t \geq 3$, and $F_k^\zeta(\mu) \neq \mathrm{Maj}_k(\mu)$, then there must exist some subset $J \subseteq \{\zeta(1), \ldots, \zeta(t)\}$ including $k = \zeta(t)$ such that, if we define $w_k := \mathrm{Maj}_k(\mu)$, and define $w_j := F_j^\zeta(\mu)$ for all $j \in J \setminus \{k\}$, then $\mathbf{w} \in \{0,1\}^J$ is a forbidden fragment for X. By choosing J to be a minimal subset with this property, we can assume $\mathbf{w}$ is critical.

Let $\{j_1, \ldots, j_N\}$ be the set of all elements of $J \setminus \{k\}$ such that $\mathrm{Maj}_{j_n}(\mu) \neq w_{j_n}$. By induction hypothesis, each j_n is the root of a μ-activated forbidden tree T_n such that

(b1′) ζ covers every other element of $\mathrm{supp}(T_n)$ before reaching j_n; and

(b2′) if $\mathbf{w}^n$ is the wood of T_n and $J_n := \mathrm{supp}(\mathbf{w}^n)$, then for all $j \in J_n \setminus \{j_n\}$, we have $F_j^\zeta(\mu) = \mathrm{Maj}_j(\mu) = w_j^n$.

Let $T := (\mathbf{w}, k; T_1, \ldots, T_N)$; we claim T is a forbidden tree. (T1) is true because $\mathbf{w}$ is a critical fragment by construction. (T2) is true by definition of $T_1, \ldots, T_n$. To see (T3), note for all $n \in [1...N]$ that μ activates $\mathbf{w}^n$ by hypothesis, so $w^n_{j_n} = \mathrm{Maj}_{j_n}(\mu)$, whereas $w_{j_n} \neq \mathrm{Maj}_{j_n}(\mu)$ by definition of $\{j_1, \ldots, j_N\}$; thus $w^n_{j_n} = \neg w_{j_n}$. To see (T4), observe that the woods of $T_1, \ldots, T_N$ are all compatible because they are all subfamilies of $F^\zeta(\mu)$, by condition (b2′). Finally, properties (b1) and (b2) follow immediately from the definitions of J and T, and properties (b1′) and (b2′). □

Proof of Proposition A.4 "$\Longrightarrow$". Let μ, k, and ζ by as in the statement of the Proposition. Lemma A.6(b) says that k is the root of a μ-activated forbidden tree T satisfying (b1) and (b2). Let $\widetilde{\mathbf{w}}$ be the wood of T; then Lemma A.6(a) says that $\widetilde{\mathbf{w}}$ is itself a forbidden fragment. Let $\widetilde{J} := \mathrm{supp}\,(\widetilde{\mathbf{w}})$; then $k \in \widetilde{J}$, and $\widetilde{\mathbf{w}}_{\widetilde{J}\setminus\{k\}}$ is *not* forbidden, by (b2). Thus, any forbidden subfragment of $\widetilde{\mathbf{w}}$ must contain coordinate k. Since $\widetilde{J}$ is finite, there exists some $J \subseteq \widetilde{J}$ (with $k \in J$) such that $\mathbf{w} := \widetilde{\mathbf{w}}_J$ is a *minimal* forbidden subfragment —i.e. a critical fragment. At this point, (F1) follows from (b1), (F2) follows from (b2), and (F3) is true because $F^\zeta_k(\mu) \neq \mathrm{Maj}_k(\mu)$ by hypothesis. □

Proof of Theorem 4.2. We first establish two claims.

Claim 1: $\mathrm{Indet}(\mu) = \{k \in [1...K]$; *there exists a path* γ *such that* $F^\gamma_k(\mu) \neq \mathrm{Maj}_k(\mu)\}$.

Proof: "$\subseteq$" Proposition 1.3(b) says that $k \in \mathrm{Indet}(\mu)$ if and only if there are paths γ and ζ such that $F^\gamma_k(\mu) \neq F^\zeta_k(\mu)$. Either $F^\gamma_k(\mu) = \mathrm{Maj}_k(\mu)$ or $F^\zeta_k(\mu) = \mathrm{Maj}_k(\mu)$. If $F^\zeta_k(\mu) = \mathrm{Maj}_k(\mu)$, then $F^\gamma_k(\mu) \neq \mathrm{Maj}_k(\mu)$.

"$\supseteq$" Suppose $\mathrm{Maj}_k(\mu) = x$. If ζ is any path such that $\zeta(1) = k$, then $F^\zeta_k(\mu) = x$. Thus, if γ is some path such that $F^\gamma_k(\mu) \neq \mathrm{Maj}_k(\mu)$, then $F^\gamma_k(\mu) \neq F^\zeta_k(\mu)$; hence $k \in \mathrm{Indet}(\mu)$. ◇ `Claim 1`

Claim 2: *For any* $\mathbf{w} \in W(X, \mu)$ *and* $k \in \mathrm{supp}\,(\mathbf{w})$, *there is a path focusing* $\mathbf{w}$ *on* k.

Proof: Suppose $\mathrm{supp}\,(\mathbf{w}) = J := \{j_1, j_2, \ldots, j_N\}$, where $j_N = k$. Let γ be a path such that $\gamma(n) = j_n$ for all $n \in [1...N]$, after which γ traverses the rest of $[1...K]$ in some order (this gives (F1)). For all $j \in J$ we have $\mathrm{Maj}_j(\mu) = w_j$ (because $\mathbf{w}$ is activated by μ, by hypothesis). Thus, we have $F^\gamma_j(\mu) = w_j$ for all $j \in J \setminus \{k\}$ (because $\mathbf{w}_{J\setminus\{k\}}$ is *not* forbidden, because $\mathbf{w}$ is critical); this yields (F2). But then $F^\gamma_k(\mu) \neq w_k$, because $\mathbf{w}$ *is* forbidden in X; this yields (F3). ◇ `Claim 2`

Now let $B := \displaystyle\bigcup_{\mathbf{w} \in W(X,\mu)} \mathrm{supp}\,(\mathbf{w})$. We must show that $\mathrm{Indet}(\mu) = B$.

We have $\mathrm{Indet}(\mu) \subseteq B$ by Claim 1 and Proposition A.4"$\Longrightarrow$".

To see that $\mathrm{Indet}(\mu) \supseteq B$, let $\mathbf{w} \in W(X, \mu)$, and let $k \in \mathrm{supp}\,(\mathbf{w})$. Claim 2 says there is a path γ which focuses $\mathbf{w}$ on k. Then Proposition A.4"$\Longleftarrow$" implies that $F^\gamma_k(\mu) \neq \mathrm{Maj}_k(\mu)$; hence Claim 1 says $k \in \mathrm{Indet}(\mu)$. □

The following result offers a diagnosis of the diachronic unanimity violations analyzed in §3 that is more refined than that of Theorem 3.2.

Proposition A.7 *Let $X \subseteq \{0,1\}^K$.*

- **(a)** *Let $\mu \in \Delta^*(X)$ and suppose μ is unanimous in coordinate $k \in [1...K]$. There exists a path γ such that $F^\gamma(\mu)$ violates μ-unanimity in coordinate k if and only if $k \in \operatorname{supp}(\mathbf{w})$ for some $\mathbf{w} \in W(X,\mu)$ with $|\mathbf{w}| \geq 4$.*

- **(b)** *Let $\mathbf{w}$ be a critical fragment for X, with $|\mathbf{w}| \geq 4$. For every $k \in \operatorname{supp}(\mathbf{w})$, there exists some $\mu \in \Delta^*(X)$ and path γ such that $F^\gamma(\mu)$ violates μ-unanimity in coordinate k.*

Proof: **(a)** "$\Longrightarrow$" If path γ violates μ-unanimity in coordinate k, then in particular $F_k^\gamma(\mu) \neq \mathrm{Maj}_k(\mu)$. Thus, Proposition A.4 yields a μ-activated critical fragment $\mathbf{w}$ which ζ focuses on k (in particular, $k \in \operatorname{supp}(\mathbf{w})$).

