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Abstract: Any demand equation satisfying Lau's (1982) *Fundamental Theorem of Exact Aggregation* and is 0° homogeneous in prices and income will have a Gorman (1981) functional form for each income term. This property does not depend on symmetry or adding up. The implications of this result are illustrated by an extensive example.

Keywords: Demand, exact aggregation, functional form, homogeneity

Running Head: Homogeneity and Aggregation

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Lau's (1982) *Fundamental Theorem of Exact Aggregation* states that when income and demographics vary across agents, then the demand equations have a finitely additive and multiplicatively separable structure,[1]

$$q = \sum_{k=1}^{K} \alpha_k(\boldsymbol{p}) h_k(m, \boldsymbol{s}). \tag{1}$$

In (1), $q > 0$ is the quantity demanded for a single good, $\boldsymbol{p} \in \mathbb{R}^n_{++}$ is the vector of market prices for all goods, $m > 0$ is consumption expenditure (*income*), $\boldsymbol{s} \in \mathbb{R}^S$ is a vector of demographic variables and other demand shifters (*demographics*). The functions $\alpha_k : \mathbb{R}^n_{++} \to \mathbb{R}, k = 1, \cdots, K,$ are assumed to be smooth and linearly independent functions of prices and the functions $h_k : \mathbb{R}_{++} \times \mathbb{R}^S \to \mathbb{R}, k = 1, \cdots, K,$ are smooth[2] and linearly independent functions of income and demographics.

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[1] To see that (1) is *sufficient* for exact aggregation over income and demographics, denote the distribution function over $(m, \boldsymbol{s}) \in \mathbb{R}_{++} \times \mathbb{R}^S$ by $\Phi : \mathbb{R}_{++} \times \mathbb{R}^S \to [0,1].$ Then

$$\begin{aligned} \bar{q}(\boldsymbol{p}, \boldsymbol{\theta}) &= \int_{\mathcal{M} \times \mathcal{S}} q(\boldsymbol{p}, m, \boldsymbol{s}) d\Phi(m, \boldsymbol{s}) \\ &= \sum_{k=1}^{K} \alpha_k(\boldsymbol{p}) \int_{\mathcal{M} \times \mathcal{S}} h_k(m, \boldsymbol{s}) d\Phi(m, \boldsymbol{s}) \\ &= \sum_{k=1}^{K} \alpha_k(\boldsymbol{p}) \bar{h}_k(\boldsymbol{\theta}), \end{aligned}$$

where $\boldsymbol{\theta}$ is a vector of parameters for the joint distribution function $\Phi(m, \boldsymbol{s}).$ Gorman (1981) and Lau (1982) use very different techniques to show that (1) is *necessary*. Though he does not analyze $\boldsymbol{s}$ as an argument of the demand equations, Gorman (1981, p. 8) notes in passing that the $\{h_k\}$ can include $\boldsymbol{s}$.

[2] A smooth function can be differentiated any number of times. This assumption simply avoids the need to keep track of the number derivatives that have been with little loss in generality.

This letter analyzes the restrictions implied by zero degree homogeneity in prices and income on the functions $\{h_k\}$. We show that this property implies the Gorman (1981) class of functional forms for the income terms. We focus on one demand equation that is exactly aggregable and 0° homogeneous. As a consequence, symmetry, adding up, and the rank of the demand equations (Gorman 1981; Lewbel 1990) play no role in establishing the following result.

Proposition: *If the demand for q takes the form,* $q = \sum_{k=1}^{K} \alpha_k(\boldsymbol{p}) h_k(m, \boldsymbol{s})$*, with K smooth, linearly independent functions of prices,* **p***, and K smooth, linearly independent functions of income and demographics,* $(m,\boldsymbol{s})$ *and is homogeneous of degree zero in* $(\boldsymbol{p}, m)$*, then* $h_k(m,\boldsymbol{s}) = g_k(\boldsymbol{s})\tilde{h}_k(m),\ k = 1,\cdots,K,$ *and each* $\tilde{h}_k$ *is*: (*i*) $m^\kappa, \kappa \in \mathbb{R}$; (*ii*) $m^\kappa (\ln m)^j, \kappa \in \mathbb{R}$, $j \in \{1,...,K\}$; (*iii*) $m^\kappa \sin(\tau \ln m)$, $m^\kappa \cos(\tau \ln m), \kappa \in \mathbb{R}, \tau \in \mathbb{R}_+$, *appearing in pairs with the same* $\{\kappa, \tau\}$ *in each pair*; *or* (*iv*) $m^\kappa (\ln m)^j \sin(\tau \ln m), m^\kappa (\ln m)^j \cos(\tau \ln m)$, $\kappa \in \mathbb{R}$, $j \in \{1,...,[½K]\}$, $\tau \in \mathbb{R}_+$, *and K≥4, appearing in pairs with the same* $\{\kappa, j, \tau\}$ *for each pair, where* $[½K]$ *is the largest integer no greater than ½K.*