It remains to show that $|\mathbf{w}| \geq 4$. By contradiction, suppose $|\mathbf{w}| = 3$ (the case $|\mathbf{w}| = 2$ is even easier). By reordering $[1...K]$ if necessary, we can suppose $\operatorname{supp}(\mathbf{w}) = \{1,2,3\}$ and $k = 1$. Thus, $\mu_1(w_1) = 1$. Thus $\mu_2(w_2) + \mu_3(w_3) \leq 1$ (otherwise we would have $\mu_J(\mathbf{w}) > 0$, contradicting the fact that $\mu \in \Delta^*(X)$ and $\mathbf{w}$ is forbidden). Thus, either $\mu_2(w_2) < \frac{1}{2}$ or $\mu_3(w_3) < \frac{1}{2}$. Thus, either $\mathrm{Maj}_2(\mu) \neq w_2$ or $\mathrm{Maj}_3(\mu) \neq w_3$. But this contradicts the fact that μ activates $\mathbf{w}$.

"$\Longleftarrow$" After suitably reordering $[1...K]$, we can assume that $\operatorname{supp}(\mathbf{w}) = \{1,2,\ldots,J+1\}$ and $k = J+1$, for some $J \in [3\ldots K-1]$. Let ζ be the path: $(1,2,3,\ldots,J,J+1,\ldots,K)$. Then for all $j \in \{1,2,\ldots,J\}$, we have $F_j^\zeta(\mu) = w_j$ because μ activates $\mathbf{w}$. Then X-consistency requires $F_{J+1}^\zeta(\mu) = \neg w_{J+1}$ (because $\mathbf{w}$ is forbidden), which violates μ-unanimity (because $\mu_{J+1}[w_{J+1}] = 1$ by hypothesis).

(b) As in the proof of **(a)**"$\Longleftarrow$", we can assume $\operatorname{supp}(\mathbf{w}) = \{1,2,\ldots,J+1\}$ and $k = J+1$, for some $J \in [3\ldots K-1]$. Without loss of generality, suppose $\mathbf{w} = (0,0,\ldots,0)$ (negate certain coordinates of X if necessary to make this true). For all $j \in [1\ldots J]$, there exists some $\mathbf{x}^j \in X$ of the form

$$\mathbf{x}^j \quad := \quad (\underbrace{0,0,\ldots,0}_{j-1},1,\underbrace{0,\ldots,0}_{J-j},0,\underbrace{*,*,\ldots,*}_{K-J}),$$

where "$*,\ldots,*$" represents any X-admissible completion (such an $\mathbf{x}^j \in X$ exists precisely because $\mathbf{w}$ is a *minimal* forbidden fragment). Let $\mu \in \Delta^*(X)$ be the profile such that $\mu(\mathbf{x}^j) = \frac{1}{J}$ for all $j \in \{1,2,\ldots,J\}$. Then $\mu_{J+1}(1) = 0$, while for all $j \in [1...J]$, we have $\mu_j(1) = \frac{1}{J} \leq \frac{1}{3} < \frac{1}{2}$ (because $J = |\mathbf{w}| - 1 \geq 4 - 1 = 3$). Thus, μ activates $\mathbf{w}$. Now **(a)**"$\Longleftarrow$" yields a path γ which violates μ-unanimity in coordinate $J+1$. $\square$

Proof of Proposition 4.4. (a) Every coordinate of $\mathbf{x}$ defines an edge in the $\mathbf{x}$-tournament. Every X_N^{pr}-critical fragment corresponds to a cycle —the coordinates of that fragment are the edges making up the cycle. Thus,

$$\begin{aligned}\Big(\mathbf{x} \in \mathrm{Crit}(X_N^{\mathrm{pr}})\Big) \quad &\Longleftrightarrow \quad \Big(\text{Every coordinate of } \mathbf{x} \text{ is part of some } X_N^{\mathrm{pr}}\text{-critical fragment}\Big) \\ &\Longleftrightarrow \quad \Big(\text{Every edge in the } \mathbf{x}\text{-tournament is part of some cycle}\Big).\end{aligned}$$

Thus, it suffices to show that

$$\begin{aligned}&\Big(\text{Every edge in the } \mathbf{x}\text{-tournament is part of some cycle}\Big) \\ &\qquad \Longleftrightarrow \quad \Big(\mathrm{topcycle}(\mathbf{x}) \text{ contains every element of } [1...N]\Big).\end{aligned}$$

"$\Longrightarrow$" (by contrapositive) Find some $a, b \in [1...N]$ such that a is in topcycle($\mathbf{x}$), whereas b is not. Then the edge (a, b) is not contained in any cycle of the $\mathbf{x}$-tournament.

"$\Longleftarrow$" By reordering $[1...N]$ if necessary, suppose topcycle($\mathbf{x}$) $= (1 \overset{\mathbf{x}}{\prec} 2 \overset{\mathbf{x}}{\prec} 3 \overset{\mathbf{x}}{\prec} \cdots \overset{\mathbf{x}}{\prec} N \overset{\mathbf{x}}{\prec} 1)$. Let $n, m \in [1...N]$. The order $(\overset{\mathbf{x}}{\prec})$ is complete, so either $m \overset{\mathbf{x}}{\prec} n$ or $n \overset{\mathbf{x}}{\prec} m$. Suppose $n \overset{\mathbf{x}}{\prec} m$. We must construct an $\mathbf{x}$-cycle containing the link "$n \overset{\mathbf{x}}{\prec} m$". The construction is similar to the last paragraph in the proof of Example 3.3(b). If $n < m$, then we can remove the elements $\{n+1, \ldots, m-1\}$ from topcycle($\mathbf{x}$) to obtain the cycle:

$$1 \overset{\mathbf{x}}{\prec} 2 \overset{\mathbf{x}}{\prec} \cdots \overset{\mathbf{x}}{\prec} n \overset{\mathbf{x}}{\prec} m \overset{\mathbf{x}}{\prec} m+1 \overset{\mathbf{x}}{\prec} \cdots \overset{\mathbf{x}}{\prec} N \overset{\mathbf{w}}{\prec} 1.$$

If $n > m$, then we can remove the elements $\{1, \ldots, m-1\}$ and $\{n+1, \ldots, N\}$ from topcycle($\mathbf{x}$) to obtain the cycle:

$$m \overset{\mathbf{x}}{\prec} m+1 \overset{\mathbf{x}}{\prec} \cdots \overset{\mathbf{x}}{\prec} n-1 \overset{\mathbf{x}}{\prec} n \overset{\mathbf{x}}{\prec} m.$$

Either way, we get a cycle containing the the link "$n \overset{\mathbf{x}}{\prec} m$", as desired.

(b) The point $\mathbf{x}$ defines an undirected graph on $[1...N]$, where $a \overset{\mathbf{x}}{\sim} b$ iff $x_{(a,b)} = 1$. A ***broken cycle*** in $\mathbf{x}$ is a sequence $a_1 \overset{\mathbf{x}}{\sim} a_2 \overset{\mathbf{x}}{\sim} \cdots \overset{\mathbf{x}}{\sim} a_M \overset{\mathbf{x}}{\not\sim} a_1$, where $a_1, \ldots, a_M \in [1...N]$ are all distinct. We say that the pairs $(a_1, a_2), \ldots, (a_{M-1}, a_M)$ and (a_M, a_1) ***belong*** to this broken cycle. Every X_N^{eq}-critical fragment corresponds to some broken cycle. Thus,

$$\begin{aligned}&\Big(\mathbf{x} \in \mathrm{Crit}(X_N^{\mathrm{eq}})\Big) \\ &\Longleftrightarrow \quad \Big(\text{Every coordinate of } \mathbf{x} \text{ is part of some } X_N^{\mathrm{eq}}\text{-critical fragment}\Big) \\ &\Longleftrightarrow \quad \Big(\text{For all } a, b \in [1...N], \ \text{the pair } (a,b) \text{ belongs to some broken cycle in } \mathbf{x}\Big).\end{aligned}$$

Thus, it suffices to show that

$$\begin{aligned}&\Big(\text{For all } a, b \in [1...N], \ \text{the pair } (a,b) \text{ belongs to some broken cycle in } \mathbf{x}\Big) \\ &\qquad \Longleftrightarrow \quad \Big(\text{The } \mathbf{x}\text{-graph is connected but not complete}\Big).\end{aligned}$$

"$\Longrightarrow$" (*Connected*) Let $a, b \in [1...N]$. Then (a, b) belongs to a broken cycle of $\mathbf{x}$. If the edge $a \sim b$ is one of the 'unbroken' links in this cycle, then $a \overset{\mathbf{x}}{\sim} b$, so they are adjacent (hence, connected) in the $\mathbf{x}$-graph. If the edge $a \sim b$ is the 'broken' link in the cycle, then the remaining (unbroken) links of the cycle define a path from a to b in the $\mathbf{x}$-graph, so a and b are connected.

(*Incomplete* —by contrapositive) If the $\mathbf{x}$-graph was complete, then there would be no broken links, and hence, no broken cycles.

"$\Longleftarrow$" Let $a, b \in [1...N]$; we must find a broken cycle in $\mathbf{x}$ containing the link (a, b).

Case 1. ($a \overset{\mathbf{x}}{\not\sim} b$) Since the $\mathbf{x}$-graph is connected, there is some path in $\mathbf{x}$ connecting a to b; assume this path contains no repeated entries. Then this path, together with the 'broken' link $a \overset{\mathbf{x}}{\not\sim} b$, defines a broken cycle in $\mathbf{x}$ containing (a, b).

Case 2. ($a \overset{\mathbf{x}}{\sim} b$) Since the $\mathbf{x}$-graph is not complete, there exist some $c, d \in [1...K]$ such that $c \overset{\mathbf{x}}{\not\sim} d$. Since the $\mathbf{x}$-graph is connected, there is a path in $\mathbf{x}$ connecting d to a. Connecting the link $a \overset{\mathbf{x}}{\sim} b$ to this path yields a path from d to b. If $b \overset{\mathbf{x}}{\not\sim} d$, then the broken link $b \overset{\mathbf{x}}{\not\sim} d$ plus the aforementioned path yields a broken cycle containing the link (a, b).

So, assume $b \overset{\mathbf{x}}{\sim} d$. Since the $\mathbf{x}$-graph is connected, there is also a path in $\mathbf{x}$ connecting c to a. Connecting the links $a \overset{\mathbf{x}}{\sim} b$ and $b \overset{\mathbf{x}}{\sim} d$ to this path yields a path from c to d; then connecting the broken link $c \overset{\mathbf{x}}{\not\sim} d$ yields a broken cycle containing (a, b). $\square$

In the next few proofs, we will use the following terminology. Say a fragment $\mathbf{w}$ ***covers*** some coordinate $k \in [1...K]$ if $k \in \operatorname{supp}(\mathbf{w})$. Let $\mathbf{x} \in \{0, 1\}^K$. A collection $\{\mathbf{w}_j\}_{j=1}^J$ of fragments ***covers*** $\mathbf{x}$ if $\mathbf{w}_j \sqsubseteq \mathbf{x}$ for all $j \in [1 \dots J]$, and $\bigcup_{j=1}^J \operatorname{supp}(\mathbf{w}_j) = [1...K]$ (i.e. every coordinate in $[1...K]$ is covered by some fragment). If $\{\mathbf{w}_j\}_{j=1}^J$ is a collection of *critical* fragments, then this is called a ***critical cover*** of $\mathbf{x}$. (Thus, $\mathbf{x}$ is 'critical' if and only if it admits a critical cover.)

Proof of Proposition 4.9. We must show that $\operatorname{Crit}(X) = \{\mathbf{0}\}$.

Claim 1: *Let $\mathbf{w}$ be any X-critical fragment. Then $\mathbf{w}$ is all zeros.*

Proof: (by contradiction) Suppose $\mathbf{w}$ was not all zeros. By reordering $[1...K]$ if necessary, suppose that

$$\mathbf{w} \quad = \quad (\underbrace{1, 1, \dots, 1}_{N}, \underbrace{0, \dots, 0}_{M}, *, \dots, *).$$

Let $\mathbf{w}' = (*, \overbrace{1, \dots, 1}^{N-1}, \overbrace{0, \dots, 0}^{M}, *, \dots, *)$. Then $\mathbf{w}'$ is not X-forbidden (because $\mathbf{w}$ is X-critical). Thus, there exists some $\mathbf{x}' \in X$ such that $\mathbf{w}' \sqsubset \mathbf{x}$. Define $\mathbf{x} \in \{0, 1\}^K$ by $x_1 := 1$ and $x_k := x'_k$ for all $k \geq 2$. Then $\mathbf{x} \geq \mathbf{x}'$. Thus, $\mathbf{x} \in X$ because $\mathbf{x}' \in X$ and X is comprehensive. But $\mathbf{w} \sqsubset \mathbf{x}$, and $\mathbf{w}$ is X-forbidden. Contradiction. $\diamond$ `Claim 1`

Now, by nondegeneracy, for every $k \in [1...K]$ there is some critical fragment $\mathbf{w}_k \in W_3(X)$ such that $k \in \text{supp}(\mathbf{w}_k)$. By Claim 1, $\mathbf{w}_k$ is all zeros. Thus, the collection $\{\mathbf{w}_k\}_{k=1}^K$ is a critical covering of $\mathbf{0}$, so $\mathbf{0} \in \text{Crit}(X)$.

Conversely if $\mathbf{x} \in \text{Crit}(X)$, then we must have $\mathbf{x} = \mathbf{0}$, because Claim 1 says that the fragments covering $\mathbf{x}$ are all zeros. Thus, $\text{Crit}(X) = \{\mathbf{0}\}$. $\square$

The following lemma will be useful in the proofs of Propositions 4.11 and 4.13.

Lemma A.8 *Consider the committee selection problem $X^{\text{com}}_{I,J;K}$. For all $I \leq J$, we have $\text{Crit}(X^{\text{com}}_{I,J;K}) \subseteq \{\mathbf{0}, \mathbf{1}\}$. Moreover, if $I > 0$ then $\mathbf{0} \in \text{Crit}(X^{\text{com}}_{I,J;K})$, and if $J < K$ then $\mathbf{1} \in \text{Crit}(X^{\text{com}}_{I,J;K})$.*

Proof: Evidently, the critical fragments of $X^{\text{com}}_{I,J;K}$ are given as follows: if $I > 0$, then all fragments of exactly $K - I + 1$ zeros are critical; moreover, if $J < K$, then all fragments of exactly $J + 1$ ones are critical. No other fragments are critical. This implies at once that $\mathbf{0} \in \text{Crit}(X^{\text{com}}_{I,J;K})$ if $I > 0$, and $\mathbf{1} \in \text{Crit}(X^{\text{com}}_{I,J;K})$ if $J < K$. Moreover, $I \leq J$, so $(K - I + 1) + (J + 1) > K$, so no element $\mathbf{x} \in \{0,1\}^K$ different from $\mathbf{0}$ and $\mathbf{1}$ can be critical for $X^{\text{com}}_{I,J;K}$. $\square$

Proof of Proposition 4.11. By Lemma A.8, $\text{Crit}(X^{\text{com}}_{I,J;K}) = \{\mathbf{0}, \mathbf{1}\}$. First, assume that $K/2 < J$. For each $l \in [1...K]$, denote by $\mathbf{x}^l$ the element of $X^{\text{com}}_{I,J;K}$ with $x^l_k = 1$ for $k = l, l+1, \ldots, l+J \pmod{K}$, and $x_k = 0$ otherwise. Evidently, for the profile μ that assigns weight $1/K$ to each $\mathbf{x}^l$, $l \in [1...K]$, one obtains $\text{Maj}(\mu) = \mathbf{1}$. Next, assume that $I < K/2$; by a completely symmetric argument, one shows that $\mathbf{0} \in \text{Maj}(X^{\text{com}}_{I,J;K})$ in this case. Thus, in either case $X^{\text{com}}_{I,J;K}$ is issue-wise indeterminate by Theorem 4.3(b).

Now assume that $J \leq K/2 \leq I$, which is only possible if K is even and $I = J = K/2$. As is easily verified, $\text{Maj}(X^{\text{com}}_{\frac{K}{2},\frac{K}{2};K}) \cap \{\mathbf{0}, \mathbf{1}\} = \emptyset$ in this case. For instance, suppose that $\text{Maj}(\mu)$ specifies a zero in each of the first $K - 1$ coordinates, then μ must contain strictly more ones than zeros in coordinate K; hence $[\text{Maj}(\mu)]_K = 1$. Thus, $X^{\text{com}}_{\frac{K}{2},\frac{K}{2};K}$ is not issue-wise indeterminate, again by Theorem 4.3(b). $\square$

Proof of Proposition 4.7. By Proposition 6.2(b) in Nehring and Pivato (2011a), $X_{\mathcal{C}}$ is McGarvey if and only if $\mathcal{C}$ contains all singletons. Thus, it suffices to show that in this case, $\text{Crit}(X_{\mathcal{C}}) \neq \emptyset$. Let $J \subseteq [1...K]$ be a minimal subset such that $\text{conv}(J) = [1...K]$; we will show that $\mathbf{1}_J \in \text{Crit}(X_{\mathcal{C}})$. Let $k \in [1...K]$, and let $I \subseteq J$; we say that I is a *J-frame* for k if $k \in \text{conv}(I)$, but $k \notin \text{conv}(H)$ for any proper subset $H \subset I$.