***Proof*:** Homogeneity of degree zero in $(\boldsymbol{p}, m)$ is equivalent to the Euler equation

$$\sum_{k=1}^{K} \frac{\partial \alpha_k(\boldsymbol{p})}{\partial \boldsymbol{p}^\top} \boldsymbol{p} h_k(m,\boldsymbol{s}) + \sum_{k=1}^{K} \alpha_k(\boldsymbol{p}) \frac{\partial h_k(m,\boldsymbol{s})}{\partial m} m = 0. \tag{2}$$

If K=1 and $\partial h_1(m,\boldsymbol{s}) \big/ \partial m = 0$, this reduces to $\partial \alpha_1(\boldsymbol{p}) \big/ \partial \boldsymbol{p}^\top \boldsymbol{p} = 0$, so that $h_1(m,\boldsymbol{s}) = g_1(\boldsymbol{s})$ and $\alpha_1(\boldsymbol{p})$ is homogeneous of degree zero. Set $\kappa = 0$ to obtain a special case of (*i*). If K=1 and $\partial h_1(m,\boldsymbol{s}) \big/ \partial m \neq 0$, or if $K \geq 2$, neither sum in (2) can vanish without a contradiction of the linear independence of the $\{\alpha_k(\boldsymbol{p})\}_{k=1,\cdots,K}$ or of the $\{h_k(m,\boldsymbol{s})\}_{k=1,\cdots,K}$.

For any $\boldsymbol{p} \in \mathbb{R}^n_{++}$, let $\mathcal{N}(\boldsymbol{p})$ be an arbitrarily small open neighborhood of $\boldsymbol{p}$. Fix K unique vectors, $\boldsymbol{p}_\ell \in \mathcal{N}(\boldsymbol{p}),\ \ell = 1, \cdots, K,$ and define $\boldsymbol{A}(\boldsymbol{p}_1, \cdots, \boldsymbol{p}_K) = \left[\alpha_k(\boldsymbol{p}_\ell)\right]_{k,\ell=1,\cdots,K}$ and $\boldsymbol{B}(\boldsymbol{p}_1, \cdots, \boldsymbol{p}_K) = \left[\partial \alpha_k(\boldsymbol{p}_\ell) / \partial \boldsymbol{p}^\top \boldsymbol{p}_\ell\right]_{k,\ell=1,\cdots,K}$. Linear independence of the price functions implies that there exists $\{\boldsymbol{p}_\ell\}_{\ell=1,\cdots,K}$ such that $\boldsymbol{A}(\boldsymbol{p}_1, \cdots, \boldsymbol{p}_K)$ is nonsingular. Therefore,

$$\begin{aligned} \frac{\partial \boldsymbol{h}(m,\boldsymbol{s})}{\partial m} m &= -\boldsymbol{A}(\boldsymbol{p}_1, \cdots, \boldsymbol{p}_K)^{-1} \boldsymbol{B}(\boldsymbol{p}_1, \cdots, \boldsymbol{p}_K) \boldsymbol{h}(m,\boldsymbol{s}) \\ &\equiv \boldsymbol{C}(\boldsymbol{p}_1, \cdots, \boldsymbol{p}_K) \boldsymbol{h}(m,\boldsymbol{s}). \end{aligned} \tag{3}$$