Let $k \in [1...K] \setminus J$ (so $(\mathbf{1}_J)_k = 0$). By hypothesis, $k \in \text{conv}(J)$. Let $I \subseteq J$ be a J-frame for k. Then I is Carathéodory-independent (because $k \in \text{conv}(I)$ but $k \notin \text{conv}(H)$ for any $H \subsetneq I$). Thus, Lemma A.3 says the fragment $(\mathbf{1}_I, 0_k)$ is critical, and clearly $(\mathbf{1}_I, 0_k)_k = 0 = (\mathbf{1}_J)_k$. Thus, every $k \in [1...K] \setminus J$ is covered by some critical fragment compatible with $\mathbf{1}_J$.

Next, let $j \in J$ (so $(\mathbf{1}_J)_j = 1$). Let $k \in [1...K] \setminus \text{conv}(J \setminus \{j\})$ (this set is nonempty precisely because J is a *minimal* spanning set for $[1...K]$). Let $I \subseteq J$ be a J-frame for k. Then $I \not\subseteq J \setminus \{j\}$, because $k \notin \text{conv}(J \setminus \{j\})$; thus, $j \in I$. Just as in the previous paragraph, I is Carathéodory-independent. Thus, Lemma A.3 says the fragment $(\mathbf{1}_I, 0_k)$ is critical, and clearly $(\mathbf{1}_I, 0_k)_j = 1 = (\mathbf{1}_J)_j$. Thus, every $j \in J$ is covered by some critical fragment compatible with $\mathbf{1}_J$.

The previous two paragraphs combined show that $\mathbf{1}_J \in \text{Crit}(X_{\mathcal{C}})$, as claimed. □

Proof of Proposition 4.12.

(a) Let $\mathbf{w}$ be an X-critical fragment of order K. Then $\mathbf{w} \in \text{Crit}(X)$. For $k = 1, 2, 3$, obtain $\mathbf{x}^k$ from $\mathbf{w}$ by negating w_k and leaving all the other coordinates the same. Then $\mathbf{x}^k \in X$ (because $\mathbf{w}$ is critical). Define $\mu \in \Delta_3^*(X)$ by $\mu[\mathbf{x}^k] = \frac{1}{3}$ for all $k \in \{1, 2, 3\}$. Then $\text{Maj}(\mu) = \mathbf{w}$; thus, μ is issue-wise indeterminate, by Theorem 4.3(a).

(b) For all $z \in \mathbb{Z}$, let $[z]$ be the unique element of $[1...N]$ such that $z \equiv [z] \pmod{N}$. (For example if $N = 7$ and $z = 11$, then $[z] = 4$.) Let $\mathbf{x} \in \{0,1\}^K$ represent any tournament $(\stackrel{\mathbf{x}}{\prec})$ such that,

$$\forall\, n, m \in [1...N], \quad \Big(m = [n+k] \text{ for some } k \in \mathbb{N} \text{ with } k < N/2\Big) \Longrightarrow \Big(n \stackrel{\mathbf{x}}{\prec} m\Big). \quad \text{(A2)}$$

(For example, if $N = 7$, then $5 \stackrel{\mathbf{x}}{\prec} 6$, $5 \stackrel{\mathbf{x}}{\prec} 7$, and $5 \stackrel{\mathbf{x}}{\prec} 1$, because $[6-5] = 1$, $[7-5] = 2$, and $[1-5] = 3$. However, $5 \stackrel{\mathbf{x}}{\succ} 2$, $5 \stackrel{\mathbf{x}}{\succ} 3$, and $5 \stackrel{\mathbf{x}}{\succ} 4$, because $[2-5] = 4$, $[3-5] = 5$, and $[4-5] = 6$; see Figure 4(a).)

If N is odd, then condition (A2) completely determines $\mathbf{x}$. If N is even, then for any $n, m \in [1...N]$ with $m = [n + N/2]$, we also have $n = [m + N/2]$; in this case, we can set $n \stackrel{\mathbf{x}}{\prec} m$ or $m \stackrel{\mathbf{x}}{\prec} n$ arbitrarily. In any case, $\text{topcycle}(\mathbf{x}) = [1...N]$, so Proposition 4.4(a) implies that $\mathbf{x} \in \text{Crit}(X_N^{\text{pr}})$.

Let $N = 3q + r$ for some $q \in \mathbb{N}$ and $r \in \{0, 1, 2\}$. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in X_N^{\text{pr}}$ correspond to the preference orders: $(1 \prec 2 \prec \cdots \prec N)$, $((q+1) \prec (q+2) \prec \cdots \prec N \prec 1 \prec 2 \prec \cdots \prec (q-1) \prec q)$ and $((2q+1) \prec (2q+2) \prec (2q+1) \prec \cdots \prec N \prec 1 \prec 2 \prec \cdots \prec (2q-1) \prec 2q)$, respectively [see Figure 4(b)]. Define $\mu \in \Delta_3^*(X_N^{\text{pr}})$ by $\mu[\mathbf{x}_k] = \frac{1}{3}$ for $k = 1, 2, 3$. If $\mathbf{x} := \text{Maj}(\mu)$, then $\mathbf{x}$ satisfies condition (A2), so $\mathbf{x} \in \text{Crit}(X_N^{\text{pr}})$; hence μ is issue-wise indeterminate, by Theorem 4.3(a).

(c) Let $A_1, A_2, A_3 \subset [1...N]$ be three nonempty disjoint subsets such that $[1...N] = A_1 \sqcup A_2 \sqcup A_3$. Let $\mathbf{x} \in \{0,1\}^K$ represent the graph such that $n \sim m$ for all $n, m \in A_i$ and $i \in \{1, 2, 3\}$. Furthermore $n \sim m$ for all $n \in A_2$ and all $m \in A_1 \sqcup A_3$; however, $n \not\sim m$ for any $n \in A_1$ and $m \in A_3$ [see Figure 4(c)]. This graph is connected but not complete, so Proposition 4.4(b) says $\mathbf{x} \in \text{Crit}(X_N^{\text{eq}})$.

Now, let $\mathbf{x}_1$ represent the complete equivalence relation $(\stackrel{1}{\sim})$ (i.e. $n \stackrel{1}{\sim} m$ for all $n, m \in [1...N]$. Let $\mathbf{x}_2, \mathbf{x}_3 \in X_N^{\text{eq}}$ represent the equivalence relations $(\stackrel{2}{\sim})$ and $(\stackrel{3}{\sim})$, described as follows [see Figure 4(d)]. First, we have $n \stackrel{i}{\sim} m$ for all $n, m \in A_i$ and $i \in \{1, 2, 3\}$. Next, we have:

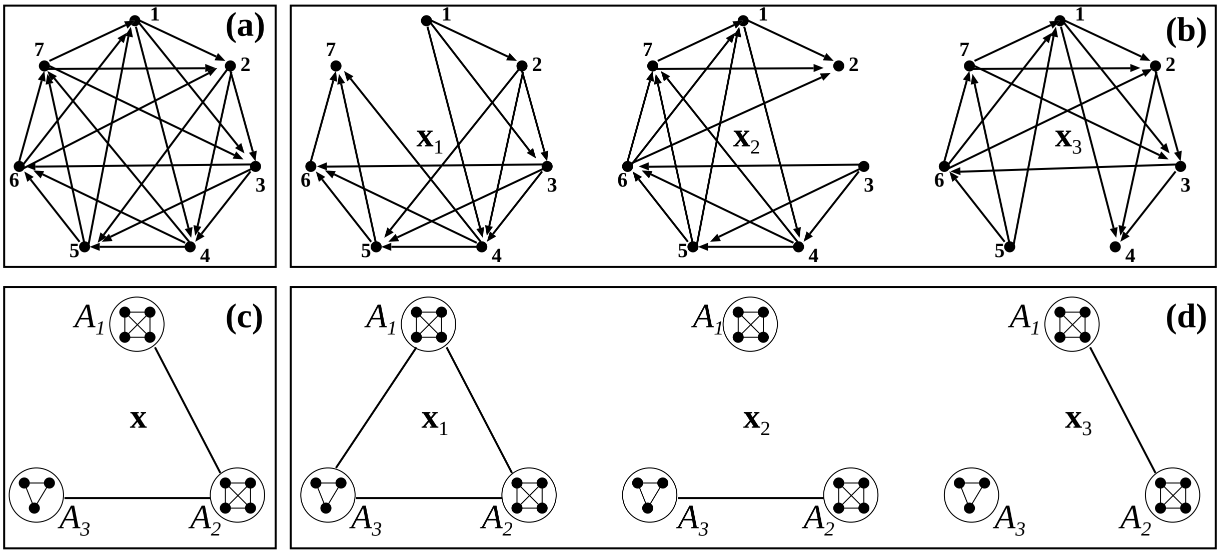

Figure 4: (a,b): The proof of Proposition 4.12(b), case $N = 7$. (c,d): The proof of Proposition 4.12(c).

- $n \overset{2}{\sim} m$ for all $n \in A_2$ and $m \in A_3$.
- $n \overset{3}{\sim} m$ for all $n \in A_1$ and $m \in A_2$.

Define $\mu \in \Delta_3^*(X_N^{\mathrm{eq}})$ by $\mu[\mathbf{x}_n] = \frac{1}{3}$ for all $n \in \{1, 2, 3\}$. Then $\mathrm{Maj}(\mu) = \mathbf{x}$; thus, μ is issue-wise indeterminate.

(d) For all $d \in \{1, 2, 3\}$, let $\mathbf{x}^d$ be the element of $X^{\vartriangle}_{M,D}$ which allocates all M dollars towards claimant d. (Thus, for all $m \in [1...M]$, we have $x^d_{d,m} = 1$, while $x^d_{c,m} = 0$ for all $c \in [1...D] \setminus \{d\}$.) Define $\mu \in \Delta_3^*(X^{\vartriangle}_{M,D})$ by $\mu[\mathbf{x}_d] = \frac{1}{3}$ for $d = 1, 2, 3$. Then $\mathrm{Maj}(\mu) = \mathbf{0}$, so Theorem 2.5 implies that $\mathrm{Cond}\big(X^{\vartriangle}_{M,D}, \mu\big) = X^{\vartriangle}_{M,D}$, hence μ is issue-wise indeterminate, by Theorem 4.3(a). $\square$

Proof of Proposition 4.13.

(a) Let $I' := K - J$ and $J' := K - I$; then $I' \leq J'$.

Claim 1: $\eta(X^{\mathrm{com}}_{I',J';K}) = \eta(X^{\mathrm{com}}_{I,J;K})$.

Proof: For any $\mathbf{x} \in \{0,1\}^K$, define $\mathbf{x}' := (\neg x_k)_{k=1}^K$. For any $\mu \in \Delta(\{0,1\}^K)$, define $\mu'(\mathbf{x}) := \mu(\mathbf{x}')$ for all $\mathbf{x} \in \{0,1\}^K$; then clearly $\mathrm{Maj}(\mu') = \mathrm{Maj}(\mu)'$. In particular, $\mathrm{Maj}(\mu) = \mathbf{0}$ if and only if $\mathrm{Maj}(\mu') = \mathbf{1}$. It is easy to check that $X^{\mathrm{com}}_{I',J';K} := \{\mathbf{x}'; \mathbf{x} \in X^{\mathrm{com}}_{I,J;K}\}$; thus, $\Delta_N^*(X^{\mathrm{com}}_{I',J';K}) := \{\mu'; \mu \in \Delta_N^*(X^{\mathrm{com}}_{I,J;K})\}$.
Lemma A.8 says that $\mathrm{Crit}(X^{\mathrm{com}}_{I,J;K}) = \{\mathbf{0}, \mathbf{1}\} = \mathrm{Crit}(X^{\mathrm{com}}_{I',J';K})$. Thus,

$$
\begin{aligned}
\eta(X^{\mathrm{com}}_{I,J;K}) \quad &= \quad \min\big\{N \in \mathbb{N}\ ;\ \exists\ \mu \in \Delta_N^*(X^{\mathrm{com}}_{I,J;K}) \text{ with } \mathrm{Maj}(\mu) = \mathbf{0} \text{ or } \mathbf{1}\big\} \\
&= \min\big\{N \in \mathbb{N}\ ;\ \exists\ \mu \in \Delta_N^*(X^{\mathrm{com}}_{I,J;K}) \text{ with } \mathrm{Maj}(\mu') = \mathbf{1} \text{ or } \mathbf{0}\big\} \\
&= \min\big\{N \in \mathbb{N}\ ;\ \exists\ \mu' \in \Delta_N^*(X^{\mathrm{com}}_{I',J';K}) \text{ with } \mathrm{Maj}(\mu') = \mathbf{1} \text{ or } \mathbf{0}\big\} \quad = \quad \eta(X^{\mathrm{com}}_{I',J';K}),
\end{aligned}
$$

as claimed. ◇ Claim 1

Now, $K - 2I' = 2J - K$ and $2J' - K = K - 2I$. Thus, $\min\left\{\frac{K}{K-2I'}, \frac{K}{2J'-K}\right\} = \min\left\{\frac{K}{K-2I}, \frac{K}{2J-K}\right\}$. This, together with Claim 1, means that the inequality (10) holds for $X^{\mathrm{com}}_{I,J;K}$, J and K if and only if it holds for $X^{\mathrm{com}}_{I',J';K}$, J' and K'; thus, we can substitute one problem for the other. Furthermore, if $K - 2I < 2J - K$, then $K - 2I' \geq 2J - K'$. Thus, by exchanging $X^{\mathrm{com}}_{I',J';K}$ for $X^{\mathrm{com}}_{I,J;K}$ if necessary, we can assume without loss of generality that $K - 2I \geq 2J - K$. Since $I \leq J$, this implies $K - 2I \geq 0$; hence $K/2 \geq I$. Thus, $\eta(X^{\mathrm{com}}_{I,J;K}) \leq \min\{N \in \mathbb{N};\ \exists\, \mu \in \Delta^*_N(X^{\mathrm{com}}_{I,J;K}) \text{ with } \mathrm{Maj}(\mu) = \mathbf{0}\}$.

For all $\mathbf{x}$ in the support of μ, $\sum_k x_k \geq I$; thus, $\sum_k \mu_k \geq I$. Thus, $\mu_k \geq I/K$, for some $k \in [1 \ldots K]$. On the other hand, $\mathbf{0} = \mathrm{Maj}(\mu)$ if and only if $\mu_k < \frac{1}{2}$ for all k. Thus, $\mathbf{0} = \mathrm{Maj}(\mu)$ if and only if

$$\frac{I}{K} \leq \max_k \mu_k < \frac{1}{2}.$$

Suppose that the number of voters N is odd. For profiles μ to exist satisfying the latter inequality, one must have $\frac{N-1}{2N} \geq \frac{I}{K}$, i.e. $N \geq \frac{K}{K-2I}$ as claimed. (The case when N is even is similar.)

(b)[i] We first illustrate the argument in the case $X^{\mathrm{com}}_{4,6;11}$. In this case, $\frac{I}{K-2I} = \frac{4}{3} \in (1, 2]$, so $N = 2$, and the theorem claims that $\eta(X^{\mathrm{com}}_{4,6;11}) \leq 5$. We will construct an issue-wise indeterminate profile with five voters. Define

$$\begin{aligned}
\mathbf{x}_1 &:= (1,1,1,1,0,0,0,0,0,0,0);\\
\mathbf{x}_2 &:= (0,0,0,0,1,1,1,1,0,0,0);\\
\mathbf{x}_3 &:= (1,0,0,0,0,0,0,0,1,1,1);\\
\mathbf{x}_4 &:= (0,1,1,1,1,0,0,0,0,0,0);\\
\text{and}\quad \mathbf{x}_5 &:= (0,0,0,0,0,1,1,1,1,0,0).
\end{aligned}$$

Define $\mu \in \Delta^*_5(X^{\mathrm{com}}_{4,6;11})$ by $\mu[\mathbf{x}_k] = \frac{1}{5}$ for $k = 1, \ldots, 5$. The clearly $\mathrm{Maj}(\mu) = \mathbf{0} \in \mathrm{Crit}(X^{\mathrm{com}}_{4,6;11})$. Thus, μ is issue-wise indeterminate, by Theorem 4.3(a).