Both $\partial \boldsymbol{h}(m,\boldsymbol{s}) / \partial m \cdot m$ and $\boldsymbol{h}(m,\boldsymbol{s})$ depend on $(m,\boldsymbol{s})$, but not on $\boldsymbol{p}$**.** On the other hand, $\boldsymbol{C}(\boldsymbol{p}_1, \cdots, \boldsymbol{p}_K)$ depends on $\boldsymbol{p}$, but not on $(m,\boldsymbol{s})$. Therefore, $\boldsymbol{C}(\boldsymbol{p}_1, \cdots, \boldsymbol{p}_K)$ is independent of $\boldsymbol{p}$ – that is, each element is an absolute constant.[3]

Thus, linear independence of the $\{\alpha_k(\boldsymbol{p})\}_{k=1,\cdots,K}$ implies that each $\partial \boldsymbol{h}(m,\boldsymbol{s}) / \partial m \cdot m$ is a linear function of $\{h_k(m,\boldsymbol{s})\}_{k=1,\cdots,K}$ with constant coefficients:

$$\frac{\partial h_k(m,\boldsymbol{s})}{\partial m} m = \sum_{\ell=1}^{K} c_{k,\ell} h_\ell(m,\boldsymbol{s}),\ \ k = 1, \cdots, K. \tag{4}$$

This is a system of K linear and homogeneous partial differential equations (pdes) of the form commonly known as Cauchy's differential equation.

Change variables from m to $x = \ln m$ to obtain a system of linear, homogeneous pdes with constant coefficients (Cohen 1933, pp. 124-125). Because $m(x) = e^x$, it follows that $m'(x) = m(x)$. Therefore, define $\tilde{h}_k(x,\boldsymbol{s}) \equiv h_k(m(x),\boldsymbol{s}),\ k = 1, ..., K,$ so that

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[3] To see this, simply differentiate with respect to $p_{k\ell}\ \ \forall\ k, \ell = 1, \cdots, K.$

$$\frac{\partial \tilde{h}_k(x,\boldsymbol{s})}{\partial x} = \sum_{\ell=1}^{K} c_{k,\ell} \tilde{h}_\ell(x,\boldsymbol{s}), \;\; k = 1,\cdots,K. \tag{5}$$

This system of first-order, linear, and homogeneous pdes has the characteristic equation $|\boldsymbol{C} - \lambda \boldsymbol{I}| = 0$. This is a K^{th} order polynomial in λ and the fundamental theorem of algebra (Gauss, 1799) implies that there are exactly K roots. These also are absolute constants with respect $(\boldsymbol{p}, m, \boldsymbol{s})$. Some roots may repeat and some may be complex conjugate pairs. Let the characteristic roots be denoted by $\lambda_k, \;\; k = 1,\cdots,K.$

The general solution to a single linear, homogeneous, differential equation of order K is the sum of K linearly independent particular solutions (Cohen 1933, Chapter 6; Boyce and DiPrima 1977, Chapter 5), where linear independence of the K functions, $\{f_1,\cdots,f_K\}$ of a variable x means that there is no non-vanishing K–vector, $(a_1,\cdots,a_K)$ such that $a_1 f_1 + \cdots + a_K f_K = 0$ for all values of the variables in an open neighborhood of any point $[x, f_1(x),\cdots, f_K(x)]$. Cohen, pp. 303-306 contains a statement of necessary and sufficient conditions. In our case, for the functions $\{\tilde{h}_k(x,\boldsymbol{s})\}_{k=1,\cdots,K}$, this property holds for any $\boldsymbol{s} \in \boldsymbol{S} \subset \mathbb{R}^S$.

Let there be $R \geq 0$ roots that repeat. Reorder the functions $\{\tilde{h}_k\}$ as necessary in the following way. Label the first repeating root (if one exists) as λ_1 and let its multiplicity be denoted by $M_1 \geq 1.$ Let the second repeating root (if one exists) be the $M_1 + 1^{st}$ root. Label this root as λ_2 and its multiplicity as $M_2 \geq 1.$ Continue in this manner until there are no more repeating roots. Let the total number of repeated roots be $\tilde{M} = \sum_{k=1}^{R} M_k.$

Label the remaining $K-\tilde{M}\geq 0$ unique roots as λ_k for $k=\tilde{M}+1,\cdots,K$. Then (without loss in generality – WLOG) the complete solution to can be written as

$$\tilde{h}_k(x,\boldsymbol{s})=\sum_{r=1}^{R}\left[\sum_{\ell=1}^{M_r} d_{k\ell}(\boldsymbol{s})x^{(\ell-1)}e^{\lambda_r x}\right]+\sum_{\ell=\tilde{M}+1}^{K} d_{k\ell}(\boldsymbol{s})e^{\lambda_\ell x},\ k=1,\cdots,K. \tag{6}$$

Substitute (6) into the demand for q and rearrange terms to obtain:

$$q=\sum_{r=1}^{R}\sum_{\ell=1}^{M_r}\sum_{k=1}^{K}\alpha_k(\boldsymbol{p})d_{k\ell}(\boldsymbol{s})m^{\lambda_r}(\ln m)^{\ell-1}+\sum_{\ell=\tilde{M}+1}^{K}\sum_{k=1}^{K}\alpha_k(\boldsymbol{p})d_{k\ell}(\boldsymbol{s})m^{\lambda_\ell}. \tag{7}$$

The terms in the triple sum give cases (*i*) and (*ii*), and case (*iv*) when $K\geq 4$ and at least one pair of complex conjugate roots repeats. The terms in the double sum on the far right give cases (*i*) for unique real roots and (*iii*) for unique pairs of complex conjugate roots when $K\geq 2$. ∎

Remarks:

1. Rearranging the terms in (6), the nature of this result can be stated transparently. There are *K* roots for (5), which can each be denoted generically by $\lambda_k=\kappa_k\pm\iota\tau_k,\ k=1,\cdots,K$. Some of these may have real parts that vanish, some may have complex parts that vanish, the complex parts always appear in pairs with opposing signs on the complex coefficient, and some roots may repeat. Associated with each term on the right-hand-side of (6) is a polynomial term of the form $(\ln m)^{j_k}$, $j_k\in\{0,1,2,\cdots\}$. Consequently, each income term can be written generically in the form $(\ln m)^{j_k}m^{\kappa_k\pm\iota\tau_k}$, $k=1,\cdots,K$. This gives the demand equation as

$$q=\sum_{k=1}^{K}\left[\sum_{\ell=1}^{K}\alpha_\ell(\boldsymbol{p})d_{\ell k}(\boldsymbol{s})\right](\ln m)^{j_k}m^{\kappa_k\pm\iota\tau_k}. \tag{8}$$

In this expression, the terms $d_{\ell k}(\boldsymbol{s})$ are "functions of integration" for the system of pdes in (5), they can depend on $\boldsymbol{s}$ in any number of flexible ways.

2. If instead of one demand equation there is a complete system, then Slutsky symmetry and the full rank property (see Gorman 1981 and Lewbel 1990) imply $K \le 3.$ Adding up further restricts one of the income terms to be m. Homogeneity, symmetry, adding up, and real-valued quantities demanded imply the following additional restrictions: all roots are either purely real or purely complex; power functions except $\kappa_k = 1 \ \forall \ k$ are ruled out if $j_k \ne 0$ for any k; and $j_k = 0$ for all k if $\kappa_k \pm \iota\tau_k = 1$ for any k (Gorman 1981).

An Illustration

The purpose of this section is to illuminate the implications of the above results through a flexible yet informative example. Begin by assuming that the utility function is a strictly concave quadratic function,

$$u(\boldsymbol{q},\boldsymbol{s}) = -\tfrac{1}{2}(\boldsymbol{q}-\boldsymbol{\alpha}-\boldsymbol{A}\boldsymbol{s})^{\top}\boldsymbol{B}^{-1}(\boldsymbol{q}-\boldsymbol{\alpha}-\boldsymbol{A}\boldsymbol{s}), \tag{9}$$

where $\partial u(\boldsymbol{q},\boldsymbol{s})/\partial \boldsymbol{q} = -\boldsymbol{B}^{-1}(\boldsymbol{q}-\boldsymbol{\alpha}-\boldsymbol{A}\boldsymbol{s}) \gg \boldsymbol{0}_n$, $m < (\boldsymbol{\alpha}+\boldsymbol{A}\boldsymbol{s})^{\top}\boldsymbol{p}$, $\boldsymbol{q} \in \mathbb{R}^n_{++}$ is an $n\times 1$ vector of consumption goods, $\boldsymbol{s} \in \mathbb{R}^L$ is an $L\times 1$ vector of demographic variables and other factors that affect preferences (possibly including random effects), $\boldsymbol{\alpha} \in \mathbb{R}^n$ is an $n\times 1$ vector of constants, $\boldsymbol{A}$ is an $n\times L$ matrix of constants, and $\boldsymbol{B} = \boldsymbol{B}^{\top}$ is a symmetric and positive definite matrix of constants. The demands functions that maximize (9) subject to $\boldsymbol{q} \in \mathbb{R}^n_{+}$ and $\boldsymbol{p}^{\top}\boldsymbol{q} \le m$ at an interior solution are,