The general construction is similar. For any $z \in \mathbb{Z}$, let $[z]$ be the unique element of $[1...K]$ such that $z \equiv [z] \pmod{K}$. Then, for all $n \in [0 \ldots 2N{+}1]$, define $k_n := [n\,I]$. Next, for all $n \in [0...2N]$, let J_n denote the 'interval between' k_n and $(k_{n+1}) - 1$ in a 'mod K' sense. That is:

- if $1 \leq k_n \leq (k_{n+1}) - 1 \leq K$, then $J_n := \{k_n, k_n + 1, \ldots, (k_{n+1}){-}1\}$.
- if $1 \leq (k_{n+1}) - 1 \leq k_n \leq K$, then $J_n := \{k_n, k_n + 1, \ldots, K\} \cup \{1, 2, \ldots, (k_{n+1}){-}1\}$.

Finally, for all $n \in [0...2N]$, let $\mathbf{x}^n := \mathbf{1}_{J_n}$. Then $\mathbf{x}^n \in X^{\mathrm{com}}_{I,J;K}$ because $\|\mathbf{x}^n\| = |J_n| = I$.

Now, define $\mu \in \Delta^*_{2N+1}(X^{\mathrm{com}}_{I,J;K})$ by $\mu[\mathbf{x}^n] = 1/(2N + 1)$ for all $n \in [0...2N]$. I claim that $\mathrm{Maj}(\mu) = \mathbf{0}$.

To see this, recall that $N := \left\lceil \dfrac{I}{K - 2I} \right\rceil \geq \dfrac{I}{K - 2I}$. Manipulating this inequality yields $(2N + 1)I \leq NK$. This means that the sequence $k_0, k_1, \ldots, k_{2N+1}$ 'wraps around' the interval $[1...K]$ at most N times. This means that for each $k \in [1...K]$, we have $|\{n \in [0...2N] \; ; \; k \in J_n\}| \leq N$. Thus, $\mu_k(1) \leq N/(2N+1) < \frac{1}{2}$. Thus, $\mathrm{Maj}_k(\mu) = 0$. This holds for all $k \in [1...K]$, so $\mathrm{Maj}(\mu) = \mathbf{0}$. But $\mathbf{0} \in \mathrm{Crit}(X^{\mathrm{com}}_{I,J;K})$; thus, μ is issue-wise indeterminate, by Theorem 4.3(a).

The proof of (b)[ii] is similar; simply reverse the roles of '0' and '1' in the proof of (b)[i]. □

Proof of Proposition 5.3. The space $\{\mathbf{1}_k\}_{k\in K} \cup \{\mathbf{1}\}$ is McGarvey by Proposition 6.2(b) of Nehring and Pivato (2011a); moreover, any superset of a McGarvey space is clearly also McGarvey. It remains to show that X does not admit a panopticon. By definition, a panopticon must lie outside X. Thus, $\mathbf{0} \in \{0,1\}^K$ cannot be a panopticon for X because any $\mathbf{1}_k$ is between $\mathbf{0}$ and $\mathbf{1}$. Thus consider $\mathbf{x} \notin X_{\mathcal{C}}$ with $\mathbf{x} \neq \mathbf{0}$. By assumption, $\mathbf{x}$ must have at least two ones, say $x_\ell = 1$ and $x_m = 1$ with $\ell \neq m$, and at least one zero, say $x_h = 0$ for $h \notin \{\ell, m\}$. But then, the element $\mathbf{1}_\ell \in X_{\mathcal{C}}$ is between the elements $\mathbf{x}$ and $\mathbf{1}_h \in X_{\mathcal{C}}$, i.e. $\mathbf{x}$ is not a panopticon. □

We will use the following simple but elegant result of Szele (1943).

Lemma A.9 *The expected number of directed Hamiltonian paths which exist in a randomly generated tournament (where all edges are independent random variables with both orientations having equal probability) is given by* $\frac{N!}{2^{N-1}}$.

Proof: There are $N!$ directed Hamiltonian chains through $[1...N]$. For any such chain, and any random tournament, there is a probability $1/2^{N-1}$ that the chain can be embedded in the tournament (because each of the $N-1$ edges of the chain has probability $1/2$ of being compatible with the corresponding edge in the tournament, and these $N-1$ events are all independent random variables). □

Proof of Proposition 6.1. (a) Lemma A.9 says there exists a tournament on $[1...N]$ (i.e. an element $\mathbf{x} \in \{0,1\}^K$) with at least $\dfrac{N!}{2^{N-1}}$ distinct Hamiltonian chains. The theorem of McGarvey (1953) says there is some $\mu \in \Delta(X^{\mathrm{pr}}_N)$ with $\mathrm{Maj}(\mu) = \mathbf{x}$; then Proposition 2.1(a) says that $|\mathrm{Cond}\,(X^{\mathrm{pr}}_N, \mu)\,| \geq \dfrac{N!}{2^{N-1}}$. Thus,

$$
\begin{aligned}
h(X^{\mathrm{pr}}_N) \;\geq\; h(\mu) &= \frac{\log_2 |\mathrm{Cond}\,(X^{\mathrm{pr}}_N, \mu)\,|}{\log_2 |X^{\mathrm{pr}}_N|} \;\geq\; \frac{\log_2(N!) - \log_2(2^{N-1})}{\log_2(N!)} \\
&= 1 - \frac{N-1}{\log_2(N!)} \\
&\overset{(*)}{\approx} 1 - \frac{N-1}{(N+\frac{1}{2})\log_2(N) - N\log_2(e) + \log_2(\sqrt{2\pi})} \;\approx\; 1 - \frac{1}{\log_2(N)} \;\xrightarrow[N\to\infty]{}\; 1.
\end{aligned}
$$

Here $(*)$ is because Stirling's formula says $N! \approx \sqrt{2\pi N}\left(\frac{N}{e}\right)^N$.

(b) Fix $c \in (0,1)$. Let $L := \lfloor cN \rfloor$ and let $R := N - L$. Let $\mathbf{x} \in \{0,1\}^K$ represent the complete bipartite graph which has L 'left' vertices and R 'right' vertices, where every left vertex is linked to every right vertex (but there are no links between any two left vertices or any two right vertices). The space X_N^{eq} is McGarvey (Nehring and Pivato, 2011a, Example 3.9(a)), so there exists some $\mu \in \Delta^*(X_N^{\mathrm{eq}})$ such that $\mathrm{Maj}(\mu) = \mathbf{x}$. Let $Y \subseteq X_N^{\mathrm{eq}}$ be the set of all equivalence relations defined as follows:

- Fix $M \in [1 \ldots L]$. Partition the set of right-hand vertices into exactly M disjoint subsets $R_1, R_2, \ldots, R_M$ (some of which may be empty).
- Let the left-hand vertices be $v_1, v_2, \ldots, v_L$. For all $m \in [1 \ldots M]$, declare every element of R_m to be equivalent to v_m and equivalent to every other element of R_m.
- Declare $v_M, \ldots, v_L$ to be equivalent to one another and equivalent to every element of R_M.

Claim 1: $Y \subseteq \mathrm{Cond}\,(X_N^{\mathrm{eq}}, \mu)$.