$$\boldsymbol{q} = \left[\boldsymbol{I} - \left(\frac{\boldsymbol{Bpp}^{\top}}{\boldsymbol{p}^{\top}\boldsymbol{Bp}}\right)\right](\boldsymbol{\alpha} + \boldsymbol{As}) + \left(\frac{\boldsymbol{Bp}}{\boldsymbol{p}^{\top}\boldsymbol{Bp}}\right)m. \tag{10}$$

For any joint distribution for $(\boldsymbol{s}, m)$, this system satisfies Lau's conditions for *exact aggregation* and Gorman's conditions for the admissible functional forms for *m*. There are *L*+2 independent functions of $\boldsymbol{s}$ and *m*, $\{1, \boldsymbol{s}^{\top}, m\}$, and 2 independent functions of *m*, $\{1, m\}$. The indirect utility function is

$$v(\boldsymbol{p}, m, \boldsymbol{s}) = -\tfrac{1}{2}\frac{\left[m - (\boldsymbol{\alpha} + \boldsymbol{As})^{\top}\boldsymbol{p}\right]^{2}}{\boldsymbol{p}^{\top}\boldsymbol{Bp}}. \tag{11}$$

This model can be extended in a number of ways, while maintaining the structures implied by both Gorman and Lau. We begin with a series of transformations of the utility index. The first transformation, $\hat{v} = -\sqrt{-2v}$, generates the Gorman Polar Form (Gorman 1961) for quadratic utility,

$$\hat{v}(\boldsymbol{p}, m, \boldsymbol{s}) = \frac{\left[m - (\boldsymbol{\alpha} + \boldsymbol{As})^{\top}\boldsymbol{p}\right]}{\sqrt{\boldsymbol{p}^{\top}\boldsymbol{Bp}}}. \tag{12}$$

The negative square root is the economically relevant one since $m < (\boldsymbol{\alpha} + \boldsymbol{As})^{\top}\boldsymbol{p}$ for *u* to be increasing in $\boldsymbol{q}$, equivalently, for *v* to be increasing in *m*. This representation of *v* is equivalent to

$$\hat{u}(\boldsymbol{q}, \boldsymbol{s}) = -\sqrt{-2u(\boldsymbol{q}, \boldsymbol{s})} = -\sqrt{(\boldsymbol{q} - \boldsymbol{\alpha} - \boldsymbol{As})^{\top}\boldsymbol{B}^{-1}(\boldsymbol{q} - \boldsymbol{\alpha} - \boldsymbol{As})}. \tag{13}$$

The second transformation, $\tilde{v} = -1/\hat{v}$, is due to Howe, Pollak, and Wales (1979),

$$\tilde{v}(\boldsymbol{p}, m, \boldsymbol{s}) = \frac{-\sqrt{\boldsymbol{p}^{\top}\boldsymbol{Bp}}}{\left[m - (\boldsymbol{\alpha} + \boldsymbol{As})^{\top}\boldsymbol{p}\right]}. \tag{14}$$

This representation of indirect preferences is equivalent to the following representation for direct preferences,

$$\tilde{u}(\boldsymbol{q},\boldsymbol{s}) = \frac{1}{\sqrt{(\boldsymbol{q}-\boldsymbol{\alpha}-\boldsymbol{A}\boldsymbol{s})^{\top}\boldsymbol{B}^{-1}(\boldsymbol{q}-\boldsymbol{\alpha}-\boldsymbol{A}\boldsymbol{s})}}. \tag{15}$$

It is convenient to work with (14) to extend the demand model.

The next step follows from the fact that (14) remains 0° homogeneous after a change in coordinates from $(\boldsymbol{p},m)$ to $(\boldsymbol{p}^{\kappa},m^{\kappa})$, $\kappa \neq 0$, with $\boldsymbol{p}^{\kappa} \equiv [p_1^{\kappa}\, p_2^{\kappa} \cdots p_n^{\kappa}]^{\top}$. This generates the Price Independent Generalized Linear (PIGL) functional form for indirect preferences (Muellbauer 1975, 1976),[4]

$$v(\boldsymbol{p},m,\boldsymbol{s}) = \frac{-\sqrt{(\boldsymbol{p}^{\kappa})^{\top}\boldsymbol{B}\boldsymbol{p}^{\kappa}}}{\left[m^{\kappa}-(\boldsymbol{\alpha}+\boldsymbol{A}\boldsymbol{s})^{\top}\boldsymbol{p}^{\kappa}\right]}. \tag{16}$$

Alternatively, replacing $(\boldsymbol{p},m)$ with $(\ln\boldsymbol{p},\ln m)$ in (14) generates the Price Independent Generalized Logarithmic (PIGLOG) functional form as a nonstandard translog indirect utility function (Christensen, Jorgenson, and Lau 1975),

$$v(\boldsymbol{p},m,\boldsymbol{s}) = \frac{-\sqrt{\ln\boldsymbol{p}^{\top}\boldsymbol{B}\ln\boldsymbol{p}}}{\left[\ln m-(\boldsymbol{\alpha}+\boldsymbol{A}\boldsymbol{s})^{\top}\ln\boldsymbol{p}\right]}. \tag{17}$$

The following parameter restrictions are necessary for 0° homogeneity in (17): $\boldsymbol{\alpha}^{\top}\boldsymbol{\iota}=1$; $\boldsymbol{A}^{\top}\boldsymbol{\iota}=\boldsymbol{0}_L$; and $\boldsymbol{B}\boldsymbol{\iota}=\boldsymbol{0}_n$.[5] Since $\boldsymbol{B}=\boldsymbol{B}^{\top}$, applying these restrictions to (17) gives

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[4] After this change in coordinates, a closed form expression for $u(\boldsymbol{q},\boldsymbol{s})$ no longer exists for any $\kappa \neq 1$.

[5] These restrictions are usually applied to the Almost Ideal Demand System (Deaton and Muellbauer 1980) and the exactly aggregable form of translog indirect preferences (Jorgenson 1990) in order to satisfy the 0° homogeneity and adding up conditions.

$$v(\boldsymbol{p},m,\boldsymbol{s})=\frac{-\sqrt{(\boldsymbol{p}^{\kappa})^{\mathsf{T}}\boldsymbol{B}\boldsymbol{p}^{\kappa}}}{\left[m^{\kappa}-(\boldsymbol{\alpha}+\boldsymbol{A}\boldsymbol{s})^{\mathsf{T}}\boldsymbol{p}^{\kappa}\right]}=\frac{-\sqrt{\left(\dfrac{\boldsymbol{p}^{\kappa}-\boldsymbol{\iota}}{\kappa}\right)^{\mathsf{T}}\boldsymbol{B}\left(\dfrac{\boldsymbol{p}^{\kappa}-\boldsymbol{\iota}}{\kappa}\right)}}{\left[\left(\dfrac{m^{\kappa}-1}{\kappa}\right)-(\boldsymbol{\alpha}+\boldsymbol{A}\boldsymbol{s})^{\mathsf{T}}\left(\dfrac{\boldsymbol{p}^{\kappa}-\boldsymbol{\iota}}{\kappa}\right)\right]},\ \forall\ \kappa\neq 0. \tag{18}$$

Since the limit as $\kappa\to 0$ of the far right-hand-side of (18) is the right-hand-side of (17), all PIGL and PIGLOG functional forms are nested within a single specification for the indirect utility function.