Proof: Given any $\mathbf{y} \in Y$, we will construct a path γ such that $F^\gamma(\mu) = \mathbf{y}$. We do this as follows:

1. For each $m \in [1 \ldots M]$, and each vertex $r \in R_m$, the path γ first visits the coordinate (v_m, r); in every one of these coordinates, the majority prevails, so we get $F^\gamma_{(v_m,r)}(\mu) = x_{(v_m,r)} = 1 = y_{(v_m,r)}$ (encoding the equivalence $v_m \sim r$).
2. At this point, for all $m \in [1 \ldots M]$ and all $r, r' \in R_m$, transitivity constraints force $r \sim r'$ —i.e. we get $F^\gamma_{(r,r')}(\mu) = 1 = y_{(r,r')}$.
3. Next, for each $m, \ell \in [1 \ldots M]$ with $m \neq \ell$, the path γ visits the coordinate (v_m, v_ℓ). Again, the majority prevails, so $F^\gamma_{(v_m,v_\ell)}(\mu) = x_{(v_m,v_\ell)} = 0 = y_{(v_m,v_\ell)}$ (encoding the nonequivalence $v_m \not\sim v_\ell$).
4. At this point, for all $m, \ell \in [1 \ldots M]$ with $m \neq \ell$, and all $r \in R_m$ and $r' \in R_\ell$, transitivity constraints force $r \not\sim r'$ —i.e. we get $F^\gamma_{(r,r')} = 0 = y_{(r,r')}$.
5. Next, for each $m, \ell \in [M \ldots L]$, the path γ visits the coordinate (v_m, v_ℓ). Again, the majority prevails, so $F^\gamma_{(v_m,v_\ell)}(\mu) = x_{(v_m,v_\ell)} = 1 = y_{(v_m,v_\ell)}$ (encoding $v_m \sim v_\ell$).
6. At this point, for all $\ell \in [M \ldots L]$ and all $r \in R_M$, transitivity constraints force $v_\ell \sim r$ —i.e. we get $F^\gamma_{(v_\ell,r)}(\mu) = 1 = y_{(v_\ell,r)}$ (encoding $v_\ell \sim r$).
7. Finally, visit the remaining coordinates in some arbitrary order. (The values for these coordinates are already completely determined by the transitivity constraints in steps 2, 4, and 6).

At this point, we have $F^\gamma_k(\mu) = y_k$ for all $k \in [1...K]$, as desired. ◇ `Claim 1`

Claim 2: $|Y| \geq L^{R+1}/(R+1)$.

Proof: For any fixed $M \in [1 \ldots L]$, let Y_M be the elements of Y obtained by partitioning the right-hand vertices into M subsets (labelled by $v_1, \ldots, v_M$, and some possibly empty). Then $|Y_M| = M^R$ (because any such partition corresponds to a function from the R right-hand vertices into $\{v_1, \ldots, v_M\}$). Now, $Y = Y_1 \sqcup Y_2 \sqcup \cdots \sqcup Y_M$, so

$$\begin{aligned} |Y| &= |Y_1| + |Y_2| + \cdots + |Y_L| &=& \quad 1^R + 2^R + \cdots + L^R \\ &\geq \int_0^L x^R \, dx &=& \quad \frac{L^{R+1}}{R+1}. \end{aligned}$$

◇ Claim 2

Claim 3: $|X_N^{\text{eq}}| \leq N^N$.

Proof: Any element of X_N^{eq} can represented by a partition of $[1...N]$ into N unlabelled subsets (some of which may be empty). We can represent a *labelled* partition by a function $f : [1...N] \longrightarrow [1...N]$. There are N^N such functions, and hence N^N labelled partitions. (Of course, many of these labelled partitions correspond to the same *un*labelled partition.) Thus, $|X_N^{\text{eq}}| \leq N^N$. ◇ Claim 3

Thus,

$$\begin{aligned} h(X_N^{\text{eq}}) \quad &\geq \quad \frac{\log_2 |\text{Cond}\,(X_N^{\text{eq}}, \mu)\,|}{\log_2 |X_N^{\text{eq}}|} \underset{(*)}{\geq} \frac{\log_2(L^{R+1}/(R+1))}{\log_2(N^N)} \\ &= \quad \frac{(R+1)\log_2(L) - \log_2(R+1)}{N \log_2(N)} \underset{(\dagger)}{\approx} \frac{[(1-c)N+1]\log_2(cN) - \log_2((1-c)N+1)}{N\log_2(N)} \\ &\approx \quad \frac{[(1-c)N+1]\,[\log_2(N) + \log(c)] - \log_2(1-c) - \log_2(N))}{N \log_2(N)} \xrightarrow[N\to\infty]{} (1-c). \end{aligned}$$

Here, $(*)$ is by Claims 1, 2 and 3, while $(\dagger)$ is because $L = \lfloor c\,N \rfloor \approx c\,N$, so that $R = N - L \approx N - c\,N = (1-c)\,N$.

However, c can be any value in $(0,1)$. By letting $c \to 0$, we conclude that $h(X_N^{\text{eq}}) \geq 1$; hence $h(X_N^{\text{eq}}) = 1$, as desired.

(c) Let $\mathcal{C}_K^{\text{line}}$ be the linear graph convexity on $[1...K]$, with the standard ordering. The convex subsets of $\mathcal{C}_K^{\text{line}}$ are the intervals $[j \ldots k]$ for any $j \leq k \in [1...K]$. Let $J := \{k \in [1...K];\ k \text{ odd}\}$. Note that $\mathbf{1}_J \in \text{Maj}(X_K^{\text{line}})$ because X_K^{line} is McGarvey by Proposition 4.7. For any $j \leq k \in [1...K]$, we have $\mathbf{1}_{[j...k]} \asymp \mathbf{1}_J$ if and only if j and k are both odd. Thus, $|X_K^{\text{line}}(\mathbf{1}_J)| \geq \frac{1}{4}|X_K^{\text{line}}|$ (since at least one quarter of the subintervals of $[1...K]$ have two odd endpoints). Thus,

$$h(X_K^{\text{line}}) \quad \geq \quad \frac{\log_2(\frac{1}{4}|X_K^{\text{line}}|)}{\log_2 |X_K^{\text{line}}|} \quad = \quad \frac{\log_2 |X_K^{\text{line}}| - 2}{\log_2 |X_K^{\text{line}}|} \quad \xrightarrow[K\to\infty]{} \quad 1.$$

Thus, X_K^{line} is asymptotically totally indeterminate.

(d) Let $D \in \mathbb{N}$, let $K = 2^D$, and let $\varphi : [1...K] \longrightarrow \{0,1\}^D$ be some bijection. Let $\mathcal{C}_D^{\square}$ be the hypercube convexity on $\{0,1\}^D$, and define $\Phi : \mathcal{P}(\{0,1\}^D) \longrightarrow \{0,1\}^K$ by $\Phi(C) := \mathbf{1}_{\varphi^{-1}(C)}$ for all $C \subseteq \{0,1\}^D$. Then define $X_D^{\square} := \Phi[\mathcal{C}_D^{\square}] \subset \{0,1\}^K$. Finally, let $E := \{\mathbf{x} \in \{0,1\}^D;\ \|\mathbf{x}\| \text{ is even}\}$, and define $\mathbf{e} := \Phi[E] \in \{0,1\}^K$.

Claim 4: *For any $C \in \mathcal{C}_D^{\square}$, if $|C| > 2$, and $\mathbf{c} := \Phi(C) \in X_D^{\square}$, then $\mathbf{c} \asymp \mathbf{e}$.*

Proof: Let $B \in \mathcal{C}_D^\square$ and let $\mathbf{b} := \Phi(B) \in X_D^\square$. Suppose $\mathbf{b}$ is strictly between $\mathbf{c}$ and $\mathbf{e}$. Then $C \cap E \subseteq B \cap E$ and $C^{\complement} \cap E^{\complement} \subseteq B^{\complement} \cap E^{\complement}$.

Now, if $C \cap E \subseteq B \cap E$, then $C \cap E \subseteq B$. Thus, $\mathrm{conv}(C \cap E) \subseteq B$ (because $B \in \mathcal{C}_D^\square$). But if $|C| > 2$, then the definition of E is such that $\mathrm{conv}(C \cap E) = C$. Thus, $C \subseteq B$. Thus, $|B| > 2$ also.

Meanwhile, if $C^{\complement} \cap E^{\complement} \subseteq B^{\complement} \cap E^{\complement}$, then $C \cap E^{\complement} \supseteq B \cap E^{\complement}$. Thus $B \cap E^{\complement} \subset C$. Thus, $\mathrm{conv}(B \cap E^{\complement}) \subseteq C$ (because $C \in \mathcal{C}_D^\square$). But if $|B| > 2$, then the definition of E is such that $\mathrm{conv}(B \cap E^{\complement}) = B$. Thus $B \subseteq C$.

We conclude that $B = C$, which means $\mathbf{b} = \mathbf{c}$, as desired. $\diamond$ `Claim 4`

Claim 5: $|X_D^\square| = 3^D$.