The last step is to extend the demand model to full rank three, which is the highest possible rank (Gorman 1981), and obtain a nested set of generalized PIGL and PIGLOG complete demand systems. This step involves choosing a function $\boldsymbol{\delta}:\mathbb{R}^{L}\to\mathbb{R}^{n}$ to satisfy $\boldsymbol{\iota}^{\mathsf{T}}\boldsymbol{\delta}(\boldsymbol{s})\equiv 0$. To simplify notation, let $\boldsymbol{x}(\boldsymbol{p})=(\boldsymbol{p}^{\kappa}-\boldsymbol{\iota})/\kappa$, let $y(m)=(m^{\kappa}-1)/\kappa$, and let $\boldsymbol{\Delta}(x_i)$ denote an $n\times n$ diagonal matrix with x_i as the i^{th} main diagonal element. Now define the indirect utility function by (see Howe, Pollak, and Wales 1979)

$$v(\boldsymbol{p},m,\boldsymbol{s})=-\frac{\sqrt{\boldsymbol{x}(\boldsymbol{p})^{\mathsf{T}}\boldsymbol{B}\,\boldsymbol{x}(\boldsymbol{p})}}{\left[y(m)-(\boldsymbol{\alpha}+\boldsymbol{A}\boldsymbol{s})^{\mathsf{T}}\boldsymbol{x}(\boldsymbol{p})\right]}-\frac{\boldsymbol{\delta}(\boldsymbol{s})^{\mathsf{T}}\boldsymbol{x}(\boldsymbol{p})}{\sqrt{\boldsymbol{x}(\boldsymbol{p})^{\mathsf{T}}\boldsymbol{B}\,\boldsymbol{x}(\boldsymbol{p})}}. \tag{19}$$

Applying Roy's Identity to (19) and carrying out a little tedious but straightforward algebra gives

$$\begin{aligned}\boldsymbol{q}=m^{1-\kappa}\Delta(p_i^{\kappa-1})\Bigg\{&\boldsymbol{\alpha}+\boldsymbol{A}\boldsymbol{s}+\left(\frac{\boldsymbol{B}\boldsymbol{x}(\boldsymbol{p})}{\boldsymbol{x}(\boldsymbol{p})^{\mathsf{T}}\boldsymbol{B}\boldsymbol{x}(\boldsymbol{p})}\right)\left[y(m)-(\boldsymbol{\alpha}+\boldsymbol{A}\boldsymbol{s})^{\mathsf{T}}\boldsymbol{x}(\boldsymbol{p})\right]\\&+\left[\boldsymbol{I}-\left(\frac{\boldsymbol{B}\boldsymbol{x}(\boldsymbol{p})\boldsymbol{x}(\boldsymbol{p})^{\mathsf{T}}}{\boldsymbol{x}(\boldsymbol{p})^{\mathsf{T}}\boldsymbol{B}\boldsymbol{x}(\boldsymbol{p})}\right)\right]\boldsymbol{\delta}(\boldsymbol{s})\frac{\left[y(m)-(\boldsymbol{\alpha}+\boldsymbol{A}\boldsymbol{s})^{\mathsf{T}}\boldsymbol{x}(\boldsymbol{p})\right]^{2}}{\boldsymbol{x}(\boldsymbol{p})^{\mathsf{T}}\boldsymbol{B}\boldsymbol{x}(\boldsymbol{p})}\Bigg\}.\end{aligned} \tag{20}$$

Distributing $m^{1-\kappa}$ within the expression in braces reveals that there are three independent functions of m, $\{m^{1-\kappa}, m, m^{1+\kappa}\}$ when $\kappa \neq 0$ and $\{m, m\ln m, m(\ln m)^2\}$ when $\kappa = 0$. Thus, depending on the value of the parameter $\kappa \in \mathbb{R}$, this demand system exhausts the full set of non-transcendental full rank 3 functional forms (Lewbel 1990). Note, however, that (20) is extremely flexible in $\boldsymbol{s}$ and its interactions with $\boldsymbol{p}$ and $\boldsymbol{m}$.

There are three sources of flexibility and generality in this type of choice for the functional form of a demand system: (1) the rank of the model can be anything from one to three, depending on the values of the estimated model parameters; (2) the exponent $\kappa \in \mathbb{R}$ creates flexible curvature of the demands with respect to $\boldsymbol{p}$ and m; and (3) the definition of $\boldsymbol{s}$ could include virtually any function of random preference parameters, demographic variables, or other variables that shift or rotate the demand functions, such as logarithms, powers, and interactions between a set of primitive variables. As a result, this class of demand models is extremely flexible in such variables. Indeed, this should be the case since economic theory says nothing about how they should affect preferences.

Thus, while the joint results of Gorman (1981) and Lau (1982) have strong – perhaps even stark – implications for the choice of functional form in a system of demand equations, the general class of admissible models can accommodate an extremely flexible and general set of demand models.

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