Proof: $|X_D^\square| = |\mathcal{C}_D^\square|$, and $|\mathcal{C}_D^\square|$ consists of all subcubes of $\{0,1\}^D$. There is a bijective correspondence between these subcubes and the set of $\{0,1\}$-valued functions whose domain is any subset of $[1...D]$. Any such function can be represented in a unique way by an element of $\{0,1,*\}^D$ (see Notation A.2). Thus, $|X_D^\square| = |\{0,1,*\}^D| = 3^D$. $\diamond$ `Claim 5`

Let $Y := \{\mathbf{x} \in X_D^\square;\ \mathbf{x}$ represents a subcube of cardinality 1 or 2 in $\{0,1\}^D$.

Claim 6: $|Y| = (1 + \frac{D}{2})\, 2^D$.

Proof: Clearly, $\{0,1\}^D$ has exactly 2^D subcubes of cardinality 1 (i.e. vertices). To obtain a subcube of cardinality 2 (i.e. an edge), we start at one of these 2^D vertices and then move to one of its D nearest neighbours. There are $D \cdot 2^D$ ways to do this, but we have then counted every edge twice, so there are actually $\frac{D}{2}\, 2^D$ edges. $\diamond$ `Claim 6`

Thus, if D is large enough, then

$$|X_D^\square(\mathbf{e})| \underset{(*)}{\geq} |X_D^\square| - |Y| \underset{(\dagger)}{=} 3^D - \left(1 + \frac{D}{2}\right) 2^D$$
$$= 3^D \left(1 - \left(1 + \frac{D}{2}\right) \cdot \left(\frac{2}{3}\right)^D\right) \underset{(\ddagger)}{\geq} \frac{1}{2}\, 3^D,$$

where $(*)$ is by Claim 4, $(\dagger)$ is by Claims 5 and 6, and $(\ddagger)$ is because $\lim_{D\to\infty} \left(1 + \frac{D}{2}\right)\left(\frac{2}{3}\right)^D = 0$. Thus,

$$\log_2 |X_D^\square(\mathbf{e})| \quad \geq \quad \log_2 \left(\frac{1}{2}\, 3^D\right) \quad = \quad D \cdot \log_2(3) - 1. \tag{A3}$$

Now, $\mathbf{e} \in \mathrm{Maj}(X_D^\square)$ because $X_D^\square$ is McGarvey by Proposition 4.7. Thus

$$h(X_D^\square) \;\geq\; \frac{\log_2 |X_D^\square(\mathbf{e})|}{\log_2 |X_D^\square|} \;\underset{(*)}{\geq}\; \frac{D \cdot \log_2(3) - 1}{\log_2 |3^D|} \;=\; \frac{D \cdot \log_2(3) - 1}{D \cdot \log_2(3)} \;\xrightarrow[D\to\infty]{}\; 1,$$

where $(*)$ is by eqn.(A3) and Claim 5. Thus, $X_D^\square$ is asymptotically totally indeterminate. $\square$

References

Barthélémy, J.-P., 1989. Social welfare and aggregation procedures: combinatorial and algorithmic aspects. In: Applications of combinatorics and graph theory to the biological and social sciences. Vol. 17 of IMA Vol. Math. Appl. Springer, New York, pp. 39–73.

Barthélémy, J.-P., Janowitz, M. F., 1991. A formal theory of consensus. SIAM J. Discrete Math. 4 (3), 305–322.

Barthélémy, J.-P., Monjardet, B., 1981. The median procedure in cluster analysis and social choice theory. Math. Social Sci. 1 (3), 235–267.

Barthélémy, J.-P., Monjardet, B., 1988. The median procedure in data analysis: new results and open problems. In: Classification and related methods of data analysis (Aachen, 1987). North-Holland, Amsterdam, pp. 309–316.

Condorcet, M. d., 1785. Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix. Paris.

Cover, T. M., Thomas, J. A., 1991. Elements of Information Theory. John Wiley & Sons, New York.

Dietrich, F., List, C., 2010. Majority voting on restricted domains. Journal of Economic Theory 145 (2), 512–543.

Dokow, E., Holzman, R., 2010. Aggregation of binary evaluations. J.Econ.Theory 145, 495–511.

Duggan, J., 2007. A systematic approach to the construction of non-empty choice sets. Social Choice and Welfare 28, 491–506.

Duggan, J., 2010. Notes on social choice rules. (preprint).

Guilbaud, G. T., 1966. Theories of the general interest, and the logical problem of aggregation. In: Lazarsfeld, P. F., Henry, N. W. (Eds.), Readings in mathematical social science. MIT press, Cambridge, MA, pp. 262–307.

Kemeny, J. G., Fall 1959. Math without numbers. Daedalus 88, 571–591.

Kornhauser, L., Sager, L., 1986. Unpacking the court. Yale Law Journal 96, 82–117.

Lindner, T., Nehring, K., Pivato, M., Puppe, C., 2010. Allocating public goods via the midpoint rule. (preprint).

List, C., 2004. A model of path-dependence in decisions over multiple propositions. American Political Science Review 98 (3), 495–513.

List, C., Pettit, P., 2002. Aggregating sets of judgements: An impossibility result. Economics and Philosophy 18, 89–110.

List, C., Puppe, C., 2009. Judgement aggregation: a survey. In: Anand, P., Pattanaik, P., Puppe, C. (Eds.), Oxford handbook of rational and social choice. Oxford University Press, Oxford, UK., Ch. 19, pp. 457–481.

May, K., 1952. A set of independent, necessary, and sufficient conditions for simple majority decision. Econometrica 20, 680–684.

McGarvey, D. C., 1953. A theorem on the construction of voting paradoxes. Econometrica 21, 608–610.

McKelvey, R., 1979. General conditions for global intransitivities in formal voting models. Econometrica 47 (5), 1085–1112.

McLean, I., 1990. The Borda and Condorcet principles: Three medieval applications. Social Choice and Welfare 7, 99–108.

Miller, A., 2009. A model of community standards. (preprint).

Moulin, H., 1988. Axioms of cooperative decision making. Cambridge University Press, Cambridge, UK.

Nehring, K., Pivato, M., 2011a. Incoherent majorities: The McGarvey problem in judgement aggregation. Discrete Applied Mathematics (to appear).
URL `http://dx.doi.org/10.1016/j.dam.2011.03.014`

Nehring, K., Pivato, M., 2011b. Majority rule without a majority. (preprint).

Nehring, K., Puppe, C., 2007. The structure of strategy-proof social choice I: General characterization and possibility results on median spaces. J.Econ.Theory 135, 269–305.

Nehring, K., Puppe, C., 2010. Abstract arrowian aggregation. J.Econ.Theory 145, 467–494.

Pivato, M., 2009. Geometric models of consistent judgement aggregation. Social Choice and Welfare 33 (4), 559–574.

Sen, A. K., 1970. Collective Choice and Social Welfare. Holden Day, San Francisco.

Shepsle, K., Weingast, B., 1984. Uncovered sets and sophisticated voting outcomes, with implications for agenda institutions. American Journal of Political Science 28 (1), 49–75.

Slater, P., 1961. Inconsistencies in a schedule of paired comparisons. Biometrica 48, 303–312.

Smith, J., 1973. Aggregation of preferences with variable elecorate. Econometrica 41, 1027–1041.

Szele, T., 1943. Kombinatorische Untersuchungen über den gerichteten vollständigen Graphen. Mat. Fiz. Lapok 50, 223–256.

van de Vel, M. L. J., 1993. Theory of convex structures. Vol. 50 of North-Holland Mathematical Library. North-Holland Publishing Co., Amsterdam.

Young, H. P., 1986. Optimal ranking and choice from pairwise decisions. In: Grofman, B., Owen, G. (Eds.), Information pooling and group decision making. JAI press, Greenwich, CT, pp. 113–122.

Young, H. P., 1988. Condorcet's theory of voting. American Political Science Review 82 (4), 1231–1244.

Young, H. P., Winter 1995. Optimal voting rules. Journal of Economic Perspectives 9 (1), 51–64.

Young, H. P., 1997. Group choice and individual judgments. In: Mueller, D. C. (Ed.), Perspectives on public choice: A handbook. Cambridge UP, Cambridge, UK, Ch. 9, pp. 181–200.

Young, H. P., Levenglick, A., 1978. A consistent extension of Condorcet's election principle. SIAM J. Appl. Math. 35 (2), 285–300